

# Pillai's problem with Padovan numbers and powers of two

El problema de Pillai con números de Padovan y potencias de dos

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ABSTRACT. Let  $(P_n)_{n \geq 0}$  be the *Padovan sequence* given by  $P_0 = 0, P_1 = P_2 = 1$  and the recurrence formula  $P_{n+3} = P_{n+1} + P_n$  for all  $n \geq 0$ . In this note we study and completely solve the Diophantine equation

$$P_n - 2^m = P_{n_1} - 2^{m_1}$$

in non-negative integers  $(n, m, n_1, m_1)$ .

*Key words and phrases.* Padovan sequence, Pillai's problem, linear forms in logarithms, reduction method.

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RESUMEN. Sea  $(P_n)_{n \geq 0}$  la *sucesión de Padovan* dada mediante  $P_0 = 0, P_1 = P_2 = 1$  y la fórmula de recurrencia  $P_{n+3} = P_{n+1} + P_n$  para todo  $n \geq 0$ . En esta nota estudiamos y resolvemos completamente la ecuación diofántica

$$P_n - 2^m = P_{n_1} - 2^{m_1}$$

en enteros no negativos  $(n, m, n_1, m_1)$ .

*Palabras y frases clave.* Sucesión de Padovan, Problema de Pillai, Formas lineales en logaritmos, método de reducción.

## 1. Introduction

Let  $a, b$  be two fixed positive integers and consider the Diophantine equation

$$a^n - b^m = a^{n_1} - b^{m_1} \tag{1}$$

in positive integers  $(n, m, n_1, m_1)$  with  $(n, m) \neq (n_1, m_1)$ . In particular, look for the integers which can be written as the difference of a power of  $a$  and a power of  $b$  in at least two distinct ways. In [11], Herschfeld proved that in the case  $(a, b) = (2, 3)$  equation (1) has finitely many solutions. In [13], Pillai extended Herschfeld's result to the case  $a, b \geq 2$  being coprime integers. In both cases the result is ineffective. In [14], Pillai conjectures that in the case  $(a, b) = (2, 3)$  all solutions of equation (1) are  $(3, 2, 1, 1)$ ,  $(5, 3, 3, 1)$  and  $(8, 5, 4, 1)$ . This conjecture remained open for about 37 years and it was confirmed by Stroeker and Tijdeman in [17] by using Baker's theory on linear forms in logarithms of algebraic numbers.

Recently, the above problem now known as the *Pillai problem* was posed in the context of linear recurrence sequences. That is, let  $\mathbf{U} := (U_n)_{n \geq 0}$  and  $\mathbf{V} := (V_m)_{m \geq 0}$  be two linearly recurrence sequences of integers and look at the diophantine equation

$$U_n - V_m = U_{n_1} - V_{m_1} \quad (2)$$

in positive integers  $(n, m, n_1, m_1)$  with  $(n, m) \neq (n_1, m_1)$ . In particular, find the integers which can be written as a difference of an element of  $\mathbf{U}$  and an element of  $\mathbf{V}$  in at least two distinct ways. This version was started in [8] by Ddamulira, Luca and Rakotomalala where they take  $\mathbf{U}$  being the Fibonacci sequence and  $\mathbf{V}$  being the sequence of powers of 2. Many other cases have been studied, see for example [5, 3, 7, 10]. In [6] it is proved that, under some natural conditions on  $\mathbf{U}$  and  $\mathbf{V}$  equation (2) has finitely many solutions and all of them are effectively computable.

Now, let  $(P_n)_{n \geq 0}$  be the *Padovan sequence*, named after the architect R. Padovan. It is a ternary recurrence sequence given by  $P_0 = 0$ ,  $P_1 = P_2 = 1$  and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \quad \text{for all } n \geq 0.$$

This is A000931 sequence in [16]. Its first few terms are

0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, ...

In this note, we study another particular case of equation (2) namely with Padovan numbers and powers of 2. More precisely, we solve the equation

$$P_n - 2^m = P_{n_1} - 2^{m_1} \quad (3)$$

in non-negative integers  $(n, m, n_1, m_1)$  with  $(n, m) \neq (n_1, m_1)$ . Since  $P_1 = P_2 = P_3 = 1$  we assume that  $n \neq 1, 2$  and  $n_1 \neq 1, 2$ . That is, whenever we think of 1 as a member of the Padovan sequence we think of it as being  $P_3$ . In the same way,  $n \neq 4$  and  $n_1 \neq 4$ . Then, with these conventions, our result is the following:

**Theorem 1.1.** *All non-negative integer solutions  $(n, m, n_1, m_1)$  of equation (3) belong to the set*

$$\left\{ \begin{array}{ccccc} (3, 1, 0, 0), & (5, 1, 3, 0), & (5, 2, 0, 1), & (6, 1, 5, 0), & (6, 2, 0, 0), \\ (6, 2, 3, 1), & (7, 1, 6, 0), & (7, 2, 3, 0), & (7, 2, 5, 1), & (7, 3, 0, 2), \\ (8, 1, 7, 0), & (8, 2, 5, 0), & (8, 2, 6, 1), & (8, 3, 3, 2), & (9, 2, 7, 0), \\ (9, 2, 8, 1), & (9, 3, 0, 0), & (9, 3, 3, 1), & (9, 3, 6, 2), & (10, 2, 9, 1), \\ (10, 3, 5, 0), & (10, 3, 6, 1), & (10, 3, 8, 2), & (10, 4, 3, 3), & (11, 2, 10, 0), \\ (11, 3, 8, 0), & (11, 4, 0, 2), & (11, 4, 7, 3), & (12, 3, 10, 0), & (12, 3, 11, 2), \\ (12, 4, 3, 0), & (12, 4, 5, 1), & (12, 4, 7, 2), & (12, 5, 0, 4), & (13, 4, 9, 1), \\ (13, 4, 10, 2), & (13, 5, 8, 4), & (14, 3, 13, 0), & (14, 4, 12, 2), & (14, 5, 0, 2), \\ (14, 5, 7, 3), & (14, 5, 11, 4), & (15, 5, 9, 1), & (15, 5, 10, 2), & (15, 5, 13, 4), \\ (15, 6, 8, 5), & (16, 4, 15, 2), & (16, 5, 13, 2), & (16, 6, 3, 4), & (17, 5, 15, 2), \\ (17, 5, 16, 4), & (17, 6, 5, 0), & (17, 6, 6, 1), & (17, 6, 8, 2), & (17, 6, 10, 3), \\ (17, 7, 3, 6), & (19, 5, 18, 2), & (19, 7, 5, 4), & (22, 9, 10, 8), & (27, 10, 17, 3), \\ (30, 12, 24, 11) \end{array} \right\}.$$

The set of integers which can be written as the difference of a Padovan number and a power of 2 in at least two distinct ways is

$$\{-1583, -247, -63, -27, -16, -15, -14, -11, -7, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 8, 12, 17, 20, 33, 57, 82\}.$$

The representations of these numbers are

$$\begin{aligned} -1583 &= P_{30} - 2^{12} = P_{24} - 2^{11}; \\ -247 &= P_{22} - 2^9 = P_{10} - 2^8; \\ -63 &= P_{17} - 2^7 = P_3 - 2^6; \\ -27 &= P_{15} - 2^6 = P_8 - 2^5; \\ -16 &= P_{12} - 2^5 = P_0 - 2^4; \\ -15 &= P_{16} - 2^6 = P_3 - 2^4; \\ -14 &= P_{19} - 2^7 = P_5 - 2^4; \\ -11 &= P_{13} - 2^5 = P_8 - 2^4; \\ -7 &= P_{10} - 2^4 = P_3 - 2^3; \\ -4 &= P_{14} - 2^5 = P_{11} - 2^4 = P_7 - 2^3 = P_0 - 2^2; \\ -3 &= P_8 - 2^3 = P_3 - 2^2; \\ -2 &= P_5 - 2^2 = P_0 - 2^1; \\ -1 &= P_9 - 2^3 = P_6 - 2^2 = P_3 - 2^1 = P_0 - 2^0; \end{aligned}$$

$$\begin{aligned}
0 &= P_{12} - 2^4 = P_7 - 2^2 = P_5 - 2^1 = P_3 - 2^0; \\
1 &= P_{17} - 2^6 = P_{10} - 2^3 = P_8 - 2^2 = P_6 - 2^1 = P_5 - 2^0; \\
2 &= P_7 - 2^1 = P_6 - 2^0; \\
3 &= P_9 - 2^2 = P_8 - 2^1 = P_7 - 2^0; \\
4 &= P_{11} - 2^3 = P_8 - 2^0; \\
5 &= P_{15} - 2^5 = P_{13} - 2^4 = P_{10} - 2^2 = P_9 - 2^1; \\
8 &= P_{12} - 2^3 = P_{11} - 2^2 = P_{10} - 2^0; \\
12 &= P_{14} - 2^4 = P_{12} - 2^2; \\
17 &= P_{16} - 2^5 = P_{13} - 2^2; \\
20 &= P_{14} - 2^3 = P_{13} - 2^0; \\
33 &= P_{17} - 2^5 = P_{16} - 2^4 = P_{15} - 2^2; \\
57 &= P_{27} - 2^{10} = P_{17} - 2^3; \\
82 &= P_{19} - 2^5 = P_{18} - 2^2.
\end{aligned}$$

## 2. Tools

In this section, we gather the tools we need to prove Theorem 1.1. The first one is a lower bound for linear forms in logarithms due to Matveev. Let  $\alpha$  be an algebraic number of degree  $d$ , let  $a > 0$  be the leading coefficient of its minimal polynomial over  $\mathbb{Z}$  and let  $\alpha = \alpha^{(1)}, \dots, \alpha^{(d)}$  denote its conjugates. The *logarithmic height* of  $\alpha$  is defined as

$$h(\alpha) = \frac{1}{d} \left( \log a + \sum_{i=1}^d \log \max \{ |\alpha^{(i)}|, 1 \} \right).$$

This height satisfies the following basic properties. For  $\alpha, \beta$  algebraic numbers and  $m \in \mathbb{Z}$  we have

- $h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log(2)$ ;
- $h(\alpha\beta) \leq h(\alpha) + h(\beta)$ ;
- $h(\alpha^m) = |m|h(\alpha)$ .

Now let  $\mathbb{L}$  be a real number field of degree  $d_{\mathbb{L}}$ ,  $\alpha_1, \dots, \alpha_{\ell} \in \mathbb{L}$  and  $b_1, \dots, b_{\ell} \in \mathbb{Z} \setminus \{0\}$ . Let  $B \geq \max\{|b_1|, \dots, |b_{\ell}|\}$  and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_{\ell}^{b_{\ell}} - 1.$$

Let  $A_1, \dots, A_\ell$  be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}} h(\alpha_i), |\log \alpha_i|, 0.16\}, \quad i = 1, 2, \dots, \ell.$$

The following result is due to Matveev in [12] (see also Theorem 9.4 in [4]).

**Theorem 2.1.** *Assume that  $\Lambda \neq 0$ . Then*

$$\log |\Lambda| > -1.4 \cdot 30^{\ell+3} \cdot \ell^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) A_1 \cdots A_\ell.$$

In this note we always use  $\ell = 3$ . Further,  $\mathbb{L} = \mathbb{Q}(\gamma)$  has degree  $d_{\mathbb{L}} = 3$ , where  $\gamma$  is defined at the beginning of Section 3. Thus, once and for all we fix the constant

$$C := 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3) \approx 2.70443 \times 10^{12}.$$

Our second tool is a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [2]. For a real number  $x$ , we write  $\|x\|$  for the distance from  $x$  to the nearest integer.

**Lemma 2.2.** *Let  $M$  be a positive integer. Let  $\tau, \mu, A > 0, B > 1$  be given real numbers. Assume that  $p/q$  is a convergent of  $\tau$  such that  $q > 6M$  and  $\varepsilon := \|\mu q\| - M\|\tau q\| > 0$ . Then there is no solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

in positive integers  $n, m$  and  $w$  satisfying

$$n \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Lemma 2.2 is a slight variation of the one given by Dujella and Pethő in [9]. Finally, the following result will be very useful. This is Lemma 7 in [15].

**Lemma 2.3.** *If  $m \geq 1, T > (4m^2)^m$  and  $T > x/(\log x)^m$ . Then*

$$x < 2^m T (\log T)^m.$$

### 3. Proof of Theorem 1.1

To start with, we recall some properties of the Padovan sequence. For a complex number  $z$  we write  $\bar{z}$  for its complex conjugate. Let  $\omega \neq 1$  be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9 + \sqrt{69}}{18}} + \sqrt[3]{\frac{9 - \sqrt{69}}{18}}, \quad \delta := \omega \sqrt[3]{\frac{9 + \sqrt{69}}{18}} + \bar{\omega} \sqrt[3]{\frac{9 - \sqrt{69}}{18}}.$$

It is clear that  $\gamma, \delta, \bar{\delta}$  are the roots of the  $\mathbb{Q}$ -irreducible polynomial  $X^3 - X - 1$ . We also have the Binet formula

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\bar{\delta}^n, \quad (4)$$

which holds for all  $n \geq 0$ , where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \quad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \quad c_3 = \bar{c}_2. \quad (5)$$

Formula (4) follows from the general theorem on linear recurrence sequences since the above polynomial is the characteristic polynomial of the Padovan sequence. We note that

$$\gamma = 1.32471\dots, \quad |\delta| = 0.86883\dots, \quad c_1 = 0.54511\dots, \quad |c_2| = 0.28241\dots$$

Further, the inequalities

$$\gamma^{n-3} \leq P_n \leq \gamma^{n-1}, \quad (6)$$

hold for all  $n \geq 1$ . These can be proved by induction.

We start with the study of equation (3) in non-negative integers  $(n, m, n_1, m_1)$  with  $(n, m) \neq (n_1, m_1)$  where, as we have said,  $n \neq 1, 2, 4$  and  $n_1 \neq 1, 2, 4$ . We note, if  $m = m_1$  then  $P_n = P_{n_1}$  which implies  $n = n_1$ , a contradiction. Thus we assume  $m > m_1$ . Rewriting equation (3) as

$$P_n - P_{n_1} = 2^m - 2^{m_1} \quad (7)$$

we observe the right-hand is positive. So, the left-hand side is also positive and therefore,  $n > n_1$ . Now, we compare both sides of (7) using (6). We have

$$\gamma^{n-8} \leq P_n - P_{n_1} = 2^m - 2^{m_1} < 2^m.$$

Indeed, the left-hand side inequality is clear if  $n_1 = 0$ . If  $n_1 = 3$ ,  $n \geq 5$ . For  $n = 5$  it is also clear and for  $n \geq 6$  we have  $P_n - P_{n_1} \geq P_n - P_{n-1} = P_{n-5} \geq \gamma^{n-8}$ . The right-hand side inequality is clear. Thus  $\gamma^{n-8} < 2^m$ . In a similar way,

$$\gamma^{n-1} \geq P_n \geq P_n - P_{n_1} = 2^m - 2^{m_1} > 2^{m-1}.$$

Thus,

$$(n-8) \frac{\log \gamma}{\log 2} \leq m \quad \text{and} \quad (n-1) \frac{\log \gamma}{\log 2} \geq m-1. \quad (8)$$

Since  $\log \gamma / \log 2 = 0.40568\dots$  we have that if  $n \leq 500$  then  $m \leq 204$ . We ran a *Mathematica* program in the range  $0 \leq n_1 < n \leq 500$ ,  $0 \leq m_1 < m \leq 204$  and, with our conventions, we obtained all the solutions listed in Theorem 1.1.

From now on we assume  $n > 500$ . Thus, from (8) we have that  $m > 199$  and also that  $n > m$ . From Binet's formula (4) we rewrite our equation as

$$|c_1\gamma^n - 2^m| < \gamma^{n_1-1} + 2|c_2||\delta|^n + 2^{m_1} < \gamma^{n_1+3} + 2^{m_1} < \max\{\gamma^{n_1+6}, 2^{m_1+1}\}.$$

Dividing through by  $2^m$  we get

$$|c_1 \gamma^n 2^{-m} - 1| < \max\{\gamma^{n_1-n+14}, 2^{m_1-m+1}\}, \tag{9}$$

where we have used  $\gamma^{n-8} < 2^m$ . Let  $\Lambda$  be the expression inside the absolute value in the left-hand side of (9). Observe that  $\Lambda \neq 0$ . To see this, we consider the  $\mathbb{Q}$ -automorphism  $\sigma$  of the Galois extension  $\mathbb{K} := \mathbb{Q}(\gamma, \delta)$  over  $\mathbb{Q}$  defined by  $\sigma(\gamma) := \delta$  and  $\sigma(\delta) := \gamma$ . We note  $\sigma(\bar{\delta}) = \bar{\delta}$ . If  $\Lambda = 0$  then  $\sigma(\Lambda) = 0$  and we get

$$2^m = \sigma(c_1 \gamma^n) = c_2 \delta^n.$$

Thus,

$$2^m = |c_2| |\delta|^n < 1,$$

which is absurd since  $m > 199$ . Hence,  $\Lambda \neq 0$ . We apply Matveev's inequality to  $\Lambda$  by taking

$$\alpha_1 = c_1, \alpha_2 = \gamma, \alpha_3 = 2, \quad b_1 = 1, b_2 = n, b_3 = -m.$$

Thus  $B = n$ . Further  $h(\alpha_2) = \log \gamma/3$ ,  $h(\alpha_3) = \log 2$ . For  $\alpha_1$  we use the properties of the height to conclude

$$h(c_1) \leq \log \gamma + 5 \log 2.$$

So we take  $A_1 = 11.3$ ,  $A_2 = 0.3$ ,  $A_3 = 2.1$ . From Matveev's inequality we obtain

$$\log |\Lambda| > -C(1 + \log n) \cdot 11.3 \cdot 0.3 \cdot 2.1,$$

which compared with (9) yields

$$\min\{(n - n_1) \log \gamma, (m - m_1) \log 2\} \leq 1.92529 \times 10^{13} (1 + \log n).$$

Now we study each one of these two possibilities.

**Case 1.**  $\min\{(n - n_1) \log \gamma, (m - m_1) \log 2\} = (n - n_1) \log \gamma$ .

In this case, we rewrite our equation as

$$|c_1(\gamma^{n-n_1} - 1)\gamma^{n_1} - 2^m| \leq 2|c_2| |\delta|^n + 2|c_2| |\bar{\delta}|^{n_1} + 2^{m_1} < 1 + 2^{m_1} \leq 2^{m_1+1},$$

since  $n > 500$ . Thus

$$|c_1(\gamma^{n-n_1} - 1)\gamma^{n_1} 2^{-m} - 1| < \frac{1}{2^{m-m_1-1}}. \tag{10}$$

Let  $\Lambda_1$  be the expression inside the absolute value in the left-hand side of (10). We note that  $\Lambda_1 \neq 0$ . For if not, we apply  $\sigma$  as above and we have  $\sigma(\Lambda_1) = 0$ . Thus

$$2^m = |\sigma(c_1) (\delta^n - \delta^{n_1})| \leq 2|c_2| < 1,$$

which is absurd since  $m > 199$ . We apply Matveev's inequality to  $\Lambda_1$  and for this we take

$$\alpha_1 = c_1(\gamma^{n-n_1} - 1), \alpha_2 = \gamma, \alpha_3 = 2, \quad b_1 = 1, b_2 = n_1, b_3 = -m.$$

We have  $B = n$ . Further, the heights of  $\alpha_2$  and  $\alpha_3$  are already calculated. For  $\alpha_1$  we use the height properties and we get

$$\begin{aligned} h(\alpha_1) &\leq h(c_1) + (n - n_1)h(\gamma) + \log 2 \\ &\leq \frac{3 \log \gamma + 18 \log 2 + (n - n_1) \log \gamma}{3} \\ &\leq \frac{1.92530 \times 10^{13}(1 + \log n)}{3}, \end{aligned}$$

where we have used (3). Thus we take  $A_1 = 1.92530 \times 10^{13}(1 + \log n)$ ,  $A_2 = 0.3$ ,  $A_3 = 2.1$ . From Matveev's inequality we obtain

$$\log |\Lambda_1| > -C(1 + \log n) \cdot (1.92530 \times 10^{13}(1 + \log n)) \cdot 0.3 \cdot 2.1,$$

which compared with (10) gives us

$$(m - m_1) \log 2 < 3.28032 \times 10^{25}(1 + \log n)^2.$$

**Case 2.**  $\min\{(n - n_1) \log \gamma, (m - m_1) \log 2\} = (m - m_1) \log 2$ .

In this case, we rewrite our equation as

$$|c_1 \gamma^n - (2^{m-m_1} - 1)2^{m_1}| \leq \gamma^{n_1-1} + 2|c_2||\delta|^n \leq 2\gamma^{n_1} < \gamma^{n_1+3}.$$

Thus,

$$\left| 1 - \left( \frac{2^{m-m_1} - 1}{c_1} \right) \gamma^{-n} 2^{m_1} \right| < \frac{1}{\gamma^{n-n_1-6}}, \quad (11)$$

where we have used  $1 < c_1 \gamma^3$ . Let  $\Lambda_2$  be the expression inside the absolute value in the left-hand side of (11). We note that  $\Lambda_2 \neq 0$ . Indeed, if it is not the case then by applying  $\sigma$  as above we obtain  $\sigma(\Lambda_2) = 0$ . Thus

$$1 \leq (2^{m-m_1} - 1)2^{m_1} = 2^m - 2^{m_1} = |c_2||\delta|^n < |c_2| < \frac{1}{2}.$$

Now we apply Matveev's inequality to  $\Lambda_2$ . To do this we take

$$\alpha_1 = \frac{2^{m-m_1} - 1}{c_1}, \alpha_2 = \gamma, \alpha_3 = 2, \quad b_1 = 1, b_2 = -n, b_3 = m_1.$$

Thus  $B = n$ . The heights of  $\alpha_2$  and  $\alpha_3$  are already calculated. From the properties of the height for  $\alpha_1$  we obtain

$$\begin{aligned} h(\alpha_1) &\leq (m - m_1)h(2) + \log 2 + h(c_1) \\ &\leq (m - m_1) \log 2 + \log \gamma + 6 \log 2 \\ &\leq 1.92530 \times 10^{13}(1 + \log n), \end{aligned}$$



where again we use (3). Thus we take  $A_1 = 5.7759 \times 10^{13}(1 + \log n)$ ,  $A_2 = 0.3$ ,  $A_3 = 2.1$ . Thus from Matveev's inequality we obtain

$$\log |\Lambda_2| > -C(1 + \log n) \cdot (5.7759 \times 10^{13}(1 + \log n)) \cdot 0.3 \cdot 2.1,$$

which compared with (11) yields

$$(n - n_1) \log \gamma < 9.84094 \times 10^{25}(1 + \log n)^2.$$

So, from the conclusion of the two cases we have that

$$\max\{(n - n_1) \log \gamma, (m - m_1) \log 2\} < 9.84094 \times 10^{25}(1 + \log n)^2.$$

Now we get a bound on  $n$ . To do this, we rewrite our equation as

$$|c_1(\gamma^{n-n_1} - 1)\gamma^{n_1} - (2^{m-m_1} - 1)2^{m_1}| < 4|c_2||\delta|^{n_1} \leq 4|c_2| < 2.$$

Thus

$$\left| \frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1} \gamma^{n_1} 2^{-m_1} - 1 \right| < \frac{2}{2^m - 2^{m_1}} \leq \frac{4}{2^m} < \frac{4}{\gamma^{n-8}} < \frac{1}{\gamma^{n-13}}, \quad (12)$$

where we have used  $\gamma^{n-8} < 2^m$  and  $4 < \gamma^5$ . Let  $\Lambda_3$  be the expression inside the absolute value in the left-hand side of (12). If  $\Lambda_3 = 0$  we apply  $\sigma$  as above and we obtain  $\sigma(\Lambda_3) = 0$ . Then

$$1 \leq (2^{m-m_1} - 1)2^{m_1} = 2^m - 2^{m_1} = |c_2(\delta^n - \delta^{n_1})| \leq 2|c_2| \leq \frac{2}{3},$$

which is false. Thus  $\Lambda_3 \neq 0$  and we apply Matveev's inequality to it. We take

$$\alpha_1 = \frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1}, \alpha_2 = \gamma, \alpha_3 = 2, \quad b_1 = 1, b_2 = n_1, b_3 = -m_1.$$

Thus  $B = n$ . The height of  $\alpha_2$  and  $\alpha_3$  have already been calculated. For  $\alpha_1$  we use the properties of the height and we get

$$\begin{aligned} h(\alpha_1) &\leq \log \gamma + (n - n_1) \frac{\log \gamma}{3} + (m - m_1) \log 2 + 7 \log 2 \\ &< \frac{3.93639 \times 10^{26}(1 + \log n)^2}{3}. \end{aligned}$$

Thus we take  $A_1 = 3.93639 \times 10^{26}(1 + \log n)^2$ ,  $A_2 = 0.3$ ,  $A_3 = 2.1$ . From Matveev's inequality we get

$$\log |\Lambda_3| > -C \cdot (1 + \log n) \cdot (3.93639 \times 10^{26}(1 + \log n)^2) \cdot 0.3 \cdot 2.1,$$

which comparing with (12) we get  $n < 1.90805 \times 10^{40}(\log n)^3$ . Now, from Lemma 2.3 we obtain

$$n < 1.21791 \times 10^{47}.$$

Now we will reduce this upper bound on  $n$ . To do this let  $\Gamma$  be defined as

$$\Gamma = n \log \gamma - m \log 2 + \log c_1,$$

and we go to equation (9). We assume  $\min\{n - n_1, m - m_1\} \geq 20$ . Observe that  $e^\Gamma - 1 = \Lambda \neq 0$ . Therefore  $\Gamma \neq 0$ . If  $\Gamma > 0$  we have

$$0 < \Gamma < e^\Gamma - 1 = |\Lambda| < \max\{\gamma^{n_1 - n + 14}, 2^{m_1 - m + 1}\}.$$

If  $\Gamma < 0$  we then have that  $1 - e^\Gamma = |e^\Gamma - 1| < 1/2$ . Thus  $e^{|\Gamma|} < 2$  and we get

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|} |\Lambda| < 2 \max\{\gamma^{n_1 - n + 14}, 2^{m_1 - m + 1}\}.$$

So, in both cases we have

$$0 < |\Gamma| < 2 \max\{\gamma^{n_1 - n + 14}, 2^{m_1 - m + 1}\}.$$

Dividing through  $\log 2$  we obtain

$$0 < |n\tau - m + \mu| < \max\left\{\frac{148}{\gamma^{n - n_1}}, \frac{6}{2^{m - m_1}}\right\}, \quad (13)$$

where

$$\tau := \frac{\log \gamma}{\log 2}, \quad \mu := \frac{\log c_1}{\log 2}.$$

Now we apply Lemma 2.2. To do this we take  $M := 1.21791 \times 10^{47}$  which is the upper bound on  $n$ . Using *Mathematica* we find that the denominator of the convergent

$$\frac{p_{115}}{q_{115}} = \frac{2247452599136518246572247053320457964630307358519626}{5539892570194379685318407717184223926861580420931369}$$

of  $\tau$  is such that  $q_{115} > 6M$  and that  $\varepsilon = \|q_{115}\mu\| - M\|q_{115}\tau\| = 0.419327 > 0$ . Thus with  $A := 148$ ,  $B := \gamma$  or  $A := 6$ ,  $B := 2$  from Lemma 2.2 we obtain that, either

$$n - n_1 \leq 444 \quad \text{or} \quad m - m_1 \leq 175.$$

Now, we study each one of these two cases. First, we assume that  $n - n_1 \leq 444$  and  $m - m_1 \geq 20$ . In this case we consider

$$\Gamma_1 = n_1 \log \gamma - m \log 2 + \log (c_1(\gamma^{n - n_1} - 1)),$$

and we go to (10). We see that  $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$ . Thus  $\Gamma_1 \neq 0$  and, with a similar argument as the previous one we obtain

$$0 < |\Gamma_1| < \frac{4}{2^{m - m_1}}.$$

Dividing through  $\log 2$  we get

$$0 < |n_1\tau - m + \mu| < \frac{6}{2^{m-m_1}}, \tag{14}$$

where  $\tau$  is the same one as above and

$$\mu := \frac{\log(c_1(\gamma^{n-n_1} - 1))}{\log 2}.$$

We note that  $n_1 > 0$ , since otherwise we would have  $n \leq 444$  which contradicts  $n > 500$ . Thus we can apply Lemma 2.2. Consider

$$\mu_k := \frac{\log(c_1(\gamma^k - 1))}{\log 2}, \quad k = 1, 2, \dots, 444.$$

With the help of *Mathematica* we found that the same 115-th convergent of  $\tau$  well works, that is, it is such that its denominator satisfies  $q_{115} > 6M$  and  $\varepsilon_k \geq 0.000889789 > 0$  for all  $k = 1, 2, \dots, 444$ . Thus, with  $A := 6$ ,  $B := 2$  we calculated  $\log(q_{115} \cdot 6/\varepsilon_k)/\log 2$  for all  $k = 1, 2, \dots, 444$  and we found that the maximum value of them is at most 184. Therefore,  $m - m_1 \leq 184$ .

We now study the other case. Assume  $m - m_1 \leq 175$  and  $n - n_1 \geq 20$ . Consider

$$\Gamma_2 = n \log \gamma - m_1 \log 2 + \log\left(\frac{c_1}{2^{m-m_1} - 1}\right),$$

and we go to (11). We see that  $1 - e^{-\Gamma_2} = \Lambda_2 \neq 0$ . Thus  $\Gamma_2 \neq 0$ . As before, we can deduce that

$$0 < |\Gamma_2| < \frac{2\gamma^6}{\gamma^{n-n_1}}.$$

Dividing through  $\log 2$  we get

$$0 < |n\tau - m_1 + \mu| < \frac{16}{\gamma^{n-n_1}}, \tag{15}$$

where  $\tau$  is as above and

$$\mu := \frac{\log(c_1/(2^{m-m_1} - 1))}{\log 2}.$$

Note that  $m_1 > 0$ . For if not, then  $m \leq 175$  which contradicts  $m > 199$ . Thus we can apply Lemma 2.2 again. Consider

$$\mu_\ell := \frac{\log(c_1/(2^\ell - 1))}{\log 2} \quad \ell = 1, \dots, 175.$$

With *Mathematica*, we find that again the same 115-th convergent of  $\tau$  above well works, that is,  $q_{115} > 6M$  and  $\varepsilon_\ell \geq 0.0016923 > 0$  for all  $\ell = 1, 2, \dots, 175$ .

Thus, with  $A := 16$ ,  $B := \gamma$  we calculated  $\log(q_{115} \cdot 16/\varepsilon_\ell)/\log \gamma$  for all  $\ell = 1, 2, \dots, 175$  and found that the maximum of its values is  $\leq 456$ . Thus  $n - n_1 \leq 456$ .

Let us summarize what we have done. We got that either  $n - n_1 \leq 444$  or  $m - m_1 \leq 175$ . Assuming the first one, we got that  $m - m_1 \leq 184$  and, assuming the second one we got  $n - n_1 \leq 456$ . Altogether, we have that  $n - n_1 \leq 456$  and  $m - m_1 \leq 184$ . So, it remains to study this case.

To do this, we consider

$$\Gamma_3 = n_1 \log \gamma - m_1 \log 2 + \log \left( \frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1} \right),$$

and we go to (12). Note that  $e^{\Gamma_3} - 1 = \Lambda_3 \neq 0$ . Thus  $\Gamma_3 \neq 0$ . As above we can deduce

$$0 < |\Gamma_3| < \frac{2 \cdot \gamma^{13}}{\gamma^n},$$

since  $n > 500$ . Dividing through by  $\log 2$  we get

$$0 < |n_1 \tau - m_1 + \mu| < \frac{112}{\gamma^n}, \quad (16)$$

where  $\tau$  is as above and

$$\mu := \frac{\log(c_1(\gamma^{n-n_1} - 1)/(2^{m-m_1} - 1))}{\log 2}.$$

As before, we note that  $n_1 > 0$  and  $m_1 > 0$ . Thus we can apply Lemma 2.2 again. Consider

$$\mu_{k,\ell} := \frac{\log(c_1(\gamma^k - 1)/(2^\ell - 1))}{\log 2}, \quad k = 1, \dots, 456, \quad \ell = 1, \dots, 184.$$

With the help of *Mathematica* we find again that the same 115-th convergent above of  $\tau$  works also well in this case, that is,  $q_{115} > 6M$  and  $\varepsilon_{k,\ell} \geq 5.27725 \times 10^{-6} > 0$  for all  $k = 1, \dots, 456$ ,  $\ell = 1, \dots, 184$ . Thus, with  $A := 112$ ,  $B := \gamma$  we calculated  $\log(q_{115} \cdot 112/\varepsilon_{k,\ell})/\log \gamma$ , for all  $k = 1, \dots, 456$ ,  $\ell = 1, \dots, 184$  and found that the maximum value of them is  $\leq 483$ . Thus  $n \leq 483$ , which contradicts our assumption on  $n$ . This finishes the proof of Theorem 1.1.

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## References

- [1] H. Davenport A. Baker, *The equations  $3X^2 - 2 = Y^2$  and  $8X^2 - 7 = Z^2$* , Quart. J. Math. Oxford **20** (1969), no. 2, 129–137.
- [2] J. J. Bravo, C. A. Gómez, and F. Luca, *Powers of two as sums of two  $k$ -Fibonacci numbers*, Miskolc Math. Notes **17** (2016), no. 1, 85–100.
- [3] J. J. Bravo, F. Luca, and K. Yazán, *On Pillai's problem with Tribonacci numbers and Powers of 2*, Bull. Korean Math. Soc. **54** (2017), no. 3, 1069–1080.
- [4] Y. Bugeaud, M. Mignotte, and S. Siksek, *Classical and modular approaches to exponential diophantine equations I: Fibonacci and Lucas perfect powers*, Ann. of Math. **163** (2006), 969–1018.
- [5] K. C. Chim, I. Pink, and V. Ziegler, *On a variant of Pillai's problem*, Int. J. Number Theory **7** (2017), 1711–1727.
- [6] ———, *On a variant of Pillai's problem II*, J. Number Theory **183** (2018), 269–290.
- [7] M. Ddamulira, C. A. Gómez, and F. Luca, *On a problem of Pillai with  $k$ -generalized Fibonacci numbers and powers of 2*, Monatsh. Math., 2018, <https://doi.org/10.1007/s00605-018-1155-1>.
- [8] M. Ddamulira, F. Luca, and M. Rakotomalala, *On a problem of Pillai with Fibonacci and powers of 2*, Proc. Indian Acad. Sci. (Math. Sci.) **127** (2017), no. 3, 411–421.
- [9] A. Dujella and A. Petho, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford **49** (1998), no. 3, 291–306.
- [10] S. Hernández Hernández, F. Luca, and L. M. Rivera, *On Pillai's problem with the Fibonacci and Pell sequences*, Accepted in the Bol. Soc. Mat. Mexicana (2018).
- [11] A. Herschfeld, *The equation  $2^x - 3^y = d$* , Bull. Amer. Math. Soc. **42** (1936), 231–234.
- [12] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II*, Izv. Math. **64** (2000), no. 6, 1217–1269.
- [13] S. S. Pillai, *On  $a^x - b^y = c$* , J. Indian Math. Soc. **2** (1936), 119–122.
- [14] ———, *On the equation  $2^x - 3^y = 2^X + 3^Y$* , Bull. Calcutta Math. Soc. **37** (1945), 15–20.

- [15] S. Guzmán Sánchez and F. Luca, *Linear combinations of factorials and  $S$ -units in a binary recurrence sequence*, Ann. Math. Québec **38** (2014), 169–188.
- [16] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, <https://oeis.org/>.
- [17] R. J. Stroeker and R. Tijdeman, *Diophantine equations*, Computational methods in number theory, Math. Centre Tracts (**155**), Centre for Mathematics and Computer Science, Amsterdam (1982), 321–369.

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