Discriminant Structures Associated to Matrix Semantics

Estructuras Discriminantes Asociadas a Semánticas Matriciales

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ABSTRACT. In this paper we show a method to characterize logical matrices by means of a special kind of structures, called here *discriminant structures* for this purpose. Its definition is based on the *discrimination* of each truth-value of a given (finite) matrix $M = (\mathbf{A}, D)$, w.r.t. its belonging to D. From this starting point, we define a whole class $\mathcal{S}_M$ of discriminant structures. This class is characterized by a set of Boolean equations, as it is shown here. In addition, several technical results are presented, and it is emphasized the relation of the Discriminant Structures Semantics (D.S.S) with other related semantics such as Dyadic or Twist-Structure.

Key words and phrases. Matrix Logic; n-valued logics; Discriminant Structures.

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RESUMEN. En este artículo mostramos un método para caracterizar matrices lógicas por medio de una clase especial de estructuras, llamadas aquí *estructuras discriminantes*. Su definición es basada en la *discriminación* de cada valor de verdad de una matriz (finita) $M = (\mathbf{A}, D)$ dada, con respecto a su pertenencia a D. Con este punto de partida, definimos toda una clase $\mathcal{S}_M$ de estructuras discriminantes. Esta clase es caracterizada por un conjunto de ecuaciones Booleanas, según aquí se indica. Además, son demostrados diversos resultados técnicos y se enfatiza la relación de la Semántica de Estructuras Discriminantes (D.S.S) con otras semánticas relacionadas tales como las Semánticas Diádicas o las Semánticas de Estructuras Twist.

Palabras y frases clave. Lógica Matricial; lógicas n-valuadas; Estructuras Discriminantes.

1. Preliminaries

An interesting point of view against semantics based on many truth-values was established by R. Suszko in [13]. According to his opinion, it would be more faithful, under a semantic perspective, to consider just two truth-values and to obtain some new kind of interpretation from the involved formulas to this "new" values (by the way, in this case the interpretations suggested do not need to be homomorphisms). Thus, the truth-values of a given matrix are merely *reference values* (which should not be confused with truth-ones).

Following this idea, several researchers have intended to "pass from" many-valued matrices to other structures that have as basis just two truth-values. A frequent underlying idea that guides some studies about these topics is the following: the truth-values of a matrix $M = (\mathbf{A}, D)$ can be often separated (or *discriminated*), according to its belonging to D, the set of distinguished truth-values of M. Moreover, this motivation was already applied in the literature (see [12], [1] or [2], for instance).

Based on the previous motivations, in this work we present a kind of representation of many-valued matrices, in such a way that every involved truth-value can be "codified" by tuples of $0's$ or $1's$. So, for every matrix $M = (\mathbf{A}, D)$ admitting this codification, the support of its underlying algebra $\mathbf{A}$ can be understood as a subset of 2^t, for a suitable t. In addition, this characterization of M induces *a whole class $\mathcal{S}_M$ of structures* (which will be called "discriminant structures", here). Every member of $\mathcal{S}_M$ can be considered as a C-matrix, indeed. But the key point along this work is that every discriminant structure of $\mathcal{S}_M$ is defined by means of certain Boolean equations, which are determined by the original representation of M. Moreover, the class $\mathcal{S}_M$ defines in a natural way the consequence relation $\models_{\mathcal{S}_M}$ (which, in an informal way, will be frequently mentioned as the "Discriminant Structure Semantics", or D.S.S., associated to M). Moreover, for every formula α, $\models_{\mathcal{S}_M} \alpha$ iff $\models_M \alpha$, as we shall demonstrate later.

Summarizing, the representation of a given matrix $M = (\mathbf{A}, D)$ showed here induces a new kind of semantics that can be considered a sort of "middle point" between the matrix semantics and the algebraic one (in the sense that $\models_{\mathcal{S}_M}$ consists of a class of matrices characterized by equations) that, in addition, is weakly adequate w.r.t. $\models_M$. Along this paper we will show these results and another ones, that motivate and provide a reasonable initial analyisis of the scope of D.S.S. With this in mind, the organization of this paper is as follows: in the sequel we will fix the definitions and notation to be used along this article. Section 2 will explain, in informal terms for the moment, the way that D.S.S. behaves, by means of some clarifying examples. All this discussion will be extended (in a more technical mode) in Section 3. Indeed, it is in this section where D.S.S. will be presented in a formal way, together with the proof of the main technical results. In Section 4 the following interesting

fact will be established: not every finite matrix admits Discriminant Structure Semantics. This will suggest some lines of research that we shall discuss later. Finally, in the last section we will compare our proposed semantics to other ones, related with the approach of D.S.S. in some sense, as Dyadic Semantics (already indicated) and Twist-Structure Semantics. We will conclude this paper with some comments about possible future work.

With respect to the basic notation and definitions that will be used in this paper, take into account that we are studying different ways of definition of a same consequence relation. So, we choose to use the traditional formalism of Abstract Logic, applied mainly to the particular case of consequence relations defined by matrices. For that, we are based mainly on the point of view developed in [3], with some little notational changes, when necessary.

Definition 1.1. We denote by $\omega = \{0, 1, 2...\}$ the set of natural numbers; a **signature** is a set $C = \{c_i\}_{i \in I}$ together with a function $\rho : C \to \omega$. Here, every element $c \in C$ will be called a **connective of** C, being $\rho(c)$ the **arity of** c. Given a signature C, the **sentential language determined by** C is the absolutely free algebra generated by C over a countable fixed set $\mathcal{V}$ (this set will be called the set of **atomic formulas of** $L(C)$).

Every language $L(C)$ can be understood as a particular, paradigmatic case of a C-*algebra*, whose formal definition is as follows:

Definition 1.2. Given a signature C, a C-algebra is a pair $\mathbf{A} = (A, C_{\mathbf{A}})$, such that every connective c of arity $\rho(c)$ has associated an operation $c_{\mathbf{A}} \in C_{\mathbf{A}}$, with the same arity of c. The set A will be called the **support** of $\mathbf{A}^1$. Note that a C-algebra is, actually, any algebra similar to $\mathbf{L(C)} := (L(C), C)$ (in this last case $C_{\mathbf{L(C)}}$ is being identified with C, indeed).

Along this paper we will work with several signatures. However, a special signature (the Boolean one) deserves special attention:

Definition 1.3. The **Boolean signature** is the set $\overline{C} = \{\vee, \wedge, -, \overline{1}, \overline{0}\}$, with obvious arities. Every $\overline{C}$-algebra $\mathbf{B} = (B, \overline{C}_{\mathbf{B}})$ will be called a **Boolean algebra** if and only if it is, in addition, a bounded, distributive, complemented lattice. In this context, it can be defined the "secondary operation" $\to$ as usual: $a_1 \to a_2 := -a_1 \vee a_2$. As a particular case of a Boolean algebra, it will be very used in this paper the canonical two-element algebra $\mathbf{2} = (2, \overline{C}_{\mathbf{2}})$, with $2 = \{0, 1\}$. By the way, every function $f : 2^r \to 2$, with $r \in \omega$, will be called simply a **2-Boolean function**.

Recall here this reformulation of the Conjunctive Normal Form Theorem, which will be used later:

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[1]If the context is clear enough, we will indicate $c_{\mathbf{A}}$ simply as c (and the set $C_{\mathbf{A}}$ will be denoted simply as C).

Proposition 1.4. *Every* 2-*Boolean function* $f : 2^n \to 2$ *can be identified with a function* $f' : 2^n \to 2$ *(the conjunctive normal form of* f, *indeed) in such a way that* f' *is defined using the functions of* $\overline{C}_2$.

At this point, note that any signature C plays (at least) two roles: on one hand, it determines sentential languages; on the other hand it defines C-algebras. Of course, these notions are strongly related. For instance, C-algebras are the basis of *sentential logics* that are defined by means of *logical matrices*. We recall both definitions in the sequel.

Definition 1.5. An **abstract** (sentential) **logic** is a pair $\mathcal{L} = (C, \vdash)$, where C is a signature and $\vdash \,\subseteq \wp(L(C)) \times L(C)$ is a **consequence relation for** $L(C)$. That is, it satisfies, for every $\Gamma \cup \{\alpha\} \subseteq L(C)$, **extensiveness**, **monotonicity** and **transitivity** (we omit these well-known definitions).

Definition 1.6. Given a signature C, a C-**matrix** is a pair $M = (\mathbf{A}, D)$, where $\mathbf{A} = (A, C_{\mathbf{A}})$ is a C-algebra and $D \subseteq A$. The elements of D are called the **designated values of** M. In the context of C-matrices, any t-ary operation $c_{\mathbf{A}} \in C_{\mathbf{A}}$ will be called an **A-truth-function**. Besides that, **the support of** M is just the support A of $\mathbf{A}$.

Every C-matrix defines a consequence relation for $L(C)$, as usual:

Definition 1.7. Let $M = (\mathbf{A}, D)$ be a C-matrix. A M-**valuation** is a homomorphism $v : L(C) \to A$. The **consequence relation induced by** M is $\models_M$, defined in the following way: $\Gamma \models_M \varphi$ iff, for each valuation v, if $v(\Gamma) \subseteq D$, then $v(\varphi) \in D$. We say that φ is **tautology** (relatively to M) iff $\varnothing \models_M \varphi$ (denoting this as $\models_M \varphi$). The **logic induced by** M is the pair $\mathcal{L} = (C, \models_M)$. If the domain of a C-matrix M is finite we will say that M is a n-valued matrix and, by extension, that $\mathcal{L} = (C, \models_M)$ is a n-**valued logic**. This definition can be generalized to classes: if $\mathcal{K}$ is a class of C-matrices, the **consequence relation** $\models_{\mathcal{K}}$ is given by: $\Gamma \models_{\mathcal{K}} \varphi$ iff $\Gamma \models_M \varphi$ for every M in $\mathcal{K}$.

Turning back to C-matrices, we will also use the following notions:

Definition 1.8. Given two C-matrices $M_1 = (\mathbf{A_1}, D_1)$ and $M_2 = (\mathbf{A_2}, D_2)$, we say that $h : A_1 \to A_2$ is a **matrix homomorphism from** M_1 **to** M_2 iff it is an homomorphism (in the algebraic sense) from $\mathbf{A_1}$ to $\mathbf{A_2}$ verifying additionally that $h(D_1) \subseteq D_2$. If, in addition, h is surjective, we will say that it is an **matrix epimorphism**. On the other hand, $h : A_1 \to A_2$ is a **matrix isomorphism** (between M_1 and M_2) iff it is an isomorphism (in the algebraic sense) verifying additionally that $h(D_1) = D_2$ and $h(A_1 \setminus D_1) = A_2 \setminus D_2$. Notice that, if M_1 and M_2 are isomorphic C-matrices, then $\models_{M_1} \,=\, \models_{M_2}$.

We conclude this section with some comments about notation: the metavariables referred to formulas will be denoted by greek lowercase letters (with

subscripts, if neccesary). In particular, we will use the letters $\alpha, \alpha_0, \alpha_1, \ldots$, *only in the denotation of atomic formulas*. The expression $\beta = \beta(\alpha_1, \ldots, \alpha_n)$ means that the atomic formulas of β belong to the set $\{\alpha_1, \ldots, \alpha_n\}$. If $\beta = \beta(\alpha_1, \ldots, \alpha_n)$, the expression $\beta(\alpha_1/\gamma_1, \ldots, \alpha_n/\gamma_n)$ denotes the uniform substitution, in β, of the atomic formulas α_i by the formulas γ_i (with $1 \leq i \leq n$). If there is no risk of confusion, this expression will be abbrevied to $\beta(\gamma_1, \ldots, \gamma_n)$. With respect to notation related to algebras: the elements of any support A will be denoted by the letters $\{a_0, a_1, a_2, \ldots\}$ (usually as reference to "specific elements of A"), or by the letters $\{x_0, \ldots, x_n, \ldots y_0, y_1, \ldots\}$ (as an informal notation for variables ranging on A). In addition, the symbols $\vec{x}$ and $\vec{a}$ denote tuples (for instance, $\vec{a} := (a_1, \ldots, a_t) \in A^t$). In this context, the symbol π_i denotes the ith projection of any tuple. Besides that, the symbol $\cong$ will denote *isomorphism* (between algebras or between matrices, depending on the context). Any other definition or notation (or, even, notational abuse or convention) to be used in this paper will be indicated when needed.

2. Discriminant Structure Semantics: some motivating examples

We begin our definition of Discriminant Structure Semantics with several examples that will help us to understand some motivations behind the formal definition of D.S.S. By the way, some of these examples will be used with other purposes later. In all these examples the signature to be used will be the same: $C^\star = \{\supset, \neg\}$, with obvious arities.

First of all, we give the definition of a Discriminant Structure Semantics for the logic $\mathbf{I^1 P^0}$ (better known simply as the "weakly - intuitionistic logic $\mathbf{I^1}$ "), defined in [11].

Definition 2.1. The logic $\mathbf{I^1 P^0} = (C^\star, \models_{[1,0]})$ is defined by means of the $C^\star$-matrix $M_{[1,0]} = (\mathbf{A}_{[1,0]}, D_{[1,0]})$, being $D_{[1,0]} = \{T_0\}$, and being $\mathbf{A}_{[1,0]}$ the $C^\star$-algebra whose support is $A_{[1,0]} := \{F_0, F_1, T_0\}$, and whose truth-functions $\neg$ and $\supset$ are indicated in the tables below:

$\supset$	F_0	F_1	T_0
F_0	T_0	T_0	T_0
F_1	T_0	T_0	T_0
T_0	F_0	F_0	T_0

$\neg$	
F_0	T_0
F_1	F_0
T_0	F_0

For a better understanding of $M_{[1,0]}$, the truth-values F_0 and T_0 are classical truth and falsehood, respectively; on the other hand, F_1 is an "intermediate value of falsehood".

We will characterize the tautologies of $\mathbf{I^1 P^0}$ *using discriminant structures*, whose definition will be given in the sequel. This characterization (with some little changes) was shown in [8]. For our purposes we are based on Boolean algebras of the form $\mathbf{B} = (B, C_B)$ (recall Definition 1.3):

Definition 2.2. The **discriminant structure (of type** $[1,0]$**) associated to a given Boolean algebra B** (the d.s. of type $[1,0]$ associated to **B**, for brevity) is the $C^\star$-matrix $R_{[1,0]}(\mathbf{B}) = (\mathbf{A}'_{[\mathbf{1},\mathbf{0}]}(B), D'_{[1,0]}(B))$, being $\mathbf{A}'_{[\mathbf{1},\mathbf{0}]}(B)$ the $C^\star$ algebra such that:

(1) Its support is $A'_{[1,0]}(B) = \{(x_0, x_1) \in B \times B : x_0 \wedge_\mathbf{B} x_1 = \bar{0}_\mathbf{B}\}$.

(2) $(x_0, x_1) \supset (y_0, y_1) := (x_0 \rightarrow_\mathbf{B} y_0, -_\mathbf{B}(x_0 \rightarrow_\mathbf{B} y_0))$.

(3) $\neg(x_0, x_1) := (x_1, -_\mathbf{B} x_1)$.

In addition, the set of designated values of $R_{[1,0]}(\mathbf{B})$ is $D'_{[1,0]}(B) = \{(\bar{1}_\mathbf{B}, \bar{0}_\mathbf{B})\}$.

Of course, $\wedge_\mathbf{B}$, $-_\mathbf{B}$ and $\rightarrow_\mathbf{B}$ are the operations, with obvious behavior, defined in the context of every Boolean algebra **B**.

The class of all the discriminant structures of type $[1,0]$ will be denoted by $\mathcal{S}_{[1,0]}$. This class will be called the **Discriminant Structure Semantics for** $\mathbf{I^1 P^0}$ (The D.S.S. for $\mathbf{I^1 P^0}$, for short). Considering this definition, we define $\models_{\mathcal{S}_{[1,0]}} \subseteq \wp(L(C^\star)) \times L(C^\star)$ as being the matrix consequence relation indicated in Definition 1.7 (for classes of matrices). So, $\Gamma \models_{\mathcal{S}_{[1,0]}} \alpha$ iff, for every d. s. $R_{[1,0]}(\mathbf{B})$ it is valid that, for every $R_{[1,0]}(\mathbf{B})$-valuation $w : L(C^\star) \rightarrow A'_{[1,0]}(B)$, $w(\Gamma) \subseteq \{(\bar{1}_\mathbf{B}, \bar{0}_\mathbf{B})\}$ implies $w(\alpha) = (\bar{1}_\mathbf{B}, \bar{0}_\mathbf{B})$.

About the previous definition it can be easily proved that:

Proposition 2.3. *The operations $\supset$ and $\neg$ are well defined. That is, every set $A'_{[1,0]}(B)$ is closed by applications of $\supset$ and $\neg$.*

Definition 2.4. The **canonical d. s.** of type $[1,0]$ is the d. s. $R_{[1,0]}(\mathbf{2})$. From Definition 2.2 we have that, in this structure, the functions $\supset$ and $\neg$ behave as depicted in the following tables.

$\supset$	$(1,0)$	$(0,0)$	$(0,1)$
$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$
$(0,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(0,1)$	$(1,0)$	$(1,0)$	$(1,0)$

$\neg$	
$(1,0)$	$(0,1)$
$(0,0)$	$(0,1)$
$(0,1)$	$(1,0)$

It should be clear that $R_{[1,0]}(\mathbf{2}) \cong M_{[1,0]}$, cf. Definition 1.8. Besides that, it is easy to prove the following:

Proposition 2.5. *The discriminant structures of $\mathcal{S}_{[1,0]}$ verify:*

(a) If $\mathbf{B_1}$ and $\mathbf{B_2}$ are isomorphic Boolean algebras, then $R_{[1,0]}(\mathbf{B_1})$ and $R_{[1,0]}(\mathbf{B_2})$ are isomorphic matrices.

(b) Every $R_{[1,0]}(\mathbf{B})$-valuation $w : L(C^\star) \longrightarrow A'_{[1,0]}(B)$ determines a pair of (non-homomorphic) functions (w_0, w_1) (being $w_i := \pi_i \circ w$) such that, for every $\varphi \in L(C^\star)$, $w(\varphi) = (\bar{1}_\mathbf{B}, \bar{0}_\mathbf{B})$ iff $w_0(\varphi) = \bar{1}_\mathbf{B}$.

In the sequel, we will prove (in a sketched way; see the complete proofs in [8]) that $\models_{[1,0]} \varphi$ iff $\models_{\mathcal{S}_{[1,0]}} \varphi$, for every $\varphi \in L(C^\star)$. For that, we will make use of some traditional results about prime filters of Boolean algebras:

Proposition 2.6. *Let $\mathbf{B}$ be any Boolean algebra. Then:*

(a) For every $x \neq \bar{1}_{\mathbf{B}}$, there exists a prime filter ∇ such that $x \notin \nabla$.

(b) For every prime filter ∇ of $\mathbf{B}$, the binary relation $\equiv_\nabla$ defined by: $x \equiv_\nabla y$ iff $\{x \to_{\mathbf{B}} y, y \to_{\mathbf{B}} x\} \subseteq \nabla$ is a congruence, and the quotient $\mathbf{B}/\nabla$ (whose support is $B/\nabla = \{\nabla, \Delta\}$) is isomorphic to the Boolean algebra $\mathbf{2}$, being $\Delta := B - \nabla$ the prime ideal associated to ∇.

From Propositions 2.5(a) and 2.6(b) we have:

Corollary 2.7. *For every Boolean algebra $\mathbf{B}$ and every prime filter $\nabla \subseteq B$, $R_{[1,0]}(\mathbf{B}/\nabla) \cong R_{[1,0]}(\mathbf{2})$, where the operations in the algebra $\mathbf{A}'_{[\mathbf{1},\mathbf{0}]}(B/\nabla)$ are given as in Definition 2.4 (replacing 1 by ∇ and 0 by Δ).*

Proposition 2.8 (Trichotomy)**.** *Let $\mathbf{B}$ be any Boolean algebra and let ∇ be any prime filter of $\mathbf{B}$. Then, for every pair $(x_0, x_1) \in A'_{[1,0]}(B)$, one and only one of the following conditions is valid:*

- $x_0 \in \nabla$ *and* $x_1 \in \Delta$.

- $x_0 \in \Delta$ *and* $x_1 \in \Delta$.

- $x_0 \in \Delta$ *and* $x_1 \in \nabla$.

Proof. It follows from basic properties of prime filters and Definition 2.2. ☑

Proposition 2.9. *Every prime filter ∇ of a Boolean algebra $\mathbf{B}$ induces a matrix epimorphism $E_\nabla : R_{[1,0]}(\mathbf{B}) \to R_{[1,0]}(\mathbf{B}/\nabla)$ such that $E_\nabla(x_0, x_1) = (\nabla, \Delta)$ iff $x_0 \in \nabla$.*

Proof. For every $(x_0, x_1) \in R_{[1,0]}(\mathbf{B})$, we define $E_\nabla(x_0, x_1) := (e_\nabla(x_0), e_\nabla(x_1))$, being $e_\nabla(x_i) := \overline{x}_\nabla$, the equivalence class of x_i in $\mathbf{B}/\nabla$. By Proposition 2.8, E_∇ is a well-defined surjective function. Besides, $x_0 \in \nabla$ iff $\overline{x_0}_\nabla = \nabla$ iff $E_\nabla(x_0, x_1) = (\nabla, \Delta)$ (again, by Proposition 2.8). This also implies that E_∇ preserves the designated values. Finally, it is not difficult to prove that E_∇ is homomorphism. That is:

(A): $E_\nabla(\neg(x_0, x_1)) = \neg(E_\nabla(x_0, x_1))$.

(B): $E_\nabla((x_0, x_1) \supset (y_0, y_1)) = E_\nabla(x_0, x_1) \supset E_\nabla(y_0, y_1)$.

So, from (A) and (B), the proof is completed. ☑

Theorem 2.10. *For every $\varphi \in L(C^\star)$, $\models_{\mathcal{S}_{[1,0]}}\varphi$ iff $\models_{[1,0]} \varphi$.*

Proof. On one hand, since $M_{[1,0]} \cong R_{[1,0]}(\mathbf{2})$, we get that $\models_{\mathcal{S}_{[1,0]}} \varphi$ implies $\models_{[1,0]}$ φ. Conversely, suppose that there exists a Boolean algebra $\mathbf{B}$ and a $R_{[1,0]}(\mathbf{B})$-valuation $w : L(C^\star) \to A'_{[1,0]}(B)$ such that $w(\varphi) \neq (\bar{1}_\mathbf{B}, \bar{0}_\mathbf{B})$. Then, $w_0(\varphi) \neq \bar{1}_\mathbf{B}$ (see Proposition 2.5(b)). So, there is a prime filter $\nabla \subseteq B$ with $w_0(\varphi) \notin \nabla$, by Proposition 2.6(a). From this and Proposition 2.9, there exists an epimorphism $E_\nabla : A'_{[1,0]}(B) \to A'_{[1,0]}(B/\nabla)$, with $E_\nabla(w(\varphi)) \neq (\nabla, \Delta)$ (because $w_0(\varphi) \notin \nabla$). So, the $R_{[1,0]}(\mathbf{B}/\nabla)$-valuation $E_\nabla \circ w : L(C^\star) \longrightarrow A'_{[1,0]}(B/\nabla)$ verifies $(E_\nabla \circ w)(\varphi) \neq (\nabla, \Delta)$, the designated value of $R_{[1,0]}(\mathbf{B}/\nabla)$. Thus, $\not\models_{[1,0]} \varphi$, since $R_{[1,0]}(\mathbf{B}/\nabla) \cong R_{[1,0]}(\mathbf{2}) \cong M_{[1,0]}$. This concludes the proof. $\quad\checkmark$

A generalization of $\mathbf{I^1P^0}$, and simultaneously of the paraconsistent logic $\mathbf{I^0P^1}$ (which is traditionally known as $\mathbf{P^1}$, see [10]) is the logic $\mathbf{I^2P^1}$, presented in [4].

Definition 2.11. The logic $\mathbf{I^2P^1} = (C^\star, \models_{[2,1]})$ is defined by means of the $C^\star$-matrix $M_{[2,1]} = (\mathbf{A}_{[2,1]}, D_{[2,1]})$, with $D_{[2,1]}=\{T_0, T_1\}$ and such that the support of $\mathbf{A}_{[2,1]}$ is $A_{[2,1]}:= \{F_0, F_1, F_2, T_0, T_1\}$. In addition, the truth-functions $\neg$ and $\supset$ here behave as follows:

$\supset$	F_0	F_1	F_2	T_0	T_1
F_0	T_0	T_0	T_0	T_0	T_0
F_1	T_0	T_0	T_0	T_0	T_0
F_2	T_0	T_0	T_0	T_0	T_0
T_1	F_0	F_0	F_0	T_0	T_0
T_0	F_0	F_0	F_0	T_0	T_0

$\neg$	
F_0	T_0
F_1	F_0
F_2	F_1
T_1	T_0
T_0	F_0

It can be obtained a Discriminant Structure Semantics for $\mathbf{I^2P^1}$ following a procedure similar to the one applied for $\mathbf{I^1P^0}$.

Definition 2.12. For every Boolean algebra $\mathbf{B}$, the **d.s.** (of type $[2, 1]$) **associated to B** is the $C^\star$-matrix $R_{[2,1]}(\mathbf{B}) = (\mathbf{A}'_{[\mathbf{2,1}]}(\mathbf{B}), D'_{[2,1]}(\mathbf{B}))$, where:

(1) The support of $\mathbf{A}'_{[\mathbf{2,1}]}(\mathbf{B})$ is $A'_{[2,1]}(B):=$

$\{(x_0, x_1, x_2) \in B^3 : (-x_0 \wedge_\mathbf{B} -x_1) \vee_\mathbf{B} (-x_1 \wedge_\mathbf{B} x_2) \vee_\mathbf{B} (x_1 \wedge_\mathbf{B} -_\mathbf{B} x_2) = \bar{1}_\mathbf{B}\}$.

(2) $(x_0, x_1, x_2) \supset (y_0, y_1, y_2) := (x_0 \to_\mathbf{B} y_0, -_\mathbf{B}(x_0 \to_\mathbf{B} y_0), x_0 \to_\mathbf{B} y_0)$.

(3) $\neg(x_0, x_1, x_2) := (x_1, x_2, -_\mathbf{B} x_2)$.

(4) $D'_{[2,1]}(\mathbf{B}):=\{(\bar{1}_\mathbf{B}; \bar{0}_\mathbf{B}; \bar{1}_\mathbf{B}); (\bar{1}_\mathbf{B}; \bar{1}_\mathbf{B}; \bar{0}_\mathbf{B})\}$.

We define, as in Definition 2.2, the class $\mathcal{S}_{[2,1]}$ of all the discriminant structures of type $(2,1)$, which will be called the D.S.S. for $\mathbf{I^2P^1}$. This class defines $\models_{\mathcal{S}_{[2,1]}}$, in a similar way to Definition 2.2. Moreover, it is possible to prove as in Theorem 2.10:

Theorem 2.13. *For every $\varphi \in L(C^\star)$, $\models_{\mathcal{S}_{[2,1]}} \varphi$ iff $\models_{[2,1]} \varphi$.*

The proof of the theorem above is based, again, on the definition of a *canonical* discriminant structure $R_{[2,1]}(\mathbf{2}) \cong M_{[2,1]}$ (recall Definition 2.4), whose support set is $A'_{[2,1]}(2) = \{(0,1,0);(0,0,1);(0,0,0);(1,0,1);(1,1,0)\}$. The isomorphism f between both matrices is defined by:

$$
\begin{aligned}
f(F_0) &= (0,1,0) \\
f(F_1) &= (0,0,1) \\
f(F_2) &= (0,0,0) \\
f(T_0) &= (1,0,1) \\
f(T_1) &= (1,1,0).
\end{aligned}
$$

***Remark* 2.14.** At this point it is reasonable to know which are the hidden reasons that motivate, in Definitions 2.2 and 2.12, this kind of structures. As a first approximation, let us pay attention to the operation $\neg$ in the matrix $M_{[1,0]}$ (see Definition 2.2): if we consider that $D_{[1,0]} = \{T_0\}$ we can see that, when applied to every truth-value of $M_{[1,0]}$, $\neg$ verifies:

$$T_0 \in D_{[1,0]}; \ \neg T_0 \notin D_{[1,0]}.$$

This fact can be informally "codified" (interpreting "1" as "belonging to $D_{[1,0]}$", and "0" otherwise), as $T_0 \to 1$, $\neg T_0 \to 0$. So, the value T_0 is associated to the pair $(1,0)$. With the same approach, we can associate F_0 with the pair $(0,1)$, and F_1 with the pair $(0,0)$ (since $\neg F_1 = F_0 \notin D_{[1,0]}$). This interpretation allows to define the support set $A'_{[0,1]}(2)$, which will suggest the definition of every support of the form $A'_{[0,1]}(B)$, as we will see in the next section.

In the case of Definition 2.12, the truth-funcion $\neg$ *cannot discriminate* the truth-values F_1 and F_2 (since nor $\neg F_1$ nor $\neg F_2$ belongs to $D_{[2,1]}$). Anyway, if $\neg$ is applied in an iterated way, we obtain the following results:

$$
\begin{aligned}
F_0 &\notin D_{[2,1]}; \ \neg F_0 = T_0 \in D_{[2,1]}; \ \neg\neg F_0 = F_0 \notin D_{[2,1]} \\
F_1 &\notin D_{[2,1]}; \ \neg F_1 = F_0 \notin D_{[2,1]}; \ \neg\neg F_1 = T_0 \in D_{[2,1]} \\
F_2 &\notin D_{[2,1]}; \ \neg F_2 = F_1 \notin D_{[2,1]}; \ \neg\neg F_2 = F_0 \notin D_{[2,1]} \\
T_0 &\in D_{[2,1]}; \ \neg T_0 = F_0 \notin D_{[2,1]}; \ \neg\neg T_0 = T_0 \in D_{[2,1]} \\
T_1 &\in D_{[2,1]}; \ \neg T_1 = T_0 \in D_{[2,1]}; \ \neg\neg T_1 = F_0 \notin D_{[2,1]}.
\end{aligned}
$$

In other words, the iteration of the application of $\neg$ suggests the definition of the isomorphism showed right after Theorem 2.13, determining also in this case the *support of the canonical structure $R_{[2,1]}(\mathbf{2})$*.

The previous examples show that, to obtain the "canonical discriminant structure", which is a special matrix, isomorphic to $M_{[1,0]}$ (resp. $M_{[2,1]}$), we will use the truth-function $\neg$, eventually iterated, that *discriminate the truth values of the matrix analized*. However, this procedure cannot be applied in the case of logics without negation, or even without any 1-ary truth-function. So, it is necessary to adapt the previous idea to these logics and, in a more general way, to every logic characterized by finite matrices. This will be developed in the next section.

3. D.S.S: Abstract Definition and Some Results

Based on the construction for the D.S.S. for $M_{[1,0]}$ and for $M_{[2,1]}$, let us try to explain in an informal way the process that will allow us to find a D.S.S of a form $\mathcal{S}_M$ for a given C-matrix logic $M = (\mathbf{A}, D)$. For that, according with the ideas of the previous section, we would consider as a starting point the map $\chi_D : A \longrightarrow 2$, the *characteristic function referred to D* (w.r.t. the universe A). Besides that, to deal with iterations of functions, we will abbreviate the composition of truth-functions $f \circ \cdots \circ f$ (k times) by f^k. Of course, if $k = 0$, then $f^k = id$. With these conventions, the basis of a D.S.S. $\mathcal{S}_M$ is the existence of a *discriminant pair* for M, which is an adequate generalization of the truth-function $\neg$ of the previous examples, as we shall see.

Definition 3.1. Given a C-matrix $M = (\mathbf{A}, D)$, a **discriminant pair for** M is a pair $(\beta, \vec{a})$, where $\beta = \beta(\alpha_0, \alpha_1, \ldots, \alpha_m) \in L(C)$, $\vec{a} = (a_1, \ldots, a_m) \in A^m$ and the $\mathbf{A}$-truth-function $f_{(\beta,\vec{a})} : A \to A$, defined by $f_{(\beta,\vec{a})}(x) := \beta_{\mathbf{A}}(x, \vec{a})^2$, is *discriminant (by iterations)*. That is, there exists $k \in \omega$ such that the function $\overline{[\chi_D]}_k : A \to 2^{k+1}$ is injective where, for every $x \in A$.

$$\overline{[\chi_D]}_k(x) := [\chi_D(x), \chi_D(f_{(\beta,\vec{a})}(x)), \chi_D(f^2_{(\beta,\vec{a})}(x)), \ldots, \chi_D(f^k_{(\beta,\vec{a})}(x))].$$

Remark 3.2. It should be clear that, if $\beta = \beta(\alpha_0)$, then we consider that $\vec{a}$ is not effectively used. This is the case of the discriminant pairs for $M_{[1,0]}$ and $M_{[2,1]}$: for both logics we consider $\beta = \beta(\alpha_0) := \neg(\alpha_0)$. So, $f_{(\beta,\vec{a})}(x) = \neg x$, as it was suggested in Remark 2.14, actually. Indeed, note that the function f showed there is precisely $f_{(\beta,\vec{a})}$ and verifies, for both logics, the "discrimination conditions" required in Definition 3.1. The only difference between the cases of $M_{[1,0]}$ and $M_{[2,1]}$ is the number of iterations that is needed. Later in this section we will see an example of a discriminant pair $(\beta, \vec{a})$ such that $\vec{a}$ is applied in an effective way.

It is worth to anticipate here the following result: if a C-matrix $M = (\mathbf{A}, D)$ admits a discriminant pair $(\beta, \vec{a})$, then it allows to obtain, in an implicit way:

---

[2]The expression $\beta_{\mathbf{A}}(x, \vec{a})$ should be interpreted, in the context of $\mathbf{A}$, as the $m + 1$ truth-function $\beta_{\mathbf{A}}(x, x_1 \ldots, x_m)$ (associated to the formula $\beta \in L(C)$), in which the values for the variables $x_1, \ldots, x_m$ are always replaced by $\vec{a} = (a_1, \ldots, a_m)$.

- The form of the *support set* $A'_M(2)$ of a certain "canonical discriminant structure", which is a particular C-matrix that will be denoted by $R_M(\mathbf{2})$.

- In addition, $(\beta, \vec{a})$ determines the behavior of the *truth-functions* in the C-algebra $\mathbf{A}'_{\mathbf{M}}(2)$ (which is the underlying algebra referred to $R_M(\mathbf{2})$).

- Finally, $(\beta, \vec{a})$ suggests the definition of *all the discriminant structures* (relative to M) that will conform a certain class $\mathcal{S}_M$ (and a consequence relation $\models_{\mathcal{S}_M}$), which is understood as the *D.S.S.* associated to M. Moreover, $\models_M \varphi$ iff $\models_{\mathcal{S}_M} \varphi$, for every $\varphi \in L(C)$.

To begin our proof of the results indicated above, it is necessary to deal in a formal way with the languages referred to *algebras* (and, in particular, to Boolean algebras, since they are the basis of the discriminant structures, as we have seen). So, recall the notion of the (first order) *Boolean equational language*:

Definition 3.3. The (First Order) **Boolean Equational language** is the first-order language with only a binary predicate symbol "=", having as set of function symbols the own set $\overline{C} = \{\vee, \wedge, -, \overline{1}, \overline{0}\}$ (recall Definition 1.3). In this language, the variable symbols will be indicated by $x_0, x_1, \dots$. In addition, the set of logical symbols (that not should be identified with any symbol of $\overline{C}$) is $\{\overline{\wedge}, \underline{\vee}, \Rightarrow, \overline{\neg}\}$. This language will be indicated as **FOBL**. Every atomic formula of this language will be called a **Boolean equation** (by the way, the set of all the Boolean equations will be denoted by $Eq_{\overline{C}}$). Finally, an expression $eq_1 \overline{\wedge} eq_2 \overline{\wedge} \cdots \overline{\wedge} eq_n \Rightarrow eq_0$, being $eq_0, \dots, eq_n$ Boolean equations, is called a **Boolean quasi-equation**.

***Remark* 3.4.** The previous definitions are motivated by the fact that, by well-known results of Universal Algebra, every Boolean equation (quasi-equation) is valid in every Boolean algebra $\mathbf{B}$ if and only if it is valid in $\mathbf{2}$. This fact will be widely used in the next proofs, as we shall see. Besides that, it is shown in Definition 3.3 another use of the signatures: $\overline{C}$ is applied in the context of **FOBL** to the definition of the *function symbols* of this language (and this is the reason that motivates the differentiation of $\overline{C}$ w.r.t the logical symbols). In addition, note the following notational abuse in the definition of **FOBL**: we are using the same notation for the informal reference to Boolean algebras, as it has been done until now, and for the Boolean Equational Language. So, the "informal" notation $x_1, x_2, \dots$ referred to variable elements of Boolean algebras will be used too, in an formal way, to denote symbols of variables of **FOBL**. This convention is applied also to the interpretation of the symbol "=". On the other hand (and considering the previous convention), recall that every term $\tau = \tau(x_1, \dots, x_n)$ of **FOBL** determines *in every Boolean algebra* $\mathbf{B}$ an n-ary function $\tau^{\mathbf{B}} : B^n \to B$ in the usual way.

With respect to the relations between terms of **FOBL** and Boolean functions, we remark this fact arisen from Definition 1.3 and Proposition 1.4:

Definition 3.5. For every Boolean algebra $\mathbf{B}$, the function $f : B^n \to B$ is $\overline{C}$-**definable** iff f can be expressed by an (iterathed) application of the functions of $\overline{C}_B$.

Proposition 3.6. *For every Boolean algebra $\mathbf{B}$ and every $\overline{C}$-definable function $f : B^r \to B$, there exists a term τ_f of* **FOBL** *such that $\tau_f^{\mathbf{B}} = f$. Moreover, by Proposition 1.4, every 2-Boolean function $f : 2^r \to 2$ determines a $\overline{C}$-definable function f' (the c.n.f of f) [3] and, therefore, a term $\tau_{f'} = \tau_{f'}(x_1, \ldots, x_n)$ of* **FOBL** *such that $\tau_{f'}^{\mathbf{2}} = f$.*

Proposition 3.7. *Let $r \in \omega$, and let $S \subseteq 2^r$. Then S can be characterized (relatively to 2^r) by a Boolean equation eq_S. That is, $S = \{\vec{x} \in 2^r : \ \vec{x} \text{ satisfies } eq_S \}$.*

Proof. Just consider the characteristic function $\chi_S : 2^r \longrightarrow 2$: since χ_S is a 2-Boolean function, it can be identified with f'_S, the c.n.f. of χ_S, which determines the term $\tau_{f'_S}$ of **FOBL** in such a way that $\tau_{f'_S}^{\mathbf{2}} = \chi_S$, by Proposition 3.6. The equation that characterizes S is, actually, $eq_S : \tau_{f'_S}(x_1, \ldots, x_r) = \overline{1}$.   ☑

Proposition 3.8. *Let $M = (\mathbf{A}, D)$ be a C-matrix which admits a discriminant pair $(\beta, \vec{a})$, by means of k-iterations, and let $\overline{[\chi_D]}_k : A \longrightarrow 2^{k+1}$ be the function obtained by $(\beta, \vec{a})$ (see Definition 3.1). Then there is a C-matrix M', whose support is $\overline{[\chi_D]}_k(A)$, such that $M' \cong M$.*

Proof. Define $M' := (\mathbf{A}', D')$, where the support of $\mathbf{A}'$ is $A' := \overline{[\chi_D]}_k(A)$ and $D' := \overline{[\chi_D]}_k(D)$. Since $\overline{[\chi_D]}_k$ is injective, $\overline{[\chi_D]}_k(A)$ is equipotent with A and $\overline{[\chi_D]}_k(D)$ is equipotent with D. Now, for every n-ary connective $c \in C$, define $c_{\mathbf{A}'} : (A')^n \longrightarrow A'$, by: $c'(x_1, \ldots, x_n) := \overline{[\chi_D]}_k(c_{\mathbf{A}}(\overline{[\chi_D]}_k^{-1}(x_1), \ldots, \overline{[\chi_D]}_k^{-1}(x_n))$. This definition makes sense by the definition of A', and since $\overline{[\chi_D]}_k$ is injective. From this, it easily follows that $\mathbf{A}' \cong \mathbf{A}$, and thus $M' \cong M$.   ☑

Remark 3.9. Note here the following fact about $\mathbf{A}'$: for every n-ary $c \in C$, the truth-function $c_{\mathbf{A}'} : (A')^n \longrightarrow A'$ can be "explained" componentwise. In other words, for every $0 \le i \le k$, there exists a truth-function $f_i^c : 2^{n(k+1)} \longrightarrow 2$ defined as $f_i^c(\vec{x}_1, \ldots, \vec{x}_n) := \pi_i(c_{\mathbf{A}'}(\vec{x}_1, \ldots, \vec{x}_n))$ (where $\vec{x}_1, \ldots, \vec{x}_n \in 2^{k+1}$). In addition, every truth-function f_i^c can be identified with its c.n.f. $f_i^{c'} : 2^{n(k+1)} \longrightarrow 2$. This implies two facts: every function f_i^c determines a term $\tau_{f_i^{c'}}$ of **FOBL** such that $\tau_{f_i^{c'}}^{\mathbf{2}} = f_i^c$), because Proposition 3.6. On the other hand, the operations on $\mathbf{A}'$ can be defined now in this alternative way: for every n-ary $c \in C$, for every set $\{\vec{x}_1, \ldots, \vec{x}_n\} \subseteq A'$, $c_{\mathbf{A}'}(\vec{x}_1, \ldots, \vec{x}_n) := (f_0^{c'}(\vec{x}_1, \ldots, \vec{x}_n), \ldots, f_k^{c'}(\vec{x}_1, \ldots, \vec{x}_n))$

---

[3] The existence of a conjunctive normal form is not essential itself. The relevant point is that f' is $\overline{C}$-definable. We could also use disjunctive normal forms, for instance. Actually, the d.s. for $\mathbf{I^1P^0}$ and for $\mathbf{I^2P^1}$ are based on formulas that are *simplifications* of the c.n.f. suggested here.

where $f_i^{c\prime} : (A')^{n(k+1)} \longrightarrow A'$ are the respective c.n.f. of f_i^c, for every $0 \leq i \leq k$. This fact suggests the following definition:

Definition 3.10. Let $M = (\mathbf{A}, D)$ be a C-matrix admitting a discriminant pair, and let eq_A and eq_D the equations that characterize $\overline{[\chi_D]}_k(A)$ (resp. $\overline{[\chi_D]}_k(D)$), cf. Proposition 3.7. For every Boolean algebra $\mathbf{B}$, the **discriminant M-structure associated to** B is the C-matrix $R_M(\mathbf{B}) = (\mathbf{A_M}'(B), D'_M(B))$, with $D'_M(B) := \{\vec{x} \in B^{k+1} : \vec{x} \text{ satisfies } eq_D\}$, and being $\mathbf{A'_M}(B)$ the C-algebra whose support is $A'_M(B) := \{\vec{x} \in B^{k+1} : \vec{x} \text{ satisfies } eq_A\}$ [4]. In addition, for any n-ary $c \in C$, the $\mathbf{A'_M}(B)$-truth-function $c_{\mathbf{A'_M}(B)} : (A'_M(B))^n \to A'_M(B)$ is defined as: $c_{\mathbf{A'_M}(B)}(\vec{x}_1, \ldots, \vec{x}_n) := (f'_{0\,\mathbf{A'_M}(B)}(\vec{x}_1, \ldots, \vec{x}_n), \ldots, f'_{k\,\mathbf{A'_M}(B)}(\vec{x}_1, \ldots, \vec{x}_n))$ (for every $\{\vec{x}_1, \ldots, \vec{x}_n\} \subseteq A'_M(B)$) where, for every $0 \leq i \leq k$, $f'_{i\,\mathbf{A'_M}(B)}$ is an abreviation of $\tau^{\mathbf{B}}_{f_i^{c\prime}}$ (taking into account Remarks 3.9 and 3.4). In particular, the discriminant structure $R_M(\mathbf{2}) = (\mathbf{A'_M}(2), D'_M(2))$ will be called the **canonical discriminant M-structure**. The class of all discriminant M-structures will be denoted by $\mathcal{S}_M$, and it will be called the **D.S.S. associated to** M.

The following result establishes that the previous definition makes sense:

Proposition 3.11. *For every C-matrix M and every Boolean algebra $\mathbf{B}$, the set $A'_M(B)$ is closed by the truth-functions $c_{\mathbf{A'_M}(B)}$. That is, $\mathbf{A'_M}(B)$ is well defined as an algebra.*

Proof. First of all note that our claim is valid for the particular case of the Boolean algebra $\mathbf{2}$: indeed, its proof is given in an implicit way in Proposition 3.8. This fact can be interpreted in the following way, cf. Definition 3.10: for every n-ary connective c, the algebra $\mathbf{2}$ satisfies simultaneously the **set of quasi-equations** $\{qec_i(\vec{x}_1, \ldots, \vec{x}_n)\}_{1 \leq 0 \leq k}$ given by (for every $0 \leq i \leq k$):

$$qec_i(\vec{x}_1, \ldots, \vec{x}_n) := eq_A(\vec{x}_1) \,\overline{\wedge}\, \cdots \,\overline{\wedge}\, eq_A(\vec{x}_n) \Rightarrow eq_A(\tau_{f_i^{c\prime}}(\vec{x}_1, \ldots, \vec{x}_n)).$$

(where $\tau_{f_i^{c\prime}}$ is given cf. Remark 3.9). Thus, every Boolean algebra $\mathbf{B}$ satisfies, for every n-ary $c \in C$, the set of quasi-equations $\{qec_i\}$, too. That is, $A'_M(B)$ is closed by applications of $c_{\mathbf{A'_M}(B)}$, for every $c \in C$. ☑

Another (abstract) result, useful for the proof of the fundamental theorem of this section, is similar to the one given in Proposition 2.5:

Proposition 3.12. *The discriminant structures of $\mathcal{S}_M$ verify:*

(a) If $\mathbf{B_1} \cong \mathbf{B_2}$, then $R_M(\mathbf{B_1}) \cong R_M(\mathbf{B_2})$.

---

[4] If $\vec{x}$ satisfies eq_D, then $\vec{x}$ satisfies eq_A, too. Hence, $D'_M(B) \subseteq A'_M(B)$.

(b) *Every $R_M(\mathbf{B})$-valuation $w : L(C)\longrightarrow A'_M(B)$ defines $(w_0, \ldots, w_k)$, a $k+1$-tuple of non-homomorphic functions $w_i : L(C) \to A'_M(B)$, with $w_i := \pi_i \circ w$ (for every $0 \le i \le k$).*

Remark 3.13. Note that $R_M(\mathbf{2})$ is the C-matrix M' of Proposition 3.8, indeed. From this, Proposition 2.6 and Proposition 3.12 (a) we get that, for every Boolean algebra $\mathbf{B}$ and every prime filter $\nabla \subseteq B$, $\mathbf{B}/\nabla$ is a Boolean algebra (which is isomorphic to $\mathbf{2}$) and so $R_M(\mathbf{B}/\nabla) \cong M'$.

As it is expected, the class $\mathcal{S}_M$ determines a consequence relation on $L(C)$:

Definition 3.14. The class $\mathcal{S}_M$ defines the **consequence relation** $\models_{\mathcal{S}_M}$ by means of the "local consequence relations" $\models_{R_M(\mathbf{B})}$, cf. Definition 1.7. That is: if $R_M(\mathbf{B})$ is in $\mathcal{S}_M$, then $\Gamma \models_{R_M(\mathbf{B})} \varphi$ iff, for every $R_M(\mathbf{B})$-valuation $w : L(C)\longrightarrow A'_M(B)$ such that $w(\gamma)$ satisfies eq_D for every $\gamma \in \Gamma$, it is valid that $w(\varphi)$ satisfies eq_D. And $\models_{\mathcal{S}_M} := \bigcap \models_{R_M(\mathbf{B})}$ (with $R_M(\mathbf{B})$ ranging on $\mathcal{S}_M$).

In the sequel we will prove the fundamental result of our paper, as we have mentioned right after Remark 3.2. That is, $\models_M \varphi$ iff $\models_{\mathcal{S}_M} \varphi$, for every $\varphi \in L(C)$. For that, we will need some technical results. First of all, realize that every set of the form $D'_M(B)$ can be characterized in a simpler way:

Proposition 3.15. *Let $R_M(\mathbf{B})=(\mathbf{A'_M}(B), D'_M(B))$ a discriminant structure in $\mathcal{S}_M$. For every $\vec{x} = (x_0, \ldots, x_k) \in A'_M(B)$, $\vec{x} \in D'_M(B)$ iff $x_0 = \bar{1}_\mathbf{B}$.*

Proof. First of all let us prove our claim for $R_M(\mathbf{2})$: let $\vec{x}=(x_0, \ldots, x_k) \in A'_M(2) = \overline{[\chi_D]}_k(A)$, by Remark 3.13. Suppose that $\vec{x} = (x_0, \ldots, x_k) \in D'_M(2) = \overline{[\chi_D]}_k(D)$. So, there is an unique $b_0 \in D$ such that $\overline{[\chi_D]}_k(b_0) = (x_0, \ldots, x_k)$. Then, $[\chi_D(b_0), \chi_D(f_{(\beta,\vec{a})}(b_0)), \chi_D(f^2_{(\beta,\vec{a})}(b_0)), \ldots, \chi_D(f^k_{(\beta,\vec{a})}(b_0))] = (x_0, \ldots, x_k)$. Now, since $\chi_D(b_0) = 1$ we have $x_0 = 1$. On the other hand, if $\vec{x} = (x_0, \ldots, x_n) \notin D'_M(2) = \overline{[\chi_D]}_k(D)$, then there is $b_1 \in A \setminus D$ with $\overline{[\chi_D]}_k(b_1) = (x_0, \ldots, x_k)$ and therefore $x_0 = 0$ (by a similar reasoning), as it was desired.

Note that this fact can be reformulated in this way: $\mathbf{2}$ *verifies the quasi-equations* $(*)$ $eq_A(\vec{x}) \bar{\wedge} eq_D(\vec{x}) \Rightarrow (x_0 = \bar{1})$ *and* $(**)$ $eq_A(\vec{x}) \bar{\wedge} (x_0 = \bar{1}) \Rightarrow eq_D(\vec{x})$.

Thus, every Boolean algebra $\mathbf{B}$ verifies $(*)$ and $(**)$, too. In addition, the equations eq_A and eq_D characterize the sets $A'_M(B)$ and $D'_M(B)$, resp., cf. Definition 3.10. From this, our claim can be proved for every d.s. $R_M(\mathbf{B})$. $\boxtimes$

In the proof of the fundamental result of our paper we will use Proposition 2.6 again and, in particular, the canonical homomorphism $e_\nabla : B \to B/\nabla$. By the way, an essential result relating in a general way homomorphisms of Boolean Algebras to prime filters is based on Definition 3.5. That result can be proved by induction on the complexity of the terms of **FOBL**:

Proposition 3.16. *Let* $\mathbf{B_i} = (B_i, \overline{C}_{\mathbf{B}})$ *(i* $= 1, 2$*) be Boolean Algebras. For every* $\overline{C}$*-definable function* $f_1 : B_1^n \to B_1$*, for every Boolean homomorphism* $h : B_1 \to B_2$*, if we define* $f_2 := \tau_{f_1}^{\mathbf{B_2}} : B_2^n \to B_2$*, cf. Remark 3.4, it holds that* $h(f_1(x_1, \ldots, x_n)) = f_2(h(x_1), \ldots, h(\underline{x_n}))$ *. As a particular case, for every* $\overline{C}$*-definable function* $f : B^n \to B$*,* $\overline{f(x_1, \ldots, x_n)}_\nabla = e_\nabla(f(x_1, \ldots, x_n))$ $= f_{\mathbf{B}/\nabla}(e_\nabla(x_1), \ldots, e_\nabla(x_n)) = f_{\mathbf{B}/\nabla}(\overline{x_1}_\nabla, \ldots, \overline{x_n}_\nabla)$ *(being* $f_{\mathbf{B}/\nabla} := \tau_f^{\mathbf{B}/\nabla}$*, as it was previously indicated).*

Proposition 3.17. *Let* $M = (\mathbf{A}, D)$ *be a finite C-matrix. Then, for every Boolean algebra* $\mathbf{B} = (B, C_B)$*, for every prime filter* $\nabla \subseteq B$*, the map* $E_\nabla : A'_M(\mathbf{B}) \longrightarrow A'_M(\mathbf{B}/\nabla)$*, defined by:* $E_\nabla(\vec{x}) := (e_\nabla(x_0), \ldots, e_\nabla(x_k))$*, for every* $\vec{x} = (x_0, \ldots, x_k) \in A'_M(B)$*, is a matrix epimorphism from* $R_M(\mathbf{B})$ *to* $R_M(\mathbf{B}/\nabla)$*.*

Proof. Obviously, E_∇ a surjective function. Let us prove now that E_∇ is a C-homomorphism: Let $c \in C$ be a n-ary connective. We will prove that $E_\nabla(c_{\mathbf{A}'_\mathbf{M}(B)}(\vec{x_1}, \ldots, \vec{x_n})) = c_{\mathbf{A}'_\mathbf{M}(B/\nabla)}(E_\nabla(\vec{x_1}), \ldots, E_\nabla(\vec{x_n}))$. For that, recall that $c_{\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n) := (f'_{0\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n), \ldots, f'_{k\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n))$, from Definition 3.10. In addition, note that every function $f'_{i\,\mathbf{A}'_\mathbf{M}(B)}$ (with $0 \leq i \leq k$) is $\overline{C}$-definable. From this and Proposition 3.16, for every $0 \leq i \leq k$, $e_\nabla(f'_{i\,\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n)) = e_\nabla(f'_{i\,\mathbf{A}'_\mathbf{M}(B)}(x_1^0, \ldots, x_1^k, \ldots, x_n^0, \ldots, x_n^k)) = $ $f'_{i\,\mathbf{A}'_\mathbf{M}(B/\nabla)}(e_\nabla(x_1^0), \ldots, e_\nabla(x_1^k), \ldots, e_\nabla(x_n^0), \ldots, e_\nabla(x_n^k))$. Thus we have:

$E_\nabla(c_{\mathbf{A}'_\mathbf{M}(B)}(\vec{x_1}, \ldots, \vec{x_n})) = $

$E_\nabla(f'_{0\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n), \ldots, f'_{k\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n)) = $

$(e_\nabla(f'_{0\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n))), \ldots, e_\nabla(f'_{k\mathbf{A}'_\mathbf{M}(B)}(\vec{x}_1, \ldots, \vec{x}_n))) = $

$(f'_{0\mathbf{A}'_\mathbf{M}(B/\nabla)}(e_\nabla(x_1^0), \ldots, e_\nabla(x_1^k), \ldots, e_\nabla(x_n^0), \ldots, e_\nabla(x_n^k)), \ldots,$ $f'_{k\mathbf{A}'_\mathbf{M}(B/\nabla)}(e_\nabla(x_1^0), \ldots, e_\nabla(x_1^k), \ldots, e_\nabla(x_n^0), \ldots, e_\nabla(x_n^k))) = $

$c_{\mathbf{A}'_\mathbf{M}(B/\nabla)}(E_\nabla(\vec{x_1}), \ldots, E_\nabla(\vec{x_n}))$, as it was desired, taking into account the definition of $c_{\mathbf{A}'_\mathbf{M}(\mathbf{B}/\nabla)}$ (see Definition 3.10 again) and the definition of E_∇. Finally, suppose that $\vec{x} = (x_0, \ldots, x_k) \in D'_M(B)$. From Proposition 3.15, $x_0 = \overline{1}_\mathbf{B} \in \nabla$, and then $e_\nabla(x_0) = \nabla$. Hence, $E_\nabla(\vec{x}) = (\nabla, \ldots, e_\nabla(x_k)) \in D'_M(B/\nabla)$, by Proposition 3.15, again. This concludes the proof. ☑

Theorem 3.18. *If a finite C-matrix* $M = (\mathbf{A}, D)$ *admits a discriminant pair* $(\beta, \vec{a})$ *(and a D.S.S.* $\mathcal{S}_M$ *therefore), then* $\models_M \varphi$ *iff* $\models_{\mathcal{S}_M} \varphi$*, for every* $\varphi \in L(C)$*.*

Proof. Suppose that there are $\beta(\alpha_0, \alpha_1, \ldots, \alpha_m) \in L(C)$, $\vec{a} = (a_1, \ldots, a_m) \in A^m$ and $k \in \omega$ such that the function $\overline{[\chi_D]}_k : A \longrightarrow 2^{k+1}$ is injective, determining thus the D.S.S. $\mathcal{S}_M$ and the consequence relation $\models_{\mathcal{S}_M}$, cf. it was developed in Definition 3.14. We will prove that $\models_M \varphi$ iff $\models_{\mathcal{S}_M} \varphi$. First of all, taking

into account Proposition 3.8 and Remark 3.13, it is obvious that $\models_{S_M} \varphi$ implies $\models_M \varphi$. For the converse, suppose that it exists a Boolean algebra $\mathbf{B}$ and a $R_M(\mathbf{B})$-valuation $w : L(C) \longrightarrow A'_M(B)$ such that $w(\varphi) \notin D'_M(B)$. So, $w_0(\varphi) := \pi_0(w(\varphi)) \neq \overline{1}_{\mathbf{B}}$, by Proposition 3.15. Thus, there is a prime filter ∇ of B, with $w_0(\varphi) \notin \nabla$. From this and Proposition 3.17, the matrix epimorphism $E_\nabla : A'_M(B) \longrightarrow A'_M(B/\nabla)$ verifies that $E_\nabla(w(\varphi)) \notin D'_M(B/\nabla)$ (because the first component of the elements of $D'_M(B/\nabla)$ is always ∇, and $\overline{w_0(\varphi)}_\nabla \neq \nabla$). So, $E_\nabla \circ w : L(C) \longrightarrow A'_M(B/\nabla)$ is a $R_M(\mathbf{B}/\nabla)$-valuation such that $(E_\nabla \circ w)(\varphi) \notin D'_M(B/\nabla)$. Hence, $\not\models_{R_M(\mathbf{B}/\nabla)} \varphi$, and so $\not\models_M \varphi$, because $R_M(\mathbf{B}/\nabla) \cong R_M(\mathbf{2}) = M' \cong M$.   ☑

Remark 3.19. Note that Theorem 3.18 establishes a *weak* adequacity between the relations $\models_M$ and $\models_{S_M}$. It is part of a future work the proof of a *strong* adequacity (that is, that $\Gamma \models_M \alpha$ iff $\Gamma \models_{S_M} \alpha$). Indeed, it is an open problem to give strong adequacity in a general case, or even to provide conditions that warrant the strong adequacity between a certain specific matrix M and its corresponding D.S.S. We will return to this topic at the end of the paper.

To illustrate the technical results proved here we will give another kind of example of D.S.S., that will allow us to understand the generalization here proposed by means of the definition of a discriminant pair $(\beta, \vec{a})$:

Example 3.20. Let $\mathcal{L}^\star$ be the logic defined on the basis of the well-known Łukasiewicz three-valued matrix $\mathbb{L}_3$ (being its underlying signature $C^\star$ again), *but considering the intermediate truth-value as the only designated one*. Formally speaking, $\mathcal{L}^\star = (C^\star, \models_{M^\star})$, where $\models_{M^\star}$ is defined by the $C^\star$-matrix $M^\star = (\mathbf{A}^\star, D^\star)$, with $A^\star = \{0^\star, \frac{1}{2}^\star, 1^\star\}$ and $D^\star = \{\frac{1}{2}^\star\}$[5]. In $\mathbf{A}^\star$, the truth-functions $\supset$ and $\neg$ are defined as:

$\supset$	$0^\star$	$\frac{1}{2}^\star$	$1^\star$
$0^\star$	$1^\star$	$1^\star$	$1^\star$
$\frac{1}{2}^\star$	$\frac{1}{2}^\star$	$1^\star$	$1^\star$
$1^\star$	$0^\star$	$\frac{1}{2}^\star$	$1^\star$

$\neg$	
$0^\star$	$1^\star$
$\frac{1}{2}^\star$	$\frac{1}{2}^\star$
$1^\star$	$0^\star$

Of course, even when the truth-values and the operations are the same as in the matrix that defines $\mathbb{L}_3$, the change in the set of designated values produces different tautologies in both logics. In fact, $\mathcal{L}^\star$ *has not tautologies at all*.

Proposition 3.21. *For every $\varphi \in L(C^\star)$, $\not\models_{M^\star} \varphi$.*

Proof. Note that $\{0^\star, 1^\star\}$ can be viewed as (the support of) a subalgebra of $\mathbf{A}^\star$. Now, for every $\varphi = \varphi(\alpha_1, \ldots, \alpha_n) \in L(C^\star)$ consider the $M^\star$ valuation

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[5]We will denote the truth-values of $A^\star$ in a "starred way" with the aim of distinguishing them of the 2-truth-values 0 and $1 \in 2$, which are always used in the definition of any D.S.S.

$w : L(C^\star) \longrightarrow A^\star$ such that $w(\alpha_i) = 0^\star$, for every $1 \leq i \leq n$. Then $w(\varphi) \neq \frac{1}{2}^\star$, and thus $\not\models_{M^\star} \varphi$. ☑

It is worth to remark here that the last result entails the fact that $\mathcal{L}^\star$ *is not algebraizable*. We will discuss some connections between algebraizability and Discriminant Structures at the end of the paper. On the other hand, turning back to $\mathcal{L}^\star$, in order to obtain a D.S.S. for this logic, we have the following straightforward result:

Proposition 3.22. *The pair $(\beta_\star, \vec{a}_\star)$, with $\beta_\star(\alpha_0, \alpha_1):=\alpha_1 \supset \alpha_0$, $\vec{a}_\star = a_1 := \frac{1}{2}^\star$, is a discriminant pair (with one iteration) for $M^\star$. In this context, the function $\overline{[\chi_D]}_k$ verifies; $\overline{[\chi_D]}_k(0^\star) = (0, 1)$, $\overline{[\chi_D]}_k(\frac{1}{2}^\star) = (1, 0)$ and $\overline{[\chi_D]}_k(1^\star) = (0, 0)$.*

Since $\overline{[\chi_D]}_k(A^\star) = \{(0, 1); (1, 0); (0, 0)\}$, we have that this set can be characterized by the equation $eq_{A^\star} : x_0 \wedge x_1 = \overline{0}$ [6]. Moreover, it is possible to obtain a matrix $M^{\star'} \simeq M^\star$, whose truth-functions behave as it is depicted below.

$\supset$	$(0,1)$	$(1,0)$	$(0,0)$
$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,0)$	$(1,0)$	$(0,0)$	$(0,0)$
$(0,0)$	$(0,1)$	$(1,0)$	$(0,0)$

$\neg$	
$(0,1)$	$(0,0)$
$(1,0)$	$(1,0)$
$(0,0)$	$(0,1)$

These truth-functions will induce the truth-functions $\supset$ and $\neg$ *in every d.s.* $R_{M^\star}(\mathbf{B})$, as it is suggested in Remark 3.9 and formalized in Definition 3.10. The formal definitions of these structures are as follows:

Definition 3.23. For every Boolean algebra $\mathbf{B} = (B, C_B)$, the **Discriminant Structure (of type $\mathcal{L}^\star$) associated to B** is the following $C^\star$-matrix:

$$R_{M^\star}(\mathbf{B}) = \left(\mathbf{A}'_{\mathbf{M}^\star}(B), D'_{M^\star}(B) \right), \text{ with } D'_{M^\star}(B) = \{(1,0)\}, \text{ and:}$$

(1) The support of $\mathbf{A}'_{\mathbf{M}^\star}(B)$ is $A'_{M^\star}(B) = \{(x_0, x_1) \in B^2 : x_0 \wedge_{\mathbf{B}} x_1 = \overline{0}\}$

(2) $(x_0, x_1) \supset_{\mathbf{A}_{\mathbf{M}^\star}(B)} (y_0, y_1) := \big(-_{\mathbf{B}} x_1 \wedge_{\mathbf{B}} [(x_0 \leftrightarrow_{\mathbf{B}} -_{\mathbf{B}} y_0) \wedge_{\mathbf{B}} (x_0 \leftrightarrow_{\mathbf{B}} y_1)], -_{\mathbf{B}}(x_0 \vee_{\mathbf{B}} x_1 \vee_{\mathbf{B}} y_0) \wedge_{\mathbf{B}} y_1 \big)$

(3) $\neg_{\mathbf{A}_{\mathbf{M}^\star}(B)}(x_0, x_1) := \big(x_0, -_{\mathbf{B}}(x_0 \vee_{\mathbf{B}} x_1) \big)$.

The class of all the discriminant structures for $\mathcal{L}^\star$ is denoted by $\mathcal{S}^\star$. In addition, the consequence relation $\models_{\mathcal{S}^\star}$ follows the same patterns as in Definitions 2.2 and 2.12. And, as in these cases, it is possible to obtain the following results, whose detailed proofs appear in [5]:

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[6] This characterization is equivalent, but simpler, to the characterization proposed in Definition 3.10 by means of Proposition 3.7, as we have commented previously.

Proposition 3.24. *Every set $A'_{M^\star}(B)$ is closed by applications of $\supset_{\mathbf{A}_{\mathbf{M}^\star}(\mathbf{B})}$ and $\neg_{\mathbf{A}_{\mathbf{M}^\star}(\mathbf{B})}$.*

Moreover, applying the "abstract versions" of the results presented above (from Definition 3.10 to Theorem 3.18) it is possible to demonstrate.

Theorem 3.25. *For every $\varphi \in L(C^\star)$, $\models_{M_\star}\varphi$ iff $\models_{\mathcal{S}^\star}\varphi$.*

4. On the existence of Discriminant Structures

Considering the definition and general results concerning D.S.S., until now we have seen that the semantics of the form $\mathcal{S}_M$ are, in fact, a sort of "convenient representations" of certain matrix semantics M. Moreover, Theorem 3.18 establishes that this kind of semantics can be obtained when a discriminant pair is found. However we will see now, by means of the following example, that *it is not always possible* to obtain such a pair, for a given matrix M.

Example 4.1. Consider Urquhart logic Urq, characterized by the C^*-matrix $M_{Urq} = (\mathbf{A}, D)$, where $C^* = \{*\}$ (with $*$ binary), $A = \{I, II, III, IV, V\}$, $D = \{I\}$), and the truth-function $*$ is indicated below:

$*$	I	II	III	IV	V
I	V	V	I	I	V
II	V	V	II	I	V
III	V	V	V	V	V
IV	V	V	V	V	V
V	V	V	V	V	V

This logic was defined by A. Urquhart in [14], to show an example of a matrix logic that cannot be axiomatized by a finite set of structural rules. It is possible to prove that M_{Urq} does not admit a discriminant pair.

Proposition 4.2. *For every formula $\beta \notin \mathcal{V}$, for every tuple $\vec{a}$ of elements of A, the truth-function $f_{(\beta,\vec{a})} : A \to A$ (see Definition 3.1) verifies that there are two elements a_0, $a_1 \notin D$, $a_0 \neq a_1$, such that $f_{(\beta,\vec{a})}(a_0) = f_{(\beta,\vec{a})}(a_1) = V$.*

Proof. We will prove our claim by induction on n, the number of ocurrences of $*$ in β (where $n \geq 1$, because $\beta \notin \mathcal{V}$). When $n = 1$, we have these possibilities: If $f_{(\beta,\vec{a})} = x * x$ (in which case $\vec{a}$ is irrelevant), then $f_{(\beta,\vec{a})}(a) = V$, for every $a \in A$. If not, then we have two cases: $f_{(\beta,\vec{a})} = a * x$ or $f_{(\beta,\vec{a})} = x * a$ [7], being $\vec{a} = a \in A$. The five truth-functions that can be obtained for the first case are $f_{(\beta,I)} = I * x, ..., f_{(\beta,V)} = V * x$. For these functions, consider $a_0 = II$, and $a_1 = V$. For the second case, the resulting truth-functions are $g_{(\beta,I)} = x * I, ..., g_{(\beta,V)} = x * V$. Here $a_0 = IV$, $a_1 = V$. Hence, our claim is valid for all the possible

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[7]Note that $*$ is not conmutative.

cases. Suppose now that the result is valid for every $t \leq n$. Consider now $\beta = \beta(\alpha_0, \alpha_1, \ldots, \alpha_m)$ and $\vec{a} = (a_1, \ldots, a_m) \in A^m$, where β has n ocurrences of $*$. Then $\beta = \beta_1 * \beta_2$. If $\beta_1, \beta_2 \in \mathcal{V}$ we would return to $n = 1$. So, suppose (without loss of generality), that $\beta_1 \notin \mathcal{V}$. By Induction Hypothesis, there are $a_0 \neq a_1$ such that $f_{(\beta_1, \vec{a})}(a_0) = f_{(\beta_1, \vec{a})}(a_1) = V$. Since $V * x = x * V = V$ for every $x \in A$, we have that $f_{(\beta, \vec{a})}(a_0) = f_{(\beta_1, \vec{a})}(a_0) * f_{(\beta_2, \vec{a})}(a_0) = V * f_{(\beta_1, \vec{a})}(a_0) = V$. In a similar way, $f_{(\beta, \vec{a})}(a_1) = V$. Now, If $\beta_1 \in \mathcal{V}$, then $\beta_2 \notin \mathcal{V}$. For this case, adapt the previous reasoning. ☑

Corollary 4.3. *The matrix M_{Urq} does not admit discriminant pairs.*

Proof. Note that Proposition 4.2 implies that there are not pairs $(\beta, \vec{a})$ that can discriminate all the values of A by means of one iteration. But noting that V is an *absorbent element in* M_{Urq} we have that, for every pair $(\beta, \vec{a})$, for every $k \in \omega$, there are $a_0, a_1 \in A$, $a_0 \neq a_1$, such that $\overline{[\chi_D]}_k(a_0) = \overline{[\chi_D]}_k(a_1) = (0, 0, \ldots, 0)$ ($k + 1$ times), and so they cannot be discriminated. ☑

This last result shows that, despite the simplicity of the process to obtain a D.S.S., the basis of it (that is, the existence of a discriminant pair) is not trivial.

An additional problem here is referred to the *uniqueness* of a discriminant pair. It is easy to see here that, if a matrix M admits a discriminant pair $(\beta, \vec{a})$, such a pair not needs to be unique. For instance, consider again the matrix $M^\star$ of Example 3.20: an alternative discriminant pair (different to the pair showed in Proposition 3.22) is $(\beta_\star', \vec{a}_\star')$, with $\beta_\star'(\alpha_0, \alpha_1) := \alpha_0 \supset \alpha_1$, and $\vec{a}'_\star = a_1 := \frac{1}{2}^\star$. For this pair, we associate $0^\star$ with $(0, 0)$, $\frac{1}{2}^\star$ with $(1, 0)$, and $1^\star$ with $(0, 1)$. We will return to this point at the end of the paper.

5. Relations with Dyadic and Twist-Structure semantics

As it was previously commented, Discriminant Structures Semantics are motivated by a simple idea, usual in the field of many-valued logics, which can be stated as follows: the truth-values of a matrix $M = (\mathbf{A}, D)$ can be often *discriminated*, according to their belonging to D. As it was mentioned, this approach was already applied, with several purposes. For example, the separation (i.e. discrimination) of truth-values can determine whether certain formulas are *synonymous* or not (see [12], for example). Also, several definitions of two-valued (non truth-functional) semantics make use of this notion. One example of such construction is Dyadic Semantics (see [1]), which possess certain similarities with D. S. S., according to our point of view. So, we will discuss briefly here the relationship between both semantics. For that, the notation of [1] was modified, for a better comparison to the present paper.

Roughly speaking, a dyadic semantics for a given logic $\mathcal{L} = (C, \models_M)$ induced by a finite C-matrix $M = (\mathbf{A}, D)$ is built on a basis of:

- A set of formulas $\{\phi_i\}_{1 \le i \le k}$, where (for every i), $\phi_i = \phi_i(\alpha) \in L(C)$ (with $\alpha \in \mathcal{V}$).

- A function $h : A \longrightarrow 2$ such that, for every ϕ_i, for every $a \in A$, $h(\phi_i(a)) = 1$ iff $\phi_i(a) \in D$.

In addition, the truth-functions associated to the set $\{\phi_i\}_{0 \le i \le k}$ (together with h) separates, actually, the values of A, by means of $(k+1)$-tuples of elements of 2, and therefore the family $\{\phi_i\}_{0 \le i \le k}$ can be understood as a generalization of the formula β in our discriminant pair. The scope of Dyadic Semantics is, considering this aspect, stronger of the D.D.S, wherein just only formula β (eventually iterated) is allowed.

However it should be noted, from the explanation above, that the separation method based on Dyadic Semantics depends on the existence of one-variable *formulas of $L(C)$* (which determine truth-functions as usual). On the other hand, in the case of D.S.S., the discriminant pair $(\beta, \vec{a})$ induces the (one variable-depending) *truth-functions* $f_{(\beta, \vec{a})}$, which do not need to be "associated" to any particular formula in $L(C)$. This suggests that, meanwhile Dyadic Semantics is focused more strictly on the involved languages, D.S.S analyze mainly the used algebras Note, anyway, that if the algebra $\mathbf{A}$ of a given C-matrix $M = (\mathbf{A}, D)$ is functionally complete, then every $\mathbf{A}$-truth-function has an associated formula that describes it (by means of the connectives of C), and so it would be possible to "jump" from the matrices to the formal languages themselves.

Besides that, an "hybrid method" of separation of truth-values can be done. Consider simply that every truth-value of a matrix $M = (\mathbf{A}, D)$ is tested by a *set $f_{(\beta_i, \vec{a_i})}$* of discriminant (non-iterated) pairs. An informal example of this idea can be developed for the logic Urq, previously defined.

Proposition 5.1. *The truth-values of the logic Urq of Example 4.1 can be discriminated by a set $\{f_i\}_{0 \le i \le 4}$ (with $f_i = f_{(\beta_i, \vec{a_i})} : A \longrightarrow A$) of truth-functions.*

Proof. The following schema shows the formulas which discriminate every truth-value of Urq, and the indentification of each truth-value by means of the function $\overline{h}_4 : A \longrightarrow 2^5$, defined by $\overline{h}_4(x) = (\chi_D(f_i(x)))_{0 \le i \le k}$, for every $x \in A$:

$f_0 = x$	$f_1 = x * I$	$f_2 = x * II$	$f_3 = I * x$	$f_4 = II * x$	$\overline{h}_4$
I	I	I	V	V	$(1, 1, 1, 0, 0)$
II	V	I	V	V	$(0, 0, 1, 0, 0)$
III	V	V	I	V	$(0, 0, 0, 1, 0)$
IV	V	V	I	I	$(0, 0, 0, 1, 1)$
V	V	V	V	V	$(0, 0, 0, 0, 0)$

So, all the truth-values of Urq can be discriminated by the set $f_{(\beta_i, \vec{a_i})}$. ☑

It must be indicated that the method that "explains" the behavior of the operation $*$, when applied to the founded t-uples, was not developed here. Moreover, it is not clear here (as in Dyadic Semantics in general terms) in which way an algebraic treatment can be done. Indeed, Dyadic Semantics is not focused (until this moment) on algebraic considerations that allow to extend a certain "canonical dyadic semantics" to a whole class $\mathcal{K}$, as in the case of D.S.S. Of course, all these topics deserve a deeper treatment in the future.

We conclude this paper mentioning certain connections with a non-standard semantics that is a kind of "hidden motivation" for the definition of the Discriminant Structures, which is Twist-Structure Semantics. Actually, the preliminar version of this paper ([5]) remarks, in an strong way, the relations between these semantics. To show some of them we proceed as above, giving in a very informal way some key-concepts on which Twist-Structure Semantics is based.

First of all, usually a Twist-Structure Semantics for a given logic $\mathcal{L} = (C, \models)$ (which *does not need* to be defined by any C-matrix), is a *class $\mathcal{K}$ of C-algebras* such that, for every $\mathbf{H}$ in $\mathcal{K}$, it holds:

- $\mathbf{H}$ is a subalgebra of the product $\mathbf{T} \times \mathbf{T}^*$, where $\mathbf{T}$ is a certain ordered C-algebra, and $\mathbf{T}^*$ is *the dual algebra of* $\mathbf{T}$. Indeed, the "torsion" of the second axis in $\mathbf{T} \times \mathbf{T}^*$ is the fact that suggests the name *twist-structure*, for every $\mathbf{H}$.

  So, in this informal definition, $\mathbf{H}$ plays the same role that the algebras $\mathbf{A}'_M(B)$ in D.S.S. In addition, the "basis algebra $\mathbf{T}$" acts as the basic Boolean Algebra $\mathbf{B}$ that determines every d.s. $R_M(\mathbf{B})$.

- As in the case of D.S.S., The operations on every $\mathbf{H}$ are given taking into account the behavior of the "original $C_{\mathbf{A}}$-operations".

- Once the class $\mathcal{K}$ is defined, it determines a certain consequence relation $\models_{\mathcal{K}}$. Now, to prove adequacity (that is, $\models_{\mathcal{K}} \varphi$ iff $\models \varphi$) is usual to prove "representation results", mainly when already exists a previously given class $\mathcal{A}$ which determines $\models$. That is, that every algebra $\mathbf{A}$ of $\mathcal{A}$ can be represented by a Twist-Structure $\mathbf{H}$ of $\mathcal{K}$, and vice versa. Moreover, this results allows to prove *strong adequacity*, actually [8].

It is usually accepted that the constructions known as Twist-Structure Semantics appeared in the works (done in an independent way) of M. Fidel and D. Vakarelov, to provide an alternative semantics for Nelson Intuitionistic Logic with Strong Negation (see [6] and [15]). Nowadays, the constructions developed in these papers were adapted to a great number of logics (see [7], [8] or [9], for example). Thus, Twist-Structure Semantics are an object of numerous, deep and fruitful investigations.

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[8]It should be remarked, anyway, that not in all the cases adequacity proofs are based on representation results.

Note here that an essential point in Twist-Structure Semantics is missed in D.S.S.: the torsion of the second axis. What is the reason of that torsion? Mainly, twist-structures, even being considered as C-matrices, usually were analyzed w.r.t its lattice-theoretic behavior. Under this perspective, the *designated elements* of every structure of the form $\mathbf{H} \subseteq \mathbf{T} \times \mathbf{T}^*$ are usually interpreted as the *greatest elements* (according to the order relation $\leq_\mathbf{H}$, which is inherited from the "twisted order" obtained in $\mathbf{T} \times \mathbf{T}^*$). For that, to obtain adequacy, it is usually necessary to consider the second axis with an inverted order, mainly because the algebras $\mathbf{T}$ and $\mathbf{T}^*$ are not of the same kind. This is the case when $\mathbf{T}$ is a (Generalized) Heyting Algebra, for example. Actually, this torsion is often motivated with the necessity of an explanation of the behavior of a certain "negation connective", $\neg$ or $\sim$ (anyway, not every Twist-Structure Semantics is defined to explain negations: see [9], as an example applied to a logic without negations). Now, in the case of D.S.S., the required torsion has no sense: note that D.S.S. is dealing only with Boolean algebras, and every Boolean algebra $\mathbf{B}$ is isomorphic to its dual $\mathbf{B}^*$.

In addition, usually twist-structures are only considered as subsets of products of *two algebras*. On the other hand, D.S.S. is defined allowing that every discriminant structure can be embedded in an arbitrary finite product (according the number k of iterations).

On the other hand, we remark the coincidences between these two kind of semantics: in both cases certain *classes of algebras* are defined. Moreover, their underlying consequence relations are obtained by means of satisfaction of equations, as we have seen. Turning back to D.S.S., note that they are not depending neither on considerations of order relations, nor on the existence of negations. Since the treatment of these last semantics is more "algorithmic", they could be used in a general way (in the context of Matrix Logics), regardless its intuition. However, twist-structures can be viewed as more intuitive than D.S.S., mainly when they deal with certain "well-motivated" logics, where their connectives can be interpreted in a more natural sense.

6. Concluding Remarks

In this paper, we have showed a way to characterize C-matrices $M = (\mathbf{A}, D)$ by means of a process that codifies the truth-values of the support A according the characteristic function χ_D. Moreover, we have demonstrated that this decodification, originally obtained for the Boolean algebra $\mathbf{2}$, can be generalized, by means of the definition of adequated equations, to every Boolean algebra $\mathbf{B}$, in such a way that a new semantics (namely, D.S.S.) can be defined on the basis of a class $\mathcal{S}_M$ of C-matrices. So, Discriminant Structure Semantics recovers certain algebraic character of that C-matrices that can be characterized in this way. Under our point of view, this implicit algebraic characteristics of D.S.S. are important, mainly because D.S.S. can be related to some problems concerning Abstract Algebraizability. Actually, a certain matrix logic that is not

algebraizable [9], but that can be characterized by means of D.S.S., is presented in [5]. Indeed, this logic is $\mathcal{L}^\star$ of Example 3.20. So, we consider that the study of the algebraic aspects of Discriminant Structures would be an interesting topic for future researches.

Besides that, an essential fact is showed in this paper, in Example 4.1: not every finite C-matrix can be associated to an adequate D.S.S., by means of a discriminant pair $(\beta, \vec{a})$. So, a very interesting open problem that we propose here is the following: Which conditions (sufficent and/or neccesary) are needed for a C-matrix $M = (\mathbf{A}, D)$ to admit a discriminant pair $(\beta, \vec{a})$? Indeed, this kind of question is very usual in the context of Abstract Logic. For instance, in [16] (see also [3]) is given the definition of *Referential Matrix Semantics* (R.M.S). This notion, that provides a suitable matrix treatment for certain modal logics (mainly for the "based on possible world semantics" ones), is one interesting example of the scope of the general Theory of Matrices. So, it is natural to suppose that the field of action of R.M.S. is connected with the motivation and applications of D.D.S. By the way, one of the main results of [16] establishes that *a given logic admits* R.M.S *iff it is Self-Extensional* [10]. We expect to obtain such kind of results, adapted to the case of D.S.S., in future works. Moreover, one of the topics of research that deserves attention is to relate D.S.S with self-extensionality (and, therefore, with R.M.S), and with other notions intrinsic to Abstract (Algebraic) Logic, as was previously commented.

Also, the *problem of uniqueness* (suggested at the end of Section 4) deserves a deep analysis: by the way, it is worth to note that, even when can exist two different discriminant pairs for a given matrix M, the matrices that both pairs determine are isomorphic. Anyway, this problem can be studied from different points of view: for instance, the relation between the D.S.S determined by each pair is an interesting matter of study. The connection between uniqueness of discriminant pairs and reduced matrices deserves a future research, too.

Finally, we remark the obvious relations of D.S.S. with Dyadic Semantics and with Twist-Structures again: these natural connections deserve a deeper study. For all the exposed, we think that this initial study of Discriminant Structure Semantics can be expanded to several interesting research lines.

Acknowledgements.

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[9]The notion of algebraizability is essential in the context of Abstract Algebraic Logic: a logic without these properties would have a poor algebraic character, in some sense. See [3] to a deeper study of these topics.

[10]As in the case of some previous abstract concepts, the notion of self-extensionality is out of the scope of this paper.

one report of the referees. In addition, the bibliography relative to Referential Matrix Semantics indicated below (also suggested in the reports of this work), seems to be a very interesting field for future investigations.

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