Revista Colombiana de Matemáticas
Volumen 52(2018)1, páginas 65-86

Two Posets of Noncrossing Partitions Coming From Undesired Parking Spaces

Dos posets de particiones sin cruces provenientes de espacios de parqueo prohibidos

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Abstract. Consider the noncrossing set partitions of an n-element set which, either do not use the block $\{n-1, n\}$ or which do not use both the singleton block $\{n\}$ and a block containing 1 and $n-1$. In this article we study the subposet of the noncrossing partition lattice induced by these elements, and show that it is a supersolvable lattice, and therefore lexicographically shellable. We give a combinatorial model for the NBB bases of this lattice and derive an explicit formula for the value of its Möbius function between least and greatest element.

This work is motivated by a recent article by M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo, and I. Nicolas, in which they introduce a subposet of the noncrossing partition lattice that is determined by parking functions with certain forbidden entries. In particular, they conjecture that the resulting poset always has a contractible order complex. We prove this conjecture by embedding their poset into ours, and showing that it inherits the lexicographic shellability.

Key words and phrases. noncrossing partition, supersolvable lattice, left-modular lattice, parking function, lexicographic shellability, NBB base, Möbius function.

2010 Mathematics Subject Classification. 05A18, 06A07.

Resumen. Considere las particiones sin cruces de un conjunto de n elementos que no usan el bloque $\{n-1, n\}$, ni usan a la vez el bloque $\{n\}$ y un bloque que contenga a 1 y $n-1$. En este artículo estudiamos el subposet del retículo de particiones sin cruces inducido por estos elementos. Probamos que este

retículo es supersoluble, y por lo tanto es lexicográficamente descascarable. También damos un modelo combinatorio de las bases NBB de este retículo y derivamos una fórmula explicita para el valor de su función de Möbius entre el elemento mínimo y el máximo.

Este trabajo es motivado por un artículo reciente de M. Bruce, M. Dougherty, M. Hlavacek, R. Kudo, e I. Nicolas en el cual introducen un subposet del retículo de particiones sin cruces que es determinado por funciones de parqueo con ciertas entradas prohibidas. En particular, ellos conjeturan que el poset resultante siempre tiene un complejo de orden contráctil. En este artículo probamos esta conjetura, sumergiendo su poset en el nuestro y mostrando que esta inmersión hereda la descascarabilidad lexicográfica.

Palabras y frases clave. Particiones sin cruces, retículo supersoluble, retículo modular izquierdo, funciones de parqueo, descascarabilidad lexicográfica, bases NBB, función Möbius.

1. Introduction

A set partition of $[n] = \{1, 2, \dots, n\}$ is noncrossing if there are no indices $i < j < k < l$ such that i, k and j, l belong to distinct blocks. Let us denote the set of all noncrossing set partitions by NC_n. We can partially order noncrossing set partitions by dual refinement, meaning that $\mathbf{x} \in NC_n$ is smaller than $\mathbf{y} \in NC_n$ if every block of $\mathbf{x}$ is contained in a block of $\mathbf{y}$. Let us denote this partial order by $\leq_{\mathrm{dref}}$.

The lattice $(NC_n, \leq_{\mathrm{dref}})$ of noncrossing set partitions is a remarkable poset with a rich combinatorial structure. It was introduced by G. Kreweras in the early 1970s [9], and has gained a lot of attention since then. It has, among other things, surprising ties to group theory, algebraic topology, representation theory of the symmetric group, and free probability. See [14] and [11] for surveys on these lattices.

A parking function of length n is a function on an n-element set with the property that the preimage of $[k]$ has at least k elements for every $k \leq n$. They were introduced in [8], and play an important role in the study of the spaces of diagonal harmonics, see [6] and [5, Chapter 5].

The maximal chains of $(NC_n, \leq_{\mathrm{dref}})$ are naturally in bijection with parking functions of length $n-1$, see [17]. This connection was used in [4] to define a subposet of $(NC_n, \leq_{\mathrm{dref}})$ as follows. Fix some $k \leq n$ and take the set of all parking functions which do not have k in their image, but every value larger than k. Let us consider the poset $(PE_{n,k}, \leq_{\mathrm{pchn}})$, which is the subposet of $(NC_n, \leq_{\mathrm{dref}})$ determined by the maximal chains corresponding to these parking functions. In the case where $n = k$ we simply write $(PE_n, \leq_{\mathrm{pchn}})$. For $n \leq 2$, the poset $(PE_n, \leq_{\mathrm{pchn}})$ is the empty poset.

Let $\mathbf{0}$ denote the discrete partition into singleton blocks, and let $\mathbf{1}$ denote the full partition into a single block. It is the statement of [4, Theorem C] that the Möbius function of $(PE_{n,k}, \leq_{\mathrm{pchn}})$ always vanishes between $\mathbf{0}$ and $\mathbf{1}$. It was

moreover conjectured there that the order complex of $(PE_{n,k}, \leq_{\mathrm{pchn}})$ with $\mathbf{0}$ and $\mathbf{1}$ removed is contractible. The main purpose of this article is to prove this conjecture.

In fact we prove the following result stating that $(PE_n, \leq_{\mathrm{pchn}})$ is lexicographically shellable.

Theorem 1.1. *For $n \geq 3$ the poset $(PE_n, \leq_{\mathrm{pchn}})$ is lexicographically shellable.*

The fact that the order complex of $(PE_n, \leq_{\mathrm{pchn}})$ with $\mathbf{0}$ and $\mathbf{1}$ removed is contractible, is then an immediate corollary of Theorem 1.1 and [4, Theorem C].

Corollary 1.2. *For $n \geq 3$ the order complex of $(PE_n, \leq_{\mathrm{pchn}})$ with $\mathbf{0}$ and $\mathbf{1}$ removed is contractible.*

Theorem 3.5 in [4] states that $(PE_{n,k}, \leq_{\mathrm{pchn}})$ is isomorphic to the direct product of $(PE_k, \leq_{\mathrm{pchn}})$ and the Boolean lattice of rank $n-k$. Since the latter is known to be lexicographically shellable [1, Theorem 3.7], and lexicographic shellability is preserved under taking direct products [1, Theorem 4.3], Theorem 1.1 indeed suffices to resolve the main conjecture of [4].

In order to prove Theorem 1.1, we take a detour through a slightly larger subposet of $(NC_n, \leq_{\mathrm{dref}})$. In fact, we consider the induced subposet $(PE_n, \leq_{\mathrm{dref}})$, and show that it is a supersolvable lattice.

Theorem 1.3. *For $n \geq 3$ the poset $(PE_n, \leq_{\mathrm{dref}})$ is a supersolvable lattice.*

It is well known that supersolvable lattices possess an edge-labeling that implies their lexicographic shellability [1, Theorem 3.7]. The last step in proving Theorem 1.1 is to show that the restriction of this edge-labeling to $(PE_n, \leq_{\mathrm{pchn}})$ retains its crucial properties. Observe that for $n \geq 5$, the poset $(PE_n, \leq_{\mathrm{pchn}})$ is not a lattice.

We remark that the edge-labeling coming from Theorem 1.3 differs from the usual labeling of $(NC_n, \leq_{\mathrm{dref}})$, which is defined as follows. If $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}$, then there are two blocks B, B' in $\mathbf{x}$ that are joined in $\mathbf{y}$. If the smallest element of B is smaller than the smallest element of B', then the label of this cover relation is n minus the largest element of B that is smaller than every element in B'. The restriction of this labeling to $(PE_n, \leq_{\mathrm{pchn}})$ does, however, not have the necessary properties to guarantee lexicographic shellability.

The last main result of this article is the explicit computation of the value of the Möbius function in $(PE_n, \leq_{\mathrm{dref}})$ between $\mathbf{0}$ and $\mathbf{1}$.

Theorem 1.4. *For $n \geq 3$ we have*

$$\mu_{(PE_n, \leq_{\mathrm{dref}})}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1} \frac{4}{n} \binom{2n-5}{n-4},$$

which is [15, A099376] up to sign.

We prove Theorem 1.4 by using A. Blass and B. Sagan's NBB bases [3]. In fact we give a combinatorial model in terms of trees for these NBB bases, from which we derive their enumeration.

The rest of the article is organized as follows. In Section 2 we recall the necessary lattice- and poset-theoretic notions (Section 2.1), and formally define noncrossing set partitions (Section 2.2). In Section 3 we define the poset $(PE_n, \leq_{\mathrm{dref}})$, and prove Theorem 1.3 (Section 3.1), and Theorem 1.4 (Section 3.2). In Section 4 we turn our attention to the poset $(PE_n, \leq_{\mathrm{pchn}})$ and conclude the proof of Theorem 1.1.

2. Preliminaries

2.1. Posets and Lattices

Let $\mathcal{L} = (L, \leq)$ be a finite partially ordered set (*poset* for short). If $\mathcal{L}$ has a least and a greatest element (usually denoted by $\hat{0}$ and $\hat{1}$, respectively), then $\mathcal{L}$ is *bounded*. If any two elements $x, y \in L$ have a least upper bound (their *join*; denoted by $x \vee y$) and a greatest lower bound (their *meet*; denoted by $x \wedge y$), then $\mathcal{L}$ is a *lattice*.

An element $y \in L$ *covers* another element $x \in L$ if $x < y$ and for all $z \in L$ with $x \leq z \leq y$ we have $x = z$ or $z = y$. We then write $x \lessdot y$, and we sometimes say that (x, y) is a *cover relation*. If $\mathcal{L}$ has a least element, then any element covering it is an *atom*.

A *chain* is a subset $X \subseteq L$ that can be written as $C = \{x_1, x_2, \dots, x_k\}$ such that $x_1 \leq x_2 \leq \cdots \leq x_k$. A chain is *saturated* if it can be written as $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$. A saturated chain is *maximal* if it contains a minimal and a maximal element of $\mathcal{L}$. Let $\mathscr{C}(\mathcal{L})$ denote the set of maximal chains of $\mathcal{L}$.

The *rank* of $\mathcal{L}$ is one less than the maximum size of a maximal chain; denoted by $\mathrm{rk}(\mathcal{L})$. We say that $\mathcal{L}$ is *graded* if all maximal chains have the same size. An *interval* of $\mathcal{L}$ is a set $[x, y] = \{z \mid x \leq z \leq y\}$.

Two lattice elements $x, z \in L$ form a *modular pair* if for all $y \leq z$ it holds that $(y \vee x) \wedge z = y \vee (x \wedge z)$; we then usually write xMz. Moreover, $x \in L$ is *left-modular* if xMz for all $z \in L$. If x satisfies both xMz and zMx for all $z \in L$, then x is *modular*. A maximal chain is *(left-)modular* if it consists entirely of (left-)modular elements.

A lattice is *modular* if all its elements are modular, and it is *left-modular* if it contains a left-modular chain. A lattice is *supersolvable* if it contains a maximal chain M with the property that for every chain C the sublattice generated by M and C is distributive. (In other words, the smallest sublattice containing M and C is distributive.) Chains with this property are called *M-chains*. It follows from [16, Proposition 2.1] that every element of an M-chain is modular, and supersolvable lattices are therefore left-modular. For graded lattices, these two notions actually coincide.

Theorem 2.1. ([12, Theorem 2]) *A finite graded lattice is left-modular if and only if it is supersolvable.*

For any bounded poset $\mathcal{L} = (L, \leq)$ let $\mathscr{H}(\mathcal{L}) = \{(x, y) \mid x \lessdot y\}$ denote the set of cover relations of $\mathcal{L}$. An *edge-labeling* of $\mathcal{L}$ is a map $\lambda : \mathscr{H}(\mathcal{L}) \to \Lambda$, for some poset $(\Lambda, \prec)$. For a saturated chain $C = \{x_1, x_2, \ldots, x_k\}$ we denote by $\lambda(C) = \big(\lambda(x_1, x_2), \lambda(x_2, x_3), \ldots, \lambda(x_{k-1}, x_k)\big)$ the associated sequence of edge-labels. We then say that C is *rising* if $\lambda(C)$ is strictly increasing with respect to $\prec$. An edge-labeling of $\mathcal{L}$ is an *EL-labeling* if the following two conditions hold for every interval $[x, y]$ of $\mathcal{L}$: (i) there exists a unique rising maximal chain C in $[x, y]$, and (ii) for every other maximal chain C' of $[x, y]$ we have that $\lambda(C)$ is lexicographically smaller than $\lambda(C')$. A poset that admits an EL-labeling is *EL-shellable*.

Theorem 2.2 ([10])**.** *Let $\mathcal{L} = (L, \leq)$ be a left-modular lattice of length n with left-modular chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$. The labeling*

$$\lambda(y, z) = \min\{i \mid y \vee x_i \wedge z = z\} \tag{1}$$

is an EL-labeling of $\mathcal{L}$.

The existence of an EL-labeling of $\mathcal{L}$ has further implications on the homotopy type of the *order complex* associated to $\mathcal{L}$, i.e. the simplicial complex whose faces are the chains of $\mathcal{L}$.

Theorem 2.3. [2, Theorem 5.9] *Let $\mathcal{L}$ be a bounded graded poset of rank n with $\mu(\hat{0}, \hat{1}) = k$. If $\mathcal{L}$ is EL-shellable, then the order complex of $\mathcal{L}$ with $\hat{0}$ and $\hat{1}$ removed has the homotopy type of a wedge of $|k|$-many $(n-2)$-dimensional spheres. Moreover, k is precisely the number of maximal chains of $\mathcal{L}$ with weakly decreasing label sequence.*

2.2. Noncrossing Set Partitions

A *set partition* of n is a covering $\mathbf{x} = \{B_1, B_2, \ldots, B_s\}$ of $[n]$ into non-empty, mutually disjoint sets; which we call *blocks*. Let Π_n denote the set of all set partitions of n. For $i, j \in [n]$ and $\mathbf{x} \in \Pi_n$ we write $i \sim_{\mathbf{x}} j$ if there is $B \in \mathbf{x}$ with $i, j \in B$. It is easily seen that $\sim_{\mathbf{x}}$ is an equivalence relation; in fact set partitions of $[n]$ and equivalence relations on $[n]$ are in bijection. Let $\mathbf{0}$ be the discrete partition which consists of n singleton blocks, and let $\mathbf{1}$ be the full partition which consists only of a single block.

A set partition $\mathbf{x}$ is *noncrossing* if for any four indices $1 \leq i < j < k < l \leq n$ the relations $i \sim_{\mathbf{x}} k$ and $j \sim_{\mathbf{x}} l$ imply $i \sim_{\mathbf{x}} j$. Let NC_n denote the set of noncrossing set partitions of n.

Set partitions can be partially ordered as follows. Let $\mathbf{x}, \mathbf{x}' \in \Pi_n$, and say that $\mathbf{x} = \{B_1, B_2, \ldots, B_s\}$ and $\mathbf{x}' = \{B'_1, B'_2, \ldots, B'_{s'}\}$. We have $\mathbf{x} \leq_{\text{dref}} \mathbf{x}'$ if

and only if for each $i \in [s]$ there exists $i' \in [s']$ such that $B_i \subseteq B'_{i'}$. We call $\leq_{\mathrm{dref}}$ the *dual refinement order*. It is quickly verified that $\mathbf{0}$ is the least element and $\mathbf{1}$ is the greatest element with respect to this partial order. Figure 1 shows the poset $(\Pi_4, \leq_{\mathrm{dref}})$; the non-highlighted part indicates the subposet $(NC_4, \leq_{\mathrm{dref}})$. We have omitted braces in the labeling of the vertices, and have separated blocks by vertical lines instead.

The posets $(\Pi_n, \leq_{\mathrm{dref}})$ and $(NC_n, \leq_{\mathrm{dref}})$ are in fact lattices, and we can explicitly describe the meet and join operations. The meet of two set partitions $\mathbf{x}, \mathbf{x}' \in \Pi_n$ is

$$\mathbf{x} \wedge_\Pi \mathbf{x}' = \{B \cap B' \mid B \in \mathbf{x}, B' \in \mathbf{x}', \text{ and } B \cap B' \neq \varnothing\}. \tag{2}$$

In order to describe the join of $\mathbf{x}$ and $\mathbf{x}'$, consider the bipartite graph

$$\mathbf{P}_{\mathbf{x},\mathbf{x}'} = \big([n] \uplus (\mathbf{x} \cup \mathbf{x}'), E\big),$$

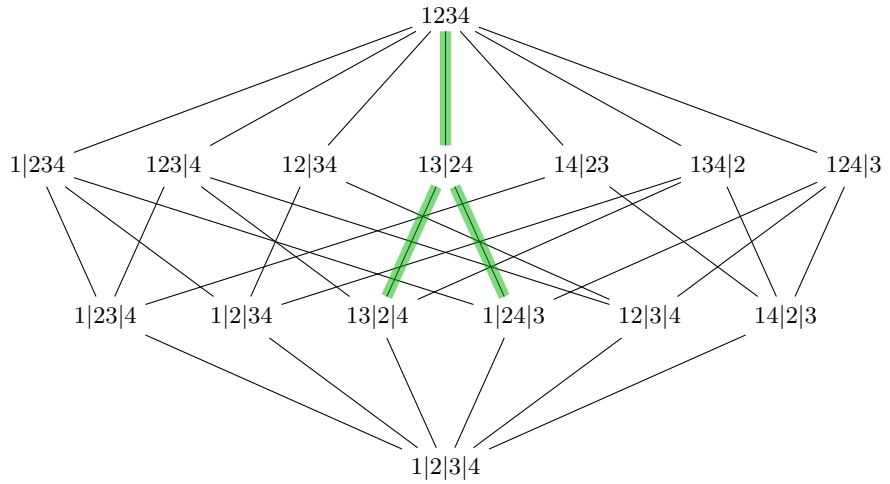


FIGURE 1. The poset $(\Pi_4, \leq_{\mathrm{dref}})$. The non-highlighted edges induce the subposet $(NC_4, \leq_{\mathrm{dref}})$.

where $(v_1, v_2) \in E$ if and only if $v_1 \in [n], v_2 \in (\mathbf{x} \cup \mathbf{x}')$, and $v_1 \in v_2$. We have

$$\mathbf{x} \vee_\Pi \mathbf{x}' = \big\{C \cap [n] \mid C \text{ is a connected component of } \mathbf{P}_{\mathbf{x},\mathbf{x}'}\big\}. \tag{3}$$

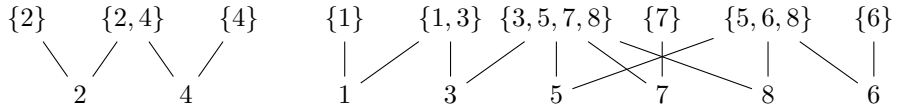
Example 2.4. Let

$$\mathbf{x} = \big\{\{1\},\{2\},\{4\},\{3,5,7,8\},\{6\}\big\} \quad \text{and} \quad \mathbf{x}' = \big\{\{1,3\},\{2,4\},\{5,6,8\},\{7\}\big\}.$$

We observe that $\mathbf{x}$ is non-crossing, while $\mathbf{x}'$ is not, since $1 \sim_{\mathbf{x}'} 3$ and $2 \sim_{\mathbf{x}'} 4$, but $1 \not\sim_{\mathbf{x}'} 2$. Their meet is

$$\mathbf{x} \wedge_\Pi \mathbf{x}' = \big\{\{1\},\{2\},\{3\},\{4\},\{5,8\},\{6\},\{7\}\big\}.$$

The graph $\mathbf{P}_{\mathbf{x},\mathbf{x}'}$ is



which implies $\mathbf{x} \vee_\Pi \mathbf{x}' = \big\{\{1,3,5,6,7,8\},\{2,4\}\big\}$.

For $\mathbf{x} \in \Pi_n$ denote by $\overline{\mathbf{x}}$ the *noncrossing closure* of $\mathbf{x}$, which is defined by successively joining crossing blocks. It is immediate that $\mathbf{x} \leq_{\mathrm{dref}} \overline{\mathbf{x}}$, and [9, Théorème 1] states that $\overline{\mathbf{x}}$ is the smallest noncrossing partition (weakly) above $\mathbf{x}$. The meet of two noncrossing set partitions $\mathbf{x}, \mathbf{x}' \in NC_n$ is then

$$\mathbf{x} \wedge_{NC} \mathbf{x}' = \mathbf{x} \wedge_\Pi \mathbf{x}', \tag{4}$$

while their join is

$$\mathbf{x} \vee_{NC} \mathbf{x}' = \overline{\mathbf{x} \vee_\Pi \mathbf{x}'}. \tag{5}$$

Example 2.5. Let $\mathbf{x}'$ be the crossing set partition from Example 2.4. We obtain

$$\overline{\mathbf{x}'} = \big\{\{1,2,3,4\},\{5,6,8\},\{7\}\big\},$$

and $\mathbf{x} \wedge_{NC} \overline{\mathbf{x}'} = \mathbf{x} \wedge_\Pi \mathbf{x}'$ and $\mathbf{x} \vee_{NC} \overline{\mathbf{x}'} = \mathbf{1}$.

Let us summarize this in a theorem.

Theorem 2.6. (Folklore, [9, Théorèmes 2 and 3]) *For $n \geq 1$, the posets $(\Pi_n, \leq_{\mathrm{dref}})$ and $(NC_n, \leq_{\mathrm{dref}})$ are graded lattices. The rank of a (noncrossing) set partition is given by n minus the number of its blocks.*

For $i \in [n]$ define $\mathbf{x}_i$ to be the noncrossing partition with the unique non-singleton block $[i-1] \cup \{n\}$. We thereby understand $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_n = \mathbf{1}$. It follows that

$$C = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \tag{6}$$

is a maximal chain in $(NC_n, \leq_{\mathrm{dref}})$.

Proposition 2.7. *For $i \in [n]$ the element $\mathbf{x}_i$ is left-modular in $(NC_n, \leq_{\mathrm{dref}})$.*

Proof. Let $X = [i-1] \cup \{n\}$ be the unique non-singleton block of $\mathbf{x}_i$, and let $\mathbf{z} \in NC_n$.

We show that $\mathbf{x}_i M \mathbf{z}$. Pick $\mathbf{y} \leq_{\mathrm{dref}} \mathbf{z}$, and let B be a block of $\mathbf{y}$. There exists a unique block B' of $\mathbf{z}$ with $B \subseteq B'$. Let $A = B' \cap X$. We distinguish two cases.

(i) $B \cap X = \varnothing$. It follows that B is a block of $\mathbf{y} \vee_{NC} \mathbf{x}_i$, and it is thus a block of $(\mathbf{y} \vee_{NC} \mathbf{x}_i) \wedge_{NC} \mathbf{z}$, too. In $\mathbf{x}_i \wedge_{NC} \mathbf{z}$ we see that A is a block, while $B' \setminus A$ is

split into singleton blocks. By assumption $B \subseteq (B' \setminus A)$, and we conclude that B is a block of $\mathbf{y} \vee_{NC} (\mathbf{x}_i \wedge_{NC} \mathbf{z})$.

(ii) $B \cap X \neq \varnothing$. It follows that $B \cup X$ is a block of $\mathbf{y} \vee_{NC} \mathbf{x}_i$, and that therefore $A \cup B$ is a block of $(\mathbf{y} \vee_{NC} \mathbf{x}_i) \wedge_{NC} \mathbf{z}$. In $\mathbf{x}_i \wedge_{NC} \mathbf{z}$ we see that A is a block, while $B' \setminus A$ is split into singleton blocks. By assumption $B \cap A \neq \varnothing$, and we thus obtain that $A \cup B$ is a block of $\mathbf{y} \vee_{NC} (\mathbf{x}_i \wedge_{NC} \mathbf{z})$. ☑

Corollary 2.8. *The chain in* (6) *is a left-modular chain in* $(NC_n, \leq_{\mathrm{dref}})$*, which is thus a supersolvable lattice.*

Proof. Proposition 2.7 implies that every element in (6) is left-modular, and Theorem 2.6 implies that $(NC_n, \leq_{\mathrm{dref}})$ is graded. In view of Theorem 2.1 we conclude that $(NC_n, \leq_{\mathrm{dref}})$ is supersolvable. ☑

The fact that $(NC_n, \leq_{\mathrm{dref}})$ is supersolvable was established before in [7, Theorem 4.3.2].

Corollary 2.9. *For* $n \geq 1$*, the lattice* $(NC_n, \leq_{\mathrm{dref}})$ *is EL-shellable.*

Proof. This follows from Theorem 2.2 and Corollary 2.8. ☑

The fact that $(NC_n, \leq_{\mathrm{dref}})$ is EL-shellable was established before in [1, Example 2.9].

3. A Subposet of $(NC_n, \leq_{\mathrm{dref}})$

Let us define two subsets $L_1, L_2 \subseteq NC_n$ by

$$L_1 = \big\{\mathbf{x} \in NC_n \mid \{n-1, n\} \in \mathbf{x}\big\},$$
$$L_2 = \big\{\mathbf{x} \in NC_n \mid 1 \sim_{\mathbf{x}} n-1 \text{ and } \{n\} \in \mathbf{x}\big\}.$$

Finally, for $n \geq 3$ define

$$PE_n = NC_n \setminus \big(L_1 \cup L_2\big). \tag{7}$$

Lemma 3.1 ([4])**.** *We have* $\big|PE_3\big| = 3$*, and for* $n \geq 4$ *we have*

$$\Big|PE_n\Big| = \left(\frac{5}{n+1} + \frac{9}{n-3}\right)\binom{2n-4}{n-4},$$

which is [15, A071718] with offset 2*.*

Proof. Define the n^{th} Catalan number to be $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. It was observed in [4] that

$$\Big|PE_n\Big| = \mathrm{Cat}(n) - 2\mathrm{Cat}(n-2).$$

We can therefore immediately verify the claim for $n = 3$. For $n \geq 4$, we obtain

$$\begin{aligned}\left|PE_n\right| &= \mathrm{Cat}(n) - 2\mathrm{Cat}(n-2)\\ &= \frac{1}{n+1}\binom{2n}{n} - \frac{2}{n-1}\binom{2n-4}{n-2}\\ &= \left(\frac{4(2n-1)(2n-3)}{(n+1)(n-2)(n-3)} - \frac{2n}{(n-2)(n-3)}\right)\binom{2n-4}{n-4}\\ &= \left(\frac{14n^2-34n+12}{(n+1)(n-2)(n-3)}\right)\binom{2n-4}{n-4}\\ &= \left(\frac{14n-6}{(n+1)(n-3)}\right)\binom{2n-4}{n-4}\\ &= \left(\frac{5}{n+1} + \frac{9}{n-3}\right)\binom{2n-4}{n-4}.\end{aligned}$$

☑

3.1. $(PE_n, \leq_{\mathrm{dref}})$ is a Supersolvable Lattice

Let us now investigate a few properties of the poset $(PE_n, \leq_{\mathrm{dref}})$. Our first main result establishes that this poset is in fact a lattice.

Theorem 3.2. *For $n \geq 3$, the poset $(PE_n, \leq_{\mathrm{dref}})$ is a lattice.*

Proof. Let $\mathbf{x}, \mathbf{x}' \in PE_n$. Let $\mathbf{w} = \mathbf{x} \wedge_{NC} \mathbf{x}'$, and write $\mathbf{w} = \{B_1, B_2, \ldots, B_s\}$. If $\mathbf{w} \in PE_n$, define $\mathbf{x} \wedge_{PE} \mathbf{x}' = \mathbf{w}$. If $\mathbf{w} \notin PE_n$, then there are two options.

(i) $\{n-1, n\} \in \mathbf{w}$. Without loss of generality say that $B_s = \{n-1, n\}$. Define $\mathbf{w}' = \big\{B_1, B_2, \ldots, B_{s-1}, \{n-1\}, \{n\}\big\}$. Then, $\mathbf{w}' \in PE_n$, and $\mathbf{w}' \leq_{\mathrm{dref}} \mathbf{w}$, which in particular implies that $\mathbf{w}' \leq_{\mathrm{dref}} \mathbf{x}$ and $\mathbf{w}' \leq_{\mathrm{dref}} \mathbf{x}'$. Let $\mathbf{z} \in PE_n$ with $\mathbf{z} \leq_{\mathrm{dref}} \mathbf{x}$ and $\mathbf{z} \leq_{\mathrm{dref}} \mathbf{x}'$. We must thus have $\mathbf{z} \leq_{\mathrm{dref}} \mathbf{w}$, and $\{n-1, n\} \notin \mathbf{z}$, which implies $\{n-1\}, \{n\} \in \mathbf{z}$ and every block of $\mathbf{z}$ is contained in some B_i for $i \in [s]$. It follows that $\mathbf{z} \leq_{\mathrm{dref}} \mathbf{w}'$. We thus put $\mathbf{x} \wedge_{PE} \mathbf{x}' = \mathbf{w}'$ for this case.

(ii) $\{n\} \in \mathbf{w}$ and $1 \sim_{\mathbf{w}} n-1$. Without loss of generality we can assume that $B_s = \{n\}$. By definition we must have $1 \sim_{\mathbf{x}} n-1$ and $1 \sim_{\mathbf{x}'} n-1$. Since $\mathbf{x}, \mathbf{x}' \in PE_n$ we conclude that there are indices $i \neq j$ with $i \sim_{\mathbf{x}} n$ and $j \sim_{\mathbf{x}'} n$. Since $\{n\} \in \mathbf{w}$ we conclude $1 < i, j < n-1$, which contradicts $\mathbf{x}, \mathbf{x}' \in NC_n$. It follows that this case cannot occur.

Now let $\mathbf{w} = \mathbf{x} \vee_{NC} \mathbf{x}'$, and write $\mathbf{w} = \{B_1, B_2, \ldots, B_s\}$. If $\mathbf{w} \in PE_n$, define $\mathbf{x} \vee_{PE} \mathbf{x}' = \mathbf{w}$. If $\mathbf{w} \notin PE_n$, then there are two options again.

(i) $\{n-1, n\} \in \mathbf{w}$. In view of (3) we conclude $\{n-1, n\} \in \mathbf{x}, \mathbf{x}'$, which contradicts $\mathbf{x}, \mathbf{x}' \in PE_n$. It follows that this case cannot occur.

(ii) $\{n\} \in \mathbf{w}$ and $1 \sim_{\mathbf{w}} n-1$. Without loss of generality let $1, n-1 \in B_1$, and let $B_s = \{n\}$. Define $\mathbf{w}' = \{B_1 \cup B_s, B_2, \ldots, B_{s-1}\}$. We then have $\mathbf{w} \leq_{\mathrm{dref}} \mathbf{w}'$,

and consequently $\mathbf{x} \leq_{\mathrm{dref}} \mathbf{w}'$ and $\mathbf{x}' \leq_{\mathrm{dref}} \mathbf{w}'$. Let $\mathbf{z} \in PE_n$ with $\mathbf{x} \leq_{\mathrm{dref}} \mathbf{z}$ and $\mathbf{x}' \leq_{\mathrm{dref}} \mathbf{z}$. Again by (3) we conclude $\{n\} \in \mathbf{x}, \mathbf{x}'$, and since $\mathbf{x}, \mathbf{x}' \in PE_n$ we see that $1 \not\sim_{\mathbf{x}} n-1$ and $1 \not\sim_{\mathbf{x}'} n-1$. Since $1 \sim_{\mathbf{w}} n-1$ there must be $i \in [n]$ with $1 \sim_{\mathbf{x}} i$ and $i \sim_{\mathbf{x}'} n-1$. We thus conclude $1 \sim_{\mathbf{z}} n-1$, and since $\mathbf{z} \in PE_n$ we further conclude $n-1 \sim_{\mathbf{z}} n$. This implies $\mathbf{w}' \leq_{\mathrm{dref}} \mathbf{z}$. We thus put $\mathbf{x} \vee_{PE} \mathbf{x}' = \mathbf{w}'$ for this case. ☑

Lemma 3.3. *For $n \geq 3$, the lattice $(PE_n, \leq_{\mathrm{dref}})$ is graded.*

Proof. Let $\mathbf{x}, \mathbf{y} \in PE_n$ with $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}$ in $(PE_n, \leq_{\mathrm{dref}})$. Assume that there is $\mathbf{z} \in NC_n$ with $\mathbf{x} <_{\mathrm{dref}} \mathbf{z} <_{\mathrm{dref}} \mathbf{y}$. It follows that $\mathbf{z} \in NC_n \setminus PE_n$. There are two cases.

(i) $\{n-1, n\}$ is a block of $\mathbf{z}$. Since $\{n-1, n\}$ is neither a block of $\mathbf{x}$, nor of $\mathbf{y}$, it must be that $n-1$ and n constitute singleton blocks in $\mathbf{x}$ and there is some $j \in [n-2]$ and some block B of $\mathbf{y}$ containing $\{j, n-1, n\}$. Consider the partition $\mathbf{w}$ that has all blocks of $\mathbf{y}$ except that B is replaced by the two blocks $B \setminus \{n-1\}$ and $\{n-1\}$. Since $\mathbf{y} \in PE_n \subseteq NC_n$ we conclude that $\mathbf{w} \in NC_n$, and we have $\mathbf{w} \lessdot_{\mathrm{dref}} \mathbf{y}$. By construction, $\mathbf{w} \in PE_n$. It follows further from $\mathbf{x} \leq_{\mathrm{dref}} \mathbf{y}$ that $\mathbf{x} <_{\mathrm{dref}} \mathbf{w}$ (since $n-1$ and n constitute singleton blocks of $\mathbf{x}$). This is a contradiction to $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}$ in $(PE_n, \leq_{\mathrm{dref}})$.

(ii) $\{n\}$ is a block of $\mathbf{z}$ and $1 \sim_{\mathbf{z}} n-1$. It follows that $1 \sim_{\mathbf{y}} n-1$, which forces $n-1 \sim_{\mathbf{y}} n$. Moreover, it follows that $\{n\}$ must be a block of $\mathbf{x}$, which implies that $1 \not\sim_{\mathbf{x}} n-1$. Let B be the block of $\mathbf{x}$ containing 1. Consider the partition $\mathbf{w}$ that consists of all the blocks of $\mathbf{x}$ except that B is replaced by $B \cup \{n\}$. Then, $\mathbf{x} \in NC_n$ implies $\mathbf{w} \in PE_n$. Moreover, $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{w} <_{\mathrm{dref}} y$, which is a contradiction to $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}$ in $(PE_n, \leq_{\mathrm{dref}})$. ☑

It follows by definition that the chain (6) belongs to $(PE_n, \leq_{\mathrm{dref}})$. It is our next goal to show that this chain is also left-modular in $(PE_n, \leq_{\mathrm{dref}})$. We first prove an auxiliary result.

Proposition 3.4. *For $i \in [n]$ and $\mathbf{y} \in PE_n$ we have $\mathbf{x}_i \wedge_{PE} \mathbf{y} = \mathbf{x}_i \wedge_{NC} \mathbf{y}$ and $\mathbf{x}_i \vee_{PE} \mathbf{y} = \mathbf{x}_i \vee_{NC} \mathbf{y}$.*

Proof. Let $\mathbf{y} \in PE_n$. If $\mathbf{x}_i \wedge_{PE} \mathbf{y} <_{\mathrm{dref}} \mathbf{x}_i \wedge_{NC} \mathbf{y}$, then it follows from the proof of Theorem 3.2 that there exists a block B of $\mathbf{x}_i$ with $\{n-1, n\} \subseteq B$. By definition this forces $i = n$, so that $\mathbf{x}_i$ is the full partition. In particular $\mathbf{y} \leq_{\mathrm{dref}} \mathbf{x}_i$, which yields the contradiction $\mathbf{y} = \mathbf{x}_i \wedge_{PE} \mathbf{y} <_{\mathrm{dref}} \mathbf{x}_i \wedge_{NC} \mathbf{y} = \mathbf{y}$.

If $\mathbf{x}_i \vee_{NC} \mathbf{y} <_{\mathrm{dref}} \mathbf{x}_i \vee_{PE} \mathbf{y}$, then it follows from the proof of Theorem 3.2 that $\{n\}$ is a block of $\mathbf{x}_i$. By definition, this forces $i = 1$, so that $\mathbf{x}_i$ is the discrete partition. In particular $\mathbf{x}_i \leq_{\mathrm{dref}} \mathbf{y}$, which yields the contradiction $\mathbf{y} = \mathbf{x}_i \vee_{NC} \mathbf{y} <_{\mathrm{dref}} \mathbf{x}_i \vee_{PE} \mathbf{y} = \mathbf{y}$. ☑

Proposition 3.5. *For $n \geq 3$, the chain in (6) is left-modular in $(PE_n, \leq_{\mathrm{dref}})$.*

Proof. The elements $\mathbf{x}_1$ and $\mathbf{x}_n$ are the least and the greatest element of $(PE_n, \leq_{\mathrm{dref}})$, so they are trivially left-modular. Let us therefore assume that $i \in \{2, 3, \ldots, n-1\}$. In particular, $n-1 \not\sim_{\mathbf{x}_i} n$ and $\{n\}$ is not a block of $\mathbf{x}_i$. Let $\mathbf{z} \in PE_n$.

We show that $\mathbf{x}_i M \mathbf{z}$ holds in $(PE_n, \leq_{\mathrm{dref}})$. Let $\mathbf{y} \in PE_n$ with $\mathbf{y} \leq_{\mathrm{dref}} \mathbf{z}$. Proposition 3.4 implies that $\mathbf{q} = \mathbf{y} \vee_{PE} \mathbf{x}_i = \mathbf{y} \vee_{NC} \mathbf{x}_i$. Assume that $\mathbf{q} \wedge_{PE} \mathbf{z} \neq \mathbf{q} \wedge_{NC} \mathbf{z}$. The proof of Theorem 3.2 implies that this can only happen if $\{n-1, n\}$ is a block of $\mathbf{q} \wedge_{NC} \mathbf{z}$. For this to happen, we need $n-1 \sim_{\mathbf{q}} n$, which forces the existence of some $j \in [i-1] \cup \{n\}$ with $j \sim_{\mathbf{y}} n-1$. If $j < i$, then we obtain the contradiction that $\mathbf{q} \wedge_{NC} \mathbf{z}$ has a block containing $\{j, n-1, n\}$ since $i \leq n-1$. We thus have $j = n$. Since $i > 1$ we see that $\mathbf{q}$ has a block containing $\{i-1, n-1, n\}$, which forces $\mathbf{z}$ to contain the block $\{n-1, n\}$; a contradiction to $\mathbf{z} \in PE_n$. We therefore have

$$(\mathbf{y} \vee_{PE} \mathbf{x}_i) \wedge_{PE} \mathbf{z} = (\mathbf{y} \vee_{NC} \mathbf{x}_i) \wedge_{NC} \mathbf{z}. \tag{8}$$

On the other hand, Proposition 3.4 also implies that $\mathbf{q}' = \mathbf{x}_i \wedge_{PE} \mathbf{z} = \mathbf{x}_i \wedge_{NC} \mathbf{z}$. Assume that $\mathbf{y} \vee_{PE} \mathbf{q}' \neq \mathbf{y} \vee_{NC} \mathbf{q}'$. The proof of Theorem 3.2 implies that this can only happen if $\{n\}$ is a block of $\mathbf{y} \vee_{NC} \mathbf{q}'$ and $1 \sim_{\mathbf{y} \vee_{NC} \mathbf{q}'} n-1$. By definition of the join, $\{n\}$ must be a block of both $\mathbf{y}$ and $\mathbf{q}'$. Since $i < n$ we see that $\{n-1\}$ is a singleton block in $\mathbf{q}'$, which forces $1 \sim_{\mathbf{y}} n-1$; a contradiction to $\mathbf{y} \in PE_n$. We therefore have

$$\mathbf{y} \vee_{PE} (\mathbf{x}_i \wedge_{PE} \mathbf{z}) = \mathbf{y} \vee_{NC} (\mathbf{x}_i \wedge_{NC} \mathbf{z}). \tag{9}$$

Proposition 2.7 implies the equality of the right-hand sides of (8) and (9), which implies $\mathbf{x}_i M \mathbf{z}$ in $(PE_n, \leq_{\mathrm{dref}})$. ☑

We now conclude the proof of Theorem 1.3.

Proof of Theorem 1.3. It follows from Theorem 3.2, Lemma 3.3, and Proposition 3.5 that $(PE_n, \leq_{\mathrm{dref}})$ is a graded left-modular lattice. Theorem 2.1 implies then that it is supersolvable. ☑

Corollary 3.6. *For $n \geq 3$, the lattice $(PE_n, \leq_{\mathrm{dref}})$ is EL-shellable.*

Proof. This follows from Theorems 1.3 and 2.2. ☑

Figure 2 shows $(PE_4, \leq_{\mathrm{dref}})$ together with the EL-labeling coming from the left-modular chain in (6). The unique rising maximal chain from $\mathbf{0}$ to $\mathbf{1}$ is highlighted.

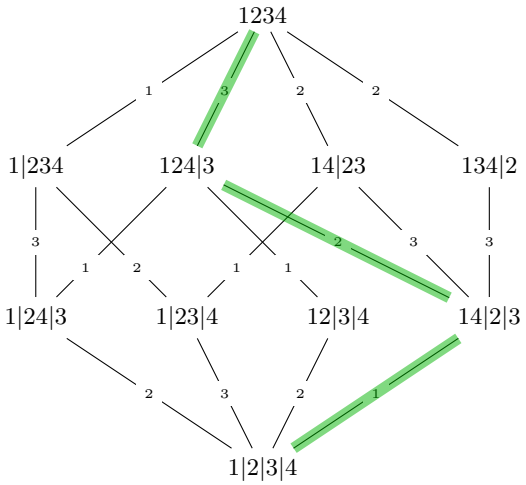


FIGURE 2. The lattice $(PE_4, \leq_{\mathrm{dref}})$. The highlighted chain is (6), and the labeling is the one defined in (1).

3.2. The Möbius Function of $(PE_n, \leq_{\mathrm{dref}})$

In this section we determine the value of the Möbius function of $(PE_n, \leq_{\mathrm{dref}})$ between $\mathbf{0}$ and $\mathbf{1}$. Recall that the *Möbius function* of a poset $\mathcal{L} = (L, \leq)$ is defined recursively by

$$\mu_{\mathcal{L}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x<z\leq y} \mu(z,y), & \text{if } x < y, \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

for all $x, y \in L$. It was shown in [3] that in a lattice $\mathcal{L}$, we can compute the value $\mu_{\mathcal{L}}(\hat{0}, x)$ for any $x \in L$ by summing over the NBB bases for x. Let us recall the necessary concepts. Let $\mathcal{A}$ denote the set of atoms of $\mathcal{L}$, and let $\trianglelefteq$ be an arbitrary partial order on $\mathcal{A}$. A set $X \subseteq \mathcal{A}$ is *bounded below* (or *BB* for short) if for every $d \in X$ there exists some $a \in \mathcal{A}$ such that $a \triangleleft d$ and $a < \bigvee X$. A set $X \subseteq \mathcal{A}$ is *NBB* if none of its nonempty subsets is BB. If X is NBB and $\bigvee X = x$, then X is an *NBB base* for x. We have the following result.

Theorem 3.7. ([3, Theorem 1.1]) *Let $\mathcal{L} = (L, \leq)$ be a finite lattice, and let $\trianglelefteq$ be any partial order on the atoms of $\mathcal{L}$. For $x \in L$ we have*

$$\mu_{\mathcal{L}}(\hat{0}, x) = \sum_{X} (-1)^{|X|},$$

where the sum is over all NBB bases for x with respect to $\trianglelefteq$.

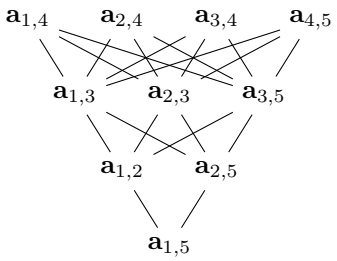


FIGURE 3. The poset $(\mathcal{A}_5, \trianglelefteq)$.

In the remainder of this section we give a combinatorial model for the NBB bases of $\mathbf{1}$ in $(PE_n, \leq_{\mathrm{dref}})$ with respect to a suitable partial order on its atoms, and conclude Theorem 1.4.

For $i, j \in [n]$ with $i < j$, define $\mathbf{a}_{i,j}$ to be the set partition whose unique non-singleton block is $\{i, j\}$. The set $\mathcal{A}_n = \{\mathbf{a}_{i,j} \mid 1 \leq i < j \leq n\}$ is the set of all atoms of $\{NC_n, \leq_{\mathrm{dref}})$. The set $\bar{\mathcal{A}}_n = \mathcal{A}_n \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$ is then the set of atoms of $(PE_n, \leq_{\mathrm{dref}})$. Consider the partition of $\mathcal{A}_n$ given by

$$A_i = \{\mathbf{a} \in \mathcal{A}_n \mid \mathbf{a} \leq_{\mathrm{dref}} \mathbf{x}_i \text{ and } \mathbf{a} \not\leq_{\mathrm{dref}} \mathbf{x}_{i-1}\}$$

for $i \in [n-1]$. Let $\bar{A}_i$ be the restriction of A_i to $\bar{\mathcal{A}}_n$. Define a partial order on $\mathcal{A}_n$ by setting $\mathbf{a} \trianglelefteq \mathbf{a}'$ if and only if $\mathbf{a} \in A_i$ and $\mathbf{a}' \in A_j$ for $i < j$. The poset $(\mathcal{A}_5, \trianglelefteq)$ is depicted in Figure 3.

Lemma 3.8. *For $j \in [n-1]$ we have $A_j = \{\mathbf{a}_{i,j} \mid 1 \leq i < j\} \cup \{\mathbf{a}_{j,n}\}$. Moreover, we have $\bar{A}_j = A_j$ for $j \in [n-2]$, and $\bar{A}_{n-1} = A_{n-1} \setminus \{\mathbf{a}_{1,n-1}, \mathbf{a}_{n-1,n}\}$.*

Proof. Let $\mathbf{a}_{i,j} \in \mathcal{A}_n$ for $1 \leq i < j \leq n$. If $j < n$, then $\mathbf{a}_{i,j} \leq_{\mathrm{dref}} \mathbf{x}_j$, but $\mathbf{a}_{i,j} \not\leq_{\mathrm{dref}} \mathbf{x}_{j-1}$. If $j = n$, then $\mathbf{a}_{i,n} \leq_{\mathrm{dref}} \mathbf{x}_i$, but $\mathbf{a}_{i,n} \not\leq_{\mathrm{dref}} \mathbf{x}_{i-1}$. ☑

Since we want to consider NBB bases in the two related posets $(NC_n, \leq_{\mathrm{dref}})$ and $(PE_n, \leq_{\mathrm{dref}})$, we use the prefixes "*NC*" and "*PE*" to indicate which lattice we consider. Theorem 3.2 implies that for $\mathbf{x}, \mathbf{y} \in PE_n$ we always have $\mathbf{x} \vee_{NC} \mathbf{y} \leq_{\mathrm{dref}} \mathbf{x} \vee_{PE} \mathbf{y}$. Therefore, if $X \subseteq \bar{\mathcal{A}}_n$ is *NC*-BB, then it is automatically *PE*-BB.

Lemma 3.9. *If $\mathbf{a} \vee_\Pi \mathbf{a}'$ is crossing, then $\{\mathbf{a}, \mathbf{a}'\}$ is NC-BB.*

Proof. Let $\mathbf{a}_{i,j}, \mathbf{a}_{k,l} \in \mathcal{A}_n$. If $\mathbf{a}_{i,j} \vee_\Pi \mathbf{a}_{k,l}$ is crossing, then $i < k < j < l$, and the join $\mathbf{a}_{i,j} \vee_{NC} \mathbf{a}_{k,l}$ has the unique non-singleton block $\{i, j, k, l\}$. We distinguish two cases.

(i) If $l < n$, then Lemma 3.8 implies $\mathbf{a}_{i,j} \in A_j$ and $\mathbf{a}_{k,l} \in A_l$. Since $j < l$ we obtain $\mathbf{a}_{i,j} \lhd \mathbf{a}_{k,l}$, and since $k < j$, Lemma 3.8 implies that $\mathbf{a}_{i,k} \lhd \mathbf{a}_{i,j}$. We clearly have $\mathbf{a}_{i,k} <_{\mathrm{dref}} \mathbf{a}_{i,j} \vee_{NC} \mathbf{a}_{k,l}$, which implies that $\{\mathbf{a}_{i,j}, \mathbf{a}_{k,l}\}$ is *NC*-BB.

(ii) If $l = n$, then Lemma 3.8 implies $\mathbf{a}_{i,j} \in A_j$ and $\mathbf{a}_{k,n} \in A_k$. Since $k < j$ we obtain $\mathbf{a}_{k,n} \lhd \mathbf{a}_{i,j}$, and since $i < k$, Lemma 3.8 implies that $\mathbf{a}_{i,n} \lhd \mathbf{a}_{k,n}$. We clearly have $\mathbf{a}_{i,n} <_{\mathrm{dref}} \mathbf{a}_{i,j} \vee_{PE} \mathbf{a}_{k,n}$, which implies that $\{\mathbf{a}_{i,j}, \mathbf{a}_{k,n}\}$ is *NC*-BB. ☑

Lemma 3.10. *If $\mathbf{a}, \mathbf{a}' \in A_j$ for $j \in [n-1]$, then $\{\mathbf{a}, \mathbf{a}'\}$ is NC-BB.*

Proof. Let $\mathbf{a}_{i,j}, \mathbf{a}_{k,j} \in A_j$. Note that $\mathbf{a}_{i,j} \vee_{NC} \mathbf{a}_{k,j}$ has the unique non-singleton block $\{i, k, j\}$. There are again two cases.

(i) If $i < j$ and $k < j$, then Lemma 3.8 implies $\mathbf{a}_{i,k} \in A_k$, and thus $\mathbf{a}_{i,k} \lhd \mathbf{a}_{i,j}$ and $\mathbf{a}_{i,k} \lhd \mathbf{a}_{k,j}$. We clearly have $\mathbf{a}_{i,k} <_{\mathrm{dref}} \mathbf{a}_{i,j} \vee_{NC} \mathbf{a}_{k,j}$, which implies that $\{\mathbf{a}_{i,j}, \mathbf{a}_{k,j}\}$ is *NC*-BB.

(ii) If $i < j$ and $k > j$. Lemma 3.8 implies that $k = n$, and that $\mathbf{a}_{i,n} \in A_i$. Therefore $\mathbf{a}_{i,n} \lhd \mathbf{a}_{i,j}$ and $\mathbf{a}_{i,n} \lhd \mathbf{a}_{j,n}$. We clearly have $\mathbf{a}_{i,n} <_{\mathrm{dref}} \mathbf{a}_{i,j} \vee_{NC} \mathbf{a}_{j,n}$, which implies that $\{\mathbf{a}_{i,j}, \mathbf{a}_{j,n}\}$ is *NC*-BB. ☑

Lemma 3.11. *Let $X \subseteq \bar{\mathcal{A}}_n$ satisfy $\bigvee_{PE} X = \mathbf{1}$. If $|X| < n-1$, then X is PE-BB.*

Proof. Suppose that $|X| = k$. Observe that if X is a set of pairwise non-crossing atoms, then $\bigvee_{NC} X = \bigvee_{\Pi} X$. By (3) $\bigvee_{\Pi} X$ has exactly $n-k$ blocks. Moreover, by Theorem 3.2 the number of blocks of $\bigvee_{PE} X$ is either $n-k$ or $n-k-1$. Since we assumed $\bigvee_{PE} X = \mathbf{1}$, we conclude that $k \in \{n-2, n-1\}$. Let $\mathbf{z} = \bigvee_{NC} X$.

If $k = n-2$, then we conclude that $1 \sim_{\mathbf{z}} n-1$, and $\{n\}$ is a block of $\mathbf{z}$. It follows that $\mathbf{a}_{1,n} \notin X$, which in view of Lemma 3.8 implies $\mathbf{a}_{1,n} \lhd \mathbf{a}$ for all $\mathbf{a} \in X$. Since $\mathbf{a}_{1,n} <_{\mathrm{dref}} \mathbf{1}$, we conclude that X is *PE*-BB. ☑

Let us denote by $\mathcal{B}_n$ the set of all *NC*-NBB bases for $\mathbf{1}$, and let $\bar{\mathcal{B}}_n$ denote the set of all *PE*-NBB bases for $\mathbf{1}$. By construction we have $\bar{\mathcal{B}}_n \subseteq \mathcal{B}_n$.

Corollary 3.12. *Every element of $\mathcal{B}_n$ has cardinality $n-1$. Consequently the same is true for the elements of $\bar{\mathcal{B}}_n$.*

Proof. The claim for the cardinality of the elements in $\mathcal{B}_n$ follows directly from (5) and Lemmas 3.9 and 3.10.

The claim for the cardinality of the elements in $\bar{\mathcal{B}}_n$ can be verified directly using Lemmas 3.9–3.11. ☑

For the moment, let us focus on the elements of $\mathcal{B}_n$. In view of Corollary 3.12 these elements are certain maximal chains of $(\mathcal{A}_n, \trianglelefteq)$. We can naturally associate a graph with $X \in \mathcal{B}_n$ by connecting the vertices i and j if and only if $\mathbf{a}_{i,j} \in X$. Denote the resulting graph by $\tau(X)$.

Lemma 3.13. *If $X \in \mathcal{B}_n$, then $\tau(X)$ is a tree.*

Proof. Since $\bigvee_{NC} X = \mathbf{1}$ it follows from (3) that $\tau(X)$ is connected. Now suppose that $\tau(X)$ contains a cycle $C = (\mathbf{a}_{i_1,i_2}, \mathbf{a}_{i_2,i_3}, \ldots, \mathbf{a}_{i_s,i_1})$. We then have $i_1 < i_2 < \cdots < i_s$, and $3 \leq s < n$.

If $i_s < n$, then $\mathbf{a}_{i_{s-1},i_s}, \mathbf{a}_{i_1,i_s} \in A_{i_s}$, which contradicts Lemma 3.10. If $i_s = n$, then $\mathbf{a}_{i_{s-2},i_{s-1}}, \mathbf{a}_{i_{s-1},i_s} \in A_{i_{s-1}}$, which contradicts Lemma 3.10. ☑

Since $\mathbf{a}_{1,n}$ is the least element in $(\mathcal{A}_n, \trianglelefteq)$ any of the trees in Lemma 3.13 contains an edge between 1 and n.

Lemma 3.14. *Let $X \in \mathcal{B}_n$, and let $\tau(X)$ be the corresponding tree. If we remove the edge between 1 and n, we obtain two trees τ_1 and τ_2, where τ_1 has vertex set $[k]$ and τ_2 has vertex set $\{k+1, k+2, \ldots, n\}$ for some $k \in [n-1]$.*

Proof. Suppose that τ_1 and τ_2 are the two trees obtained by removing the edge connecting 1 and n in $\tau(X)$. The claim is certainly true for $n \leq 3$, so suppose that $n > 3$. Assume that there is a vertex k in τ_1 such that there exists $i \in [k-1]$ which is a vertex of τ_2, and choose k minimal with this property. Since τ_1 is a tree, there is a unique path from 1 to k, and let k' be the predecessor of k along this path. It follows that $\mathbf{a}_{k',k} \in X$, and thus $k' < k$. The minimality of k implies that there is l in $\{k'+1, k'+2, \ldots, k-1\}$ which is a vertex of τ_2. Let $l = l_0 < l_1 < \cdots < l_s = n$ denote the elements (in order) on the unique path from l to n in τ_2. Again by construction we have $\mathbf{a}_{l_{i-1},l_i} \in X$ for $i \in [s]$. Moreover, there exists a unique index $i \in [s]$ such that $l_{i-1} < k$ and $l_i > k$. Then, however, Lemma 3.9 implies that $\{\mathbf{a}_{k',k}, \mathbf{a}_{l_{i-1},l_i}\}$ is *NC*-BB, which contradicts the fact that X is an *NC*-NBB base for $\mathbf{1}$. This completes the proof. ☑

We say that the trees occurring as $\tau(X)$ for some $X \in \mathcal{B}_n$ are *noncrossing*. Recall that the *Catalan numbers* are defined by $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$, and they satisfy the recurrence relation

$$\mathrm{Cat}(n+1) = \sum_{k=0}^{n} \mathrm{Cat}(k)\mathrm{Cat}(n-k), \tag{11}$$

with initial condition $\mathrm{Cat}(0) = 1$ [13].

Corollary 3.15. *For $n \geq 1$ we have $\left|\mathcal{B}_n\right| = Cat(n-1)$.*

Proof. Let $C_n = \left|\mathcal{B}_n\right|$. Lemma 3.14 implies that $C_n = \sum_{k=1}^{n-1} C_k C_{n-k}$, and it is quickly verified that $C_1 = 1$. Therefore the numbers C_n and $\mathrm{Cat}(n-1)$ satisfy the same recurrence relation and the same initial condition and must thus be equal. ☑

In view of Theorem 3.7 we obtain the following well-known corollary.

Corollary 3.16. ([9, Théorème 6]) *For $n \geq 1$ we have*

$$\mu_{(NC_n, \leq_{\mathrm{dref}})}(\mathbf{0}, \mathbf{1}) = (-1)^{n-1}\mathrm{Cat}(n-1).$$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. In view of Corollary 3.15 it remains to determine the size of $\mathcal{B}_n \setminus \bar{\mathcal{B}}_n$. Essentially this set consists of three types of elements; those that contain $\mathbf{a}_{1,n-1}$, those that contain $\mathbf{a}_{n-1,n}$, and those that (after removal of $\mathbf{a}_{1,n}$) join to $\mathbf{1}$ in $(PE_n, \leq_{\mathrm{dref}})$. Since every element of $\mathcal{B}_n$ contains $\mathbf{a}_{1,n}$, Lemma 3.13 implies that $X \in \mathcal{B}_n$ cannot contain both of $\mathbf{a}_{1,n-1}$ and $\mathbf{a}_{n-1,n}$.

Let $\mathcal{S}_n^{(1)} = \{X \in \mathcal{B}_n \mid \mathbf{a}_{1,n-1} \in X\}$ and $\mathcal{S}_n^{(2)} = \{X \in \mathcal{B}_n \mid \mathbf{a}_{n-1,n} \in X\}$, and let

$$\mathcal{R}_n = \Big\{X \in \mathcal{B}_n \mid \bigvee\nolimits_{PE}(X \setminus \{\mathbf{a}_{1,n}\}) = \mathbf{1}\Big\}.$$

By construction we have $\bar{\mathcal{B}}_n = \mathcal{B}_n \setminus \Big(\mathcal{S}_n^{(1)} \cup \mathcal{S}_n^{(2)} \cup \mathcal{R}_n\Big)$.

The proof of Theorem 3.2 implies that for $X \in \mathcal{R}_n$ the only vertex adjacent to n in the corresponding tree $\tau(X)$ is 1. As a consequence $\mathcal{S}_n^{(1)} \subseteq \mathcal{R}_n$, and $\mathcal{S}_n^{(2)} \cap \mathcal{R}_n = \varnothing$. It therefore suffices to determine the cardinalities of $\mathcal{S}_n^{(2)}$ and $\mathcal{R}_n$.

Let $X \in \mathcal{S}_n^{(2)}$, and let $\tau(X)$ be the corresponding noncrossing tree. Lemma 3.14 implies that there is some $k \in [n-1]$ such that after removing the edge between 1 and n we are left with a noncrossing tree τ_1 on vertex set $[k]$ and a noncrossing tree τ_2 on vertex set $\{k+1, k+2, \ldots, n\}$ which has an edge between $n-1$ and n. As a consequence, $k < n-1$ and we can view τ_2 as a noncrossing tree on $n-k-1$ vertices. We obtain $\Big|\mathcal{S}_1^{(2)}\Big| = 1$, and

$$\Big|\mathcal{S}_n^{(2)}\Big| = \sum_{k=1}^{n-2} \Big|\mathcal{B}_k\Big| \cdot \Big|\mathcal{B}_{n-k-1}\Big|,$$

which in view of (11) implies $\Big|\mathcal{S}_n^{(2)}\Big| = \mathrm{Cat}(n-2)$.

Let $X \in \mathcal{R}_n$. We have seen already that in $\tau(X)$ the only edge adjacent to n is 1. It follows that the elements of $\mathcal{R}_n$ correspond bijectively to noncrossing trees on $n-1$ vertices. Corollary 3.15 then implies that $\Big|\mathcal{R}_n\Big| = \mathrm{Cat}(n-2)$.

We thus obtain

$$\begin{aligned}\left|\bar{\mathcal{B}}_n\right| &= \mathrm{Cat}(n-1) - 2\mathrm{Cat}(n-2)\\ &= \frac{1}{n}\binom{2n-2}{n-1} - \frac{2}{n-1}\binom{2n-4}{n-2}\\ &= \left(\frac{4(2n-3)}{n(n-3)} - \frac{4}{n-3}\right)\binom{2n-5}{n-4}\\ &= \frac{4}{n}\binom{2n-5}{n-4},\end{aligned}$$

and the claim follows from Theorem 3.7. ☑

Figure 4 illustrates the proof of Theorem 1.4 for $n = 5$. It displays the noncrossing trees corresponding to the elements of $\mathcal{B}_5$. We have crossed out the trees corresponding to elements of $\mathcal{S}_5^{(2)}$ in red, to elements of $\mathcal{S}_5^{(1)}$ in blue, and to elements of $\mathcal{R}_5$ in green.

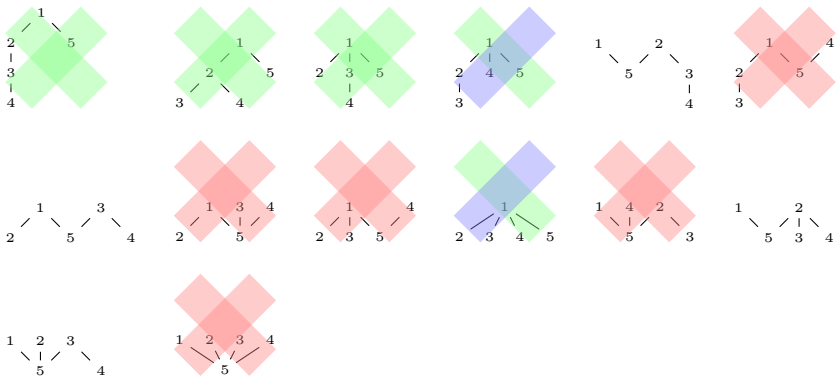


FIGURE 4. The noncrossing trees corresponding to the *NC*-NBB bases for **1** in $(NC_5, \leq_{\mathrm{dref}})$. We have crossed out certain trees as indicated in the proof of Theorem 1.4.

We can use the combinatorial model from above to compute *NC*-NBB bases for any element of NC_n, by simply picking at most one element of each rank of $(\mathcal{A}_n, \trianglelefteq)$ keeping the restriction that their join in the partition lattice is again noncrossing. This process works since every interval in $(NC_n, \leq_{\mathrm{dref}})$ is a direct product of smaller noncrossing partition lattices. The analogous procedure for $(PE_n, \leq_{\mathrm{dref}})$ does not work, due to the extra condition for *PE*-NBB bases (Lemma 3.11). Moreover, the subintervals of $(PE_n, \leq_{\mathrm{dref}})$ do not

factor nicely into direct products of smaller lattices. Consider the interval $[\mathbf{a}_{n-2,n-1}, \mathbf{1}]$ in $(PE_n, \leq_{\mathrm{dref}})$. The cardinalities of these intervals are given by $\mathrm{Cat}(n-1) - \mathrm{Cat}(n-3)$ as observed by J. Lewis; see [15, A280891]. We observe that large prime factors appear in this sequence. It seems, however, that every proper interval of $(PE_n, \leq_{\mathrm{dref}})$ can be written as a direct product of an interval of the previous form and some noncrossing partition lattice.

4. A Subposet of $(PE_n, \leq_{\mathrm{dref}})$

Now we consider a subposet of $(PE_n, \leq_{\mathrm{dref}})$ that was introduced in [4]. To that end recall that a function $f : [n] \to [n]$ is a *parking function* if for all $k \in [n]$ the cardinality of $f^{-1}\big([k]\big)$ is at least k. It is a classical result that the number of parking functions of length n is $(n+1)^{n-1}$ [6, Proposition 2.6.1].

For two noncrossing partitions $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}$, there are two unique blocks B_1 and B_2 of $\mathbf{x}$ such that $B_1 \cup B_2$ is a block of $\mathbf{y}$. Suppose without loss of generality that $\min B_1 < \min B_2$, and define

$$\pi(\mathbf{x}, \mathbf{y}) = \max\{j \in B_1 \mid j \leq i \text{ for all } i \in B_2\}. \tag{12}$$

Clearly π extends to an edge-labeling of $(NC_n, \leq_{\mathrm{dref}})$; the *parking labeling*. Let $\mathscr{C}_n$ denote the set of maximal chains of $(NC_n, \leq_{\mathrm{dref}})$. For any $X \in \mathscr{C}_n$ the sequence $\pi(X)$ is a parking function of length $n-1$, and every such parking function arises in this way [17, Theorem 3.1]. As a consequence $\big|\mathscr{C}_n\big| = n^{n-2}$.

Now let $\mathscr{D}_n = \big\{X \in \mathscr{C}_n \mid n-1 \notin \pi(X)\big\}$ be the set of all maximal chains of $(NC_n, \leq_{\mathrm{dref}})$ whose parking labeling does not contain the value $n-1$. Let $\mathcal{L}_n$ be the subposet of $(NC_n, \leq_{\mathrm{dref}})$ whose maximal chains are precisely $\mathscr{D}_n$, see [4, Definition 3.3].

Proposition 4.1. ([4, Proposition 3.4]) *For $n \geq 3$, the ground set of $\mathcal{L}_n$ is precisely PE_n.*

We can therefore write $\mathcal{L}_n = (PE_n, \leq_{\mathrm{pchn}})$, where $\leq_{\mathrm{pchn}}$ is a subset of $\leq_{\mathrm{dref}}$. Figure 6 shows the poset $(PE_4, \leq_{\mathrm{pchn}})$. This poset was extensively studied in [4]. For our purposes the next statement is the most relevant.

Theorem 4.2. ([4, Theorem C]) *For $n \geq 3$ we have $\mu_{(PE_n, \leq_{\mathrm{pchn}})}(\mathbf{0}, \mathbf{1}) = 0$.*

The main goal of this section is to prove Theorem 1.1 and Corollary 1.2, which essentially proves the conjecture in [4]. To that end we show that the restriction of the EL-labeling of $(PE_n, \leq_{\mathrm{dref}})$ coming from the left-modular chain (6) is an EL-labeling of $(PE_n, \leq_{\mathrm{pchn}})$. First we need to show that the property of being an EL-labeling is preserved under removing particular cover relations.

Proposition 4.3. *Let $\mathcal{L} = (L, \leq)$ be a bounded graded poset with an EL-labeling λ. Let $x, y \in L \setminus \{\hat{0}, \hat{1}\}$ with $x \lessdot y$. Let $\mathcal{L}'$ be the poset that arises from*

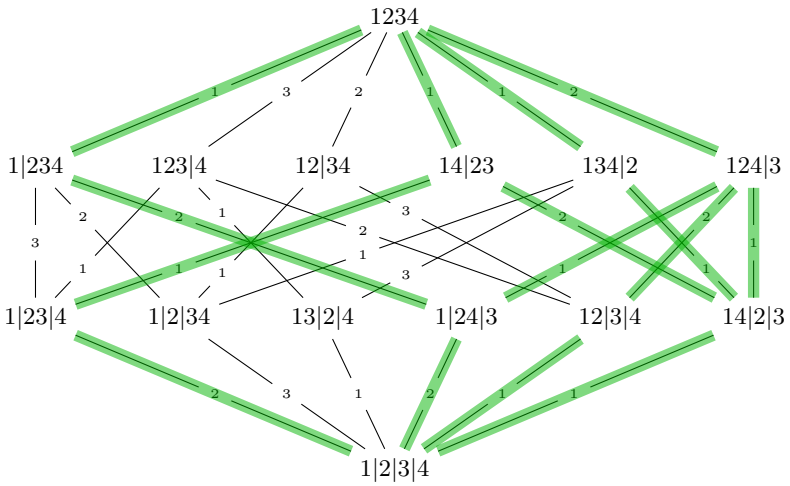


FIGURE 5. The lattice $(NC_4, \leq_{\text{dref}})$ with its parking labeling. The highlighted chains form $\mathscr{D}_4$.

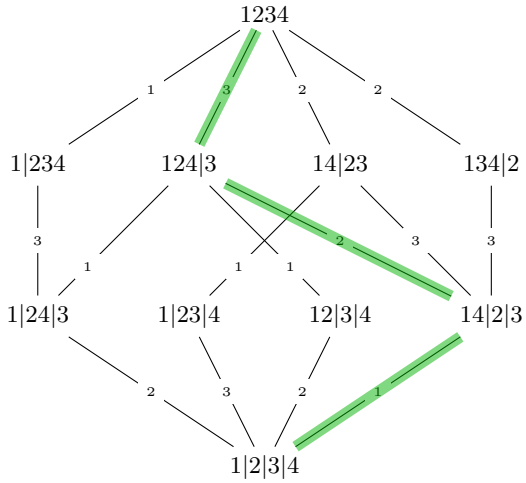


FIGURE 6. The poset $(PE_4, \leq_{\text{pchn}})$. The labeling is inherited from $(PE_4, \leq_{\text{dref}})$, see Figure 2.

$\mathcal{L}$ by removing the cover relation (x, y). If there is some $y' \in L$ with $x \lessdot y'$ and $\lambda(x, y) \succ \lambda(x, y')$, then the restriction of λ to $\mathcal{L}'$ is again an EL-labeling.

Proof. Let λ' denote the restriction of λ to $\mathcal{L}'$, and let x, y be the elements from the statement. We proceed by contradiction and suppose that λ' is not an EL-labeling of $\mathcal{L}'$.

Note that $\mathscr{C}(\mathcal{L}') \subseteq \mathscr{C}(\mathcal{L})$, and for $X \in \mathscr{C}(\mathcal{L}')$ we have $\lambda'(X) = \lambda(X)$. Since λ' is not an EL-labeling of $\mathcal{L}'$, there must be some interval I' in $\mathcal{L}'$ in which the EL-property of λ' fails. We conclude that $x, y \in I'$ (since otherwise $\lambda \equiv \lambda'$ on I', which is a contradiction). We can moreover assume without loss of generality that x is the least element of I', i.e. $I' = [x, z]$ for some z. Let I be the corresponding interval in $\mathcal{L}$. There are three possibilities for λ' to fail to be an EL-labeling of I'. The existence of more than one rising maximal chain in I' contradicts the assumption that λ is an EL-labeling of I, and the same holds for the assumption that the unique rising chain of I' is not lexicographically first. It follows that there does not exist a rising maximal chain in I'. Since there is a rising maximal chain X in I, we conclude that $x, y \in X$; in particular x is the first and y is the second element of X. Since λ is an EL-labeling of $\mathcal{L}$, we conclude that $\lambda(x, y) \preceq \lambda(x, y')$ for all $y' \in L$ with $x \lessdot y'$. ☑

By definition $(PE_n, \leq_{\mathrm{pchn}})$ is obtained from $(PE_n, \leq_{\mathrm{dref}})$ by removing certain cover relations, and the next result states that these satisfy the condition from Proposition 4.3.

Proposition 4.4. *Let $\mathbf{x}, \mathbf{y} \in PE_n$ such that $\pi(\mathbf{x}, \mathbf{y}) = n-1$, where π is the labeling defined in* (12)*. There exists $\mathbf{y}' \in PE_n$ with $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}'$ such that $\pi(\mathbf{x}, \mathbf{y}') < n-1$ and $\lambda(\mathbf{x}, \mathbf{y}) > \lambda(\mathbf{x}, \mathbf{y}')$, where λ is the EL-labeling of $(PE_n, \leq_{\mathrm{dref}})$ coming from the left-modular chain* (6)*.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be as desired. Since $\pi(\mathbf{x}, \mathbf{y}) = n-1$, there must be a block B of $\mathbf{x}$ containing $n-1$, and $\{n\}$ must be a singleton block of $\mathbf{x}$. Moreover, $\mathbf{y}$ must contain the block $B \cup \{n\}$. Since $\mathbf{x} \in PE_n$ we conclude that $B \neq \{n-1\}$ and $1 \notin B$; in particular $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{1}$. Let A be the block of $\mathbf{x}$ containing 1. Let $\mathbf{y}'$ be the partition that contains all blocks of $\mathbf{x}$ except that A and $\{n\}$ are replaced by $A \cup \{n\}$. Since $\mathbf{x} \in PE_n$, the blocks A and B cannot be crossing, which implies that $\mathbf{y}' \in PE_n$. Moreover, we have $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}'$. We claim that $\mathbf{y}'$ is the desired element.

First of all $\pi(\mathbf{x}, \mathbf{y}') < n-1$, since $n-1 \notin A$, so that the cover relation $\mathbf{x} \lessdot_{\mathrm{dref}} \mathbf{y}'$ is still present in $(PE_n, \leq_{\mathrm{pchn}})$.

Recall that the left-modular chain (6) of $(PE_n, \leq_{\mathrm{dref}})$ consists of the elements $\mathbf{x}_i$ given by the unique non-singleton block $[i-1] \cup \{n\}$.

Observe that $\mathbf{x}_2 \leq_{\mathrm{dref}} \mathbf{y}'$, since $\{1, n\}$ is the unique non-singleton block of $\mathbf{x}_2$, and $1 \sim_{\mathbf{y}'} n$. Since $1 \not\sim_{\mathbf{x}} n$, we conclude $\mathbf{x}_2 \not\leq_{\mathrm{dref}} \mathbf{x}$. We thus obtain

$\mathbf{x} \vee_{PE} \mathbf{x}_2 \wedge_{PE} \mathbf{y}' = \mathbf{y}'$, which implies $\lambda(\mathbf{x}, \mathbf{y}') = 1$. (Observe the shift in the indices of (6) compared to the definition of the labeling λ in (1).) On the other hand, $1 \not\sim_{\mathbf{y}} n$ implies $\mathbf{x} \vee_{PE} \mathbf{x}_2 \wedge_{PE} \mathbf{y} <_{\mathrm{dref}} \mathbf{y}$. We thus have $\lambda(\mathbf{x}, \mathbf{y}) > 1 = \lambda(\mathbf{x}, \mathbf{y}')$. (In fact we have $\lambda(\mathbf{x}, \mathbf{y}) = k$, where $k = \min B$.) ☑

We conclude this article with the remaining proofs.

Proof of Theorem 1.1. This follows by construction from Propositions 4.3 and 4.4. ☑

Proof of Corollary 1.2. This follows from Theorem 1.1 and Theorems 4.2 and 2.3. ☑

Acknowledgements

I thank Joel Lewis for finding the formula for the cardinalities of the intervals $[\mathbf{a}_{n-2,n-1}, \mathbf{1}]$ in $(PE_n, \leq_{\mathrm{dref}})$ considered at the end of Section 3. Further thanks go to an anonymous referee for helpful comments and suggestions on the exposition of this paper.

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(Recibido en marzo de 2017. Aceptado en enero de 2018)

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