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Blow up and globality of solutions for a nonautonomous semilinear heat equation with Dirichlet condition

Explosión y globalidad de soluciones para una ecuación de calor semilineal no autónoma con condición de Dirichlet

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Abstract. In this paper we prove the local existence of a nonnegative mild solution for a nonautonomous semilinear heat equation with Dirichlet condition, and give sufficient conditions for the globality and for the blow up in finite time of the mild solution. Our approach for the global existence goes back to the Weissler's technique and for the finite time blow up we uses the intrinsic ultracontractivity property of the semigroup generated by the diffusion operator.

Key words and phrases. Reaction-diffusion equations, finite time blow up, Lévy processes, Dirichlet problem, ultracontractive semigroup, killed process.

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Resumen. En este artículo demostramos la existencia local de una solución "mild"no negativa para una ecuación de calor semilineal no autónoma con condición de Dirichlet, y damos condiciones suficientes para la globalidad y la explosión en tiempo finito de la solución "mild". Nuestro enfoque para la existencia global se remonta a la técnica de Weissler y para la explosión en tiempo finito utilizamos la ultracontractividad intrínseca del semigrupo generado por el operador de difusión.

Palabras y frases clave. Ecuaciones de reacción-difusión, explosión en tiempo finito, procesos de Lévy, problema de Dirichlet, semigrupo ultracontractivo, proceso matado.

1. Introduction

Consider a semilinear heat equations of the type

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= k(t)\mathcal{A}u(t,x) + h(t)\eta(u), \qquad (t,x) \in (0,T] \times D, \\ u(0,x) &= f(x), \qquad x \in D, \\ u(t,x) &= 0, \qquad (t,x) \in (0,T] \times D^c, \end{aligned} \tag{1}$$

where $D \subset \mathbb{R}^d$ is an open set, $k, h : [0,\infty) \to [0,\infty)$ are continuous and not identically zero functions, f is a continuous function on D, $\mathcal{A}$ is the infinitesimal generator of a symmetric Lévy process and the nonlinearity $\eta(u)$ is assumed to satisfy $\eta(0) = 0$ and $\eta(u) > 0$ for $u > 0$. Reaction-diffusion equations of this prototype model a great number of molecular biology, physic and engineering problems (see [2], [29], [31]). The most common interpretation is to consider u as the temperature of a substance in a recipient $D \subset \mathbb{R}^d$ subjet to a chemical reaction; in this case η represents a heat source due to an exothermic reaction.

In contrast to linear equations, the solutions of nonlinear parabolic equations can develop singularities in finite time, no matter how smooth the initial data are. It is well known that solutions of many differential equations of the type (1) with or without Dirichlet conditions, can become unbounded in finite time (phenomenon known as blow up in finite time). Rigorously, a solution u of the semilinear heat equation (1) blows up in finite time if there exists a time $T_e < \infty$ such that u is bounded for all $T < T_e$ and $\lim_{t\to T_e} \|u(\cdot,t)\|_\infty = \infty$; T_e is called blow up time. If u exists for all $T > 0$ and $\|u(\cdot,t)\|_\infty < \infty$ for all $t \geq 0$, u is called a global solution.

After de pioneering works of Kaplan [17] and Fujita [11, 12], many authors have studied global existence and blow up in finite time of positive solutions for semilinear heat equations (1) (with and without Dirichlet conditions) when $k \equiv 1$ and $A = \Delta$, the Laplacian operator, for several types of nonlinearities η. The articles [1, 7, 8, 9, 10, 13, 23, 24, 25, 35, 36] are only a few examples. The articles [5, 28] address these topics for non-local diffusions. The cases where $\mathcal{A} = -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional power of the Laplacian, $0 < \alpha < 2$, have been used in models of anomalous growth of certain fractal interfaces [16]. The articles [3, 15, 18, 20, 27, 32, 33, 34] are only a few examples for the study of global existence and blow up in finite time of positive solutions.

In this paper we consider the semilinear heat equation (1) for $\eta(u) = u^\beta$ with $\beta > 1$. This equation is a scalar version of the system studied in [21], but here, unlike [21], we have considered a time dependent coefficient, $h(t)$, for the reaction term u^β.

2. Some preliminaries and assumptions

In the semilinear heat equation

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= k(t)\mathcal{A}u(t,x) + h(t)u^{\beta}(t,x), \qquad t>0,\ x\in D, \\ u(0,x) &= f(x), \qquad x\in D, \\ u(t,x) &= 0, \qquad t>0,\ x\in D^c,\end{aligned} \tag{2}$$

$D\subset\mathbb{R}^d$, $d\geq 1$, is a bounded nonempty open set, $\mathcal{A}$ is the infinitesimal generator of a symmetric Lévy process $\{Z_t\}_{t\geq 0}$, $\beta>1$ is a constant, the initial value f is a nonnegative function in the space $C_0(D)$ of continuous functions on D vanishing on D^c and the time dependent coefficients $k,h:[0,\infty)\to[0,\infty)$ are continuous and not identically zero.

Recall [30] that the Lévy process $Z\equiv\{Z_t\}_{t\geq 0}$ is called symmetric when Z_t and $-Z_t$ have the same distribution for all $t\geq 0$. The probability law of Z is uniquely determined by the probability measure $\mu(B):=P[Z\in B]$, $B\in\mathcal{B}\left(\mathbb{R}^d\right)$, where $\mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-field on $\mathbb{R}^d$. This probability measure is infinitely divisible and therefore, by the Lévy-Khintchine formula, its characteristic function $\hat{\mu}$ admits the representation

$$\hat{\mu}(z)=\exp\left[-\frac{1}{2}\langle z,Az\rangle+i\langle\gamma,z\rangle+\int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle}-1-i\langle z,x\rangle 1_{\{x:|x|\leq 1\}}(x)\right)\nu(dx)\right],$$

$z\in\mathbb{R}^d,\ i=\sqrt{-1}$ where $A=(a_{jk})$ is a symmetric nonnegative-definite matrix, $\gamma=(\gamma_1,\dots,\gamma_d)\in\mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$ such that $\nu(\{0\})=0$ and $\int_{\mathbb{R}^d}\left(|x|^2\wedge 1\right)\nu(dx)<\infty$, which is termed Lévy measure. The operator $\mathcal{A}$ is the infinitesimal generator of the strongly continuous semigroup of contractions $\{S_t\}_{t\geq 0}$ defined by $S_tf(x)=E\left[f\left(x+Z_t\right)\right]$, $f\in C_0\left(\mathbb{R}^d\right)$, and is given by

$$\begin{aligned}\mathcal{A}f(x)=&\frac{1}{2}\sum_{j,k=1}^{d}a_{jk}\frac{\partial^2 f(x)}{\partial x_j\partial x_k}+\sum_{j=1}^{d}\gamma_j\frac{\partial f(x)}{\partial x_j}\\ &+\int_{\mathbb{R}^d}\left(f(x+y)-f(x)-\sum_{j=1}^{d}y_j\frac{\partial f(x)}{\partial x_j}1_{\{y:|y|\leq 1\}}(y)\right)\nu(dy)\end{aligned}$$

for any twice continuously differentiable $f\in C_0\left(\mathbb{R}^d\right)$. Special instances of $\mathcal{A}$ include the Laplacian Δ and its fractional powers $\Delta_\alpha=-(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha\leq 2$.

As we already said in the introduction, Dirichlet boundary value problems of the type (2) in the Gaussian case ($\nu\equiv 0$), have been studied by many authors. In this paper we consider the purely non-Gaussian symmetric case in which $A=0$ and ν is a nontrivial Lévy measure, hence Z is a pure-jump process which leaves D only when it hits D^c. This gives rise to the condition

$u(t,x)=0$, $t>0$, $x\in D^c$ in (2), which is the form that the Dirichlet boundary condition takes in our setting; see [4].

Throughout this paper we assume that Z possesses a family of transition densities $p_t(x,y)\equiv p_t(x-y)$ which are continuous for every $t>0$, and that for any $\delta>0$ there exists a constant $c=c(\delta)$ such that $p_t(x)\leq c$ for all $t>0$ and all $|x|\geq\delta$. In [19], Lemma 2.5 and [4], Lemma 1.1, sufficient conditions are given for continuity on $\mathbb{R}^d\backslash\{0\}$ and for boundedness on $\mathbb{R}^d$, respectively, for every $t>0$ of the transition densities of isotropic unimodal pure-jump Lévy processes. Letting

$$K(t,s)=\int_s^t k(r)dr,\quad 0\leq s\leq t,$$

it is known (see [20], p. 3 and 4) that the time-inhomogeneous Markov process $W\equiv\{W_t\}_{t\geq 0}$, where $W_t\overset{\mathrm{D}}{=}Z_{K(t,0)}$ (here $\overset{\mathrm{D}}{=}$ means equality in distribution) has the transition probability

$$P_{s,t}(x,B)=P\left[Z_{K(t,s)}\in B-x\right]=S_{K(t,s)}1_B(x),$$

where $\{S_t\}_{t\geq 0}$ denotes the semigroup with generator $\mathcal{A}$ and 1_B is the indicator function of B. Moreover, the function $(t,x)\mapsto S_{K(t,s)}f(x)$, $(t,x)\in[s,\infty)\times\mathbb{R}^d$, is the unique solution of

$$\begin{aligned}\frac{\partial w(t,x)}{\partial t}&=k(t)\mathcal{A}w(t,x),\qquad t>s,\ x\in\mathbb{R}^d,\\ w(s,x)&=f(x),\qquad f\in C_0\left(\mathbb{R}^d\right).\end{aligned}$$

For this reason we call $\{W_t\}_{t\geq 0}$ the time-inhomogeneous Markov process corresponding to the family of generators $\{k(t)\mathcal{A}\}_{t\geq 0}$. Letting

$$p_{s,t}(x,y)=p_{K(t,s)}(x,y),\qquad 0\leq s\leq t,\ x,y\in\mathbb{R}^d,$$

we see that $p_{s,t}(x,y)$ is a transition density function for the process $\{W_t\}_{t\geq 0}$. We define

$$\tau_D=\inf\{t>0:W_t\notin D\}\quad\text{and}\quad\widehat{\tau}_D=\inf\{t>0:Z_t\notin D\}.$$

Using that $W_t\overset{\mathrm{D}}{=}Z_{K(t,0)}$ we get

$$\widehat{\tau}_D=K\left(\tau_D,0\right).\tag{3}$$

Let us consider the Z^D process killed on leaving D, which is given by

$$Z_t^D=\begin{cases}Z_t & \text{on }\{t<\widehat{\tau}_D\},\\ \partial & \text{on }\{t\geq\widehat{\tau}_D\},\end{cases}$$

where ∂ is a cemetery state. The state space of $\left\{Z_t^D\right\}_{t\geq 0}$ is the set $D_\partial = D\cup\{\partial\}$ and its transition probability is

$$P_t^D(x,\Gamma) = P_x\left[Z_t\in\Gamma; t<\widehat{\tau}_D\right], \quad t>0,\ x\in D,\ \Gamma\in\mathcal{B}(D),$$

where $\mathcal{B}(D)$ denotes the Borel σ-field on D. Here and in the sequel P_x and E_x denote, respectively, the distribution and expectation with respect to the process $\{x+Z_t\}_{t\geq 0}$ starting in $x\in\mathbb{R}^d$, but we use the same symbol $\{Z_t\}_{t\geq 0}$ for the resulting process.

Let $\left\{S_t^D\right\}_{t\geq 0}$ be the semigroup associated to the process $\left\{Z_t^D\right\}_{t\geq 0}$, and let $p_t^D(x,y)$ be the transition density function of $\left\{S_t^D\right\}_{t\geq 0}$, i.e.

$$\begin{aligned} S_t^D f(x) &= E_x\left[f\left(Z_t\right): t<\widehat{\tau}_D\right] \\ &= \int_D f(y)p_t^D(x,y)dy, \quad t>0,\ x\in D,\ f\in B^+\left(\mathbb{R}^d\right), \end{aligned}$$

where $B^+\left(\mathbb{R}^d\right)$ is the space of nonnegative bounded measurable functions on $\mathbb{R}^d$. It is known [14] that $p_t^D(x,y)=p_t^D(y,x)$ and $p_t^D(x,y)\leq p_t(x,y)$ for all $t>0$ and $x,y\in D$, and that $\left\{S_t^D\right\}_{t\geq 0}$ is a strongly continuous semigroup of contractions on the space of square-integrable functions $L^2(D)$. Any operator S_t^D is self-adjoint due to the a.e. symmetry of $p_t^D(x,y)$. Moreover, since D is bounded, the continuity of $p_t(\cdot)$ for all $t>0$ implies (see [14], p. 93) that $p_t^D(\cdot)$ is bounded for every $t>0$. Therefore S_t^D is a Hilbert-Schmidt operator, hence it is also compact, and there exists an orthonormal basis of eigenfunctions $\{\varphi_n\}_{n=0}^{\infty}$ with corresponding eigenvalues $\left\{e^{-\lambda_n t}\right\}_{n=0}^{\infty}$ satisfying $0<\lambda_0<\lambda_1\leq\lambda_2\leq\cdots$, and $\lim_{n\to\infty}\lambda_n=\infty$. All eigenfunctions φ_n are continuous and real-valued (see [14]). Let $V_r(x)=\left\{y\in\mathbb{R}^d : |y-x|<r\right\}$ be the open ball of radious $r>0$ centered at $x\in\mathbb{R}^d$. If, in addition to the above assumptions:

(H1) D is a connected open set, or

(H2) D is a bounded open set and for every $x\in\mathbb{R}^d$ and $r>0$, $\nu\left(V_r(x)\right)>0$, then the transition density $p_t^D(\cdot,\cdot)$, $t>0$ is strictly positive on $D\times D$ and the eigenfunction $\varphi_0(x)>0$ for every $x\in D$ (see [14], Proposition 2.2).

Let $\left\{W_t^D\right\}_{t\geq 0}$ be the additive process $\{W_t\}_{t\geq 0}$ killed on exiting D, namelly

$$W_t^D = \begin{cases} W_t & \text{on } \{t<\tau_D\}, \\ \partial & \text{on } \{t\geq\tau_D\}, \end{cases}$$

where ∂ is a cemetery point. The state space of $\left\{W_t^D\right\}_{t\geq 0}$ is the set $D_\partial = D\cup\{\partial\}$, and from (3) it follows that its transition function is given by

$$P_{s,t}^D(x,\Gamma) = P_x\left[Z_{K(t,s)}\in\Gamma; K(t,s)<\widehat{\tau}_D\right], \quad 0\leq s<t,\ x\in D,\ \Gamma\in\mathcal{B}(D).$$

Hence the transition density function of $\left\{W_t^D\right\}_{t\geq 0}$ is given by $p_{s,t}^D(x,y) = p_{K(t,s)}^D(x,y)$ and thus, for every $f \in L^2(D)$,

$$U_{t,s}^D f(x) \equiv \int_D f(y) p_{s,t}^D(x,y)dy = S_{K(t,s)}^D f(x), \quad 0 \leq s < t, \ x \in D. \qquad (4)$$

It is easy to see from Proposition 1 in [21] that $p_{s,t}^D(x,y)$ is a density of $P_{s,t}^D(x,\Gamma)$, which is strictly positive, symmetric and continuous on $D \times D$.

Using (4) and the fact that $\left\{S_t^D\right\}_{t\geq 0}$ is a strongly continuous semigroup of contractions on $L^2(D)$, we obtain that $\left\{U_{t.s}^D\right\}_{t\geq s\geq 0}$ is an evolution family of contractions on $L^2(D)$. In [14], Theorem 3.1 it is proved that either condition (H2) or

(H3) D is an open bounded connected Lipschitz set, and for every $x \in S$, $\gamma \in \left(0, \frac{\pi}{2}\right]$ and $r > 0$,

$$\nu\left(\Gamma_\gamma(x) \cap V_r(0)\right) > 0,$$

where S denotes the unit sphere in $\mathbb{R}^d$ and $\Gamma_\gamma(x) = \left\{y \in \mathbb{R}^d : \langle x, y\rangle > |y| \cos\gamma\right\}$, imply that $\left\{S_t^D\right\}_{t\geq 0}$ is an intrinsically ultracontractive semigroup, i.e. for all $t > 0$ there exists a positive constant $c = c(t, D)$ such that for all $f \in L^2(D)$,

$$\left|S_t^D f(x)\right| < c\varphi_0(x)\|f\|_{L^2(D)}, \qquad x \in D; \qquad (5)$$

see [6], Theorem 3.2.

3. Local existence of a mild solution in L^∞

A solution of the integral equation

$$u(t,x) = U_{t,s}^D f(x) + \int_0^t h(s) U_{t,s}^D u^\beta(s,x)ds, \qquad (6)$$

is called a mild solution of (2). It is known ([26], pp. 129-130) that any classical solution of (2) is a solution of the integral equation (6).

We are going to assume that the initial value f is a nonnegative function in $L^\infty(D)$, where $L^\infty(D)$ is the space of real-valued essentially bounded functions defined on D.

For any constant $\tau > 0$ let

$$E_\tau := \left\{u : [0,\tau] \to L^\infty(D), |||u||| < \infty\right\},$$

where $|||u||| := \sup_{0\leq t\leq\tau} \|u(t,\cdot)\|_\infty$.

The couple $(E_\tau, |||\cdot|||)$ is a Banach space and

$$C_R^+ := \left\{u \in E_\tau : |||u||| \leq R, u \geq 0\right\}$$

is a closed subset of E_τ.

Theorem 3.1. *Let $f \in L^{\infty}(D)$ be nonnegative. There exists a constant $\tau = \tau(f) > 0$ such that the integral equation* (6) *possesses a unique solution in $L^{\infty}([0,\tau] \times D) \cap C_R^+$.*

Proof. Let us define the operator Ψ on C_R^+ by

$$\Psi u(t,x) = U_{t,0}^D f(x) + \int_0^t h(s) U_{t,s}^D u^{\beta}(s,x) ds.$$

We are going to show that Ψ is a contraction on C_R^+ for suitably chosen $R > 0$ and $\tau > 0$. In fact, if $u, v \in C_R^+$, then

$$|||\Psi u - \Psi v||| \leq \sup_{0 \leq t \leq \tau} \int_0^t h(s) \|u^{\beta}(s,\cdot) - v^{\beta}(s,\cdot)\|_{\infty} ds. \tag{7}$$

Applying the elementary inequality $\left|a^{\beta} - b^{\beta}\right| \leq \beta (a \vee b)^{\beta-1} |a-b|$, $a, b > 0$, $\beta \geq 1$ in (7), we get

$$|||\Psi u - \Psi v||| \leq \beta R^{\beta-1} \int_0^{\tau} h(s) \|u(s,\cdot) - v(s,\cdot)\|_{\infty} ds. \tag{8}$$

Noticing that

$$|||\Psi u||| \leq \|f\|_{\infty} + \int_0^{\tau} h(s) R^{\beta} ds$$

and taking $R > 0$ big enough and $\tau > 0$ sufficiently small we get from (8) that Ψ is a contraction mapping on C_R^+. Hence the Banach fixed-point theorem implies that (6) has a unique solution in $L^{\infty}([0,\tau] \times D) \cap C_R^+$. ☑

Since the evolution system $\left\{U_{t,s}^D\right\}_{t \geq s \geq 0}$ preserves positivity (due to (4)) we have that

$$u_0(t,x) := U_{t,0}^D f(x) \geq 0, \quad t \geq 0, \; x \in D. \tag{9}$$

Define

$$u_n(t,x) := \Psi u_{n-1}(t,x), \quad t \geq 0, \; x \in D, \; n = 1, 2, \ldots, \tag{10}$$

where Ψ is given by

$$\Psi v(t,x) := U_{t,0}^D f(x) + \int_0^t h(s) U_{t,s}^D v^{\beta}(s,x) ds \tag{11}$$

for any nonnegative $v \in L^{\infty}(D)$. Using again that $\left\{U_{t,s}^D\right\}_{t \geq s \geq 0}$ preserves positivity it follows by induction that $u_{n-1}(t,x) \leq u_n(t,x)$, $n = 1, 2, \ldots$. Hence the limit

$$u(t,x) := \lim_{n \to \infty} u_n(t,x) \tag{12}$$

exists for all $t \geq 0$ and $x \in D$. From the monotone convergence theorem we conclude that $u(t,x)$ satisfies (6). This shows that the solution of the integral equation (6) is given by the increasing limit (12).

4. Global existence for a nonnegative initial value in $L^\infty(D)$

Here we suppose again that the initial value f is a nonnegative function in $L^\infty(D)$. Our proof of the global existence is an adaptation, to our case, of the proof given in [36].

Theorem 4.1. *Let* $\phi(t) = \|\sup_{0\le s\le t} U^D_{s,0} f\|_\infty$, $t > 0$. *If*

$$(\beta-1)\int_0^\infty h(t)\phi^{\beta-1}(t)dt < 1,$$

then the solution of the integral equation (6) *is global.*

Proof. First, we note that ϕ is a nondecreasing function on $[0,\infty)$ and that for any $t \ge 0$ and $x \in D$,

$$0 \le U^D_{t,0} f(x) \le \phi(t) \le \|f\|_\infty < \infty. \tag{13}$$

Now, let us define

$$B(t) := \left[1-(\beta-1)\int_0^t h(s)\phi^{\beta-1}(s)ds\right]^{-\frac{1}{\beta-1}}.$$

Then $B(0) = 1$ and

$$\begin{aligned} B'(t) &= -\frac{1}{\beta-1}\left[1-(\beta-1)\int_0^t h(s)\phi^{\beta-1}(s)ds\right]^{-\frac{1}{\beta-1}-1}\left[-(\beta-1)h(t)\phi^{\beta-1}(t)\right] \\ &= h(t)B^\beta(t)\phi^{\beta-1}(t), \end{aligned}$$

which gives

$$B(t) = 1 + \int_0^t h(s)B^\beta(s)\phi^{\beta-1}(s)ds. \tag{14}$$

Since the evolution system $\left\{U^D_{t,s}\right\}_{t\ge s\ge 0}$ is positivity-preserving and ϕ is nondecreasing, it is follows from (13) that for any function $v : [0,\infty)\times D \to [0,\infty)$ such that for each $t \ge 0$, $v(t,\cdot) \in L^\infty(D)$ and $v(t,x) \le B(t)\phi(t)$ for all $x \in D$, we have

$$\begin{aligned} 0 \le \Psi v(t,x) &\le \phi(t) + \int_0^t h(s)\left(B(s)\phi(s)\right)^\beta ds \\ &\le \phi(t) + \int_0^t h(s)B^\beta(s)\phi(s)\phi^{\beta-1}(s)ds \\ &\le \phi(t) + \phi(t)\int_0^t h(s)B^\beta(s)\phi^{\beta-1}(s)ds \\ &= B(t)\phi(t) \qquad \text{for all } x \in D, \end{aligned}$$

where we have used (14) in the last equality. Therefore,

$$0 \leq \Psi v(t,x) \leq B(t)\phi(t), \qquad t \geq 0,\ x \in D. \tag{15}$$

Finally, defining the sequence $u_n(t,x)$, $n = 0,1,2,\dots$ as in (9) and (10) it is follows that $u_{n-1}(t,x) \leq u_n(t,x)$, $n = 1,2,\dots$. From (15) we conclude that

$$u(t,x) := \lim_{n\to\infty} u_n(t,x) \leq B(t)\phi(t) < \infty$$

for all $t \geq 0$, $x \in D$. Hence u is a global mild solution of (2). ☑

5. Blow up in finite time for a nonnegative initial value in $C_0(D)$

Recall that φ_0 is the eigenfunction corresponding to the first eigenvalue λ_0 of the infinitesimal generator of the semigroup $\left\{S_t^D\right\}_{t\geq 0}$. Arguing as in the case of Brownian motion in a bounded domain (see [22]), it can be shown that $\varphi_0^2(x)dx$ is the unique invariant measure of the semigroup $\{Q_t\}_{t\geq 0}$ given by

$$Q_t g(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} S_t^D\left(g\varphi_0\right)(x), \quad x \in D,\ g \in C_b(D),\ t \geq 0.$$

Thus, defining

$$E[g] := \int_D g(x)\varphi_0^2(x)dx, \qquad g \in C_b(D),$$

and

$$T_{t,s}g(x) = \frac{e^{\lambda_0 K(t,s)}}{\varphi_0(x)} S_{K(t,s)}^D\left(g\varphi_0\right)(x), \quad x \in D,\ g \in C_b(D),\ t \geq s \geq 0,$$

we have that for any $t \geq s \geq 0$ and $g \in C_b(D)$,

$$E\left[Q_t g\right] = E[g] \qquad \text{and} \qquad T_{t,s}g = Q_{K(t,s)}g. \tag{16}$$

Lemma 5.1. *For any $t \geq s \geq 0$ and $g \in C_b(D)$,*

$$E[T_{t,s}g] = E[g].$$

Proof. This is a direct consequence of (16). ☑

Proposition 5.2. *Let $f = g\varphi_0$, where $g \in C_b(D)$ is nonnegative and not identically zero. If*

$$\int_D f(x)\varphi_0(x)dx > \left[\frac{1}{(\beta-1)\int_0^\infty h(s)e^{-\lambda_0(\beta-1)K(s,0)}ds}\right]^{\frac{1}{\beta-1}} \|\varphi_0\|_1, \tag{17}$$

then the mild solution of (2) blows up in finite time.

Proof. Notice that

$$\int_D f(x)\varphi_0(x)dx = E[g] > 0.$$

We define

$$w(t,x) = \frac{e^{\lambda_0 K(t,0)}u(t,x)}{\varphi_0(x)} \qquad \text{and} \qquad z(t,x) = e^{-\lambda_0 K(t,0)}\varphi_0(x), \quad x \in D,\ t \geq 0,$$

where u is the mild solution of (2), i.e., u solves the integral equation (6). Multiplying both sides of (6) by $\varphi_0^{-1}(x)e^{\lambda_0 K(t,0)}$ we get

$$\begin{aligned}
w(t,x) &= T_{t,0}g(x) + \int_0^t h(s)\frac{e^{\lambda_0 K(t,0)}}{\varphi_0(x)}U^D_{t,s}u^\beta(s,x)ds \\
&= T_{t,0}g(x) + \int_0^t h(s)\frac{e^{\lambda_0 K(t,0)}}{\varphi_0(x)}U^D_{t,s}\left(\frac{u^\beta(s,x)}{\varphi_0^{\beta-1}(x)}\varphi_0^{\beta-1}(x)\right)ds \\
&= T_{t,0}g(x) + \int_0^t h(s)e^{\lambda_0 K(s,0)}\frac{e^{\lambda_0 K(t,s)}}{\varphi_0(x)}U^D_{t,s}\left(\frac{u^\beta(s,x)}{\varphi_0^{\beta-1}(x)}\varphi_0^{\beta-1}(x)\right)ds \\
&= T_{t,0}g(x) + \int_0^t h(s)e^{\lambda_0 K(s,0)}T_{t,s}\left(\frac{u^\beta(s,x)}{\varphi_0^{\beta-1}(x)}\varphi_0^{\beta-1}(x)\right)ds \\
&= T_{t,0}g(x) + \int_0^t h(s)T_{t,s}\left(\frac{e^{\lambda_0 K(s,0)\beta}u^\beta(s,x)}{\varphi_0^\beta(x)}e^{-\lambda_0 K(s,0)(\beta-1)}\varphi_0^{\beta-1}(x)\right)ds \\
&= T_{t,0}g(x) + \int_0^t h(s)T_{t,s}w^\beta(s,x)z^{\beta-1}(s,x)ds.
\end{aligned}$$

The last equality yields

$$E\left[w(t,\cdot)\right] = E\left[T_{t,0}g\right] + \int_0^t h(s)E\left[T_{t,s}\left(w^\beta(s,\cdot)z^{\beta-1}(s,\cdot)\right)\right]ds$$

and, due to Lemma 5.1,

$$E\left[w(t,\cdot)\right] = E[g] + \int_0^t h(s)E\left[w^\beta(s,\cdot)z^{\beta-1}(s,\cdot)\right]ds.$$

It follows that for any $\epsilon > 0$,

$$E\left[w(t+\epsilon,\cdot)\right] - E\left[w(t,\cdot)\right] = \int_t^{t+\epsilon} h(s)E\left[w^\beta(s,\cdot)z^{\beta-1}(s,\cdot)\right]ds, \tag{18}$$

with

$$E\left[w^{\beta}(s,\cdot)z^{\beta-1}(s,\cdot)\right] = e^{-\lambda_0 K(s,0)(\beta-1)}\int_D \left[w(s,x)\varphi_0(x)\right]^{\beta}\varphi_0(x)dx \qquad (19)$$
$$\geq e^{-\lambda_0 K(s,0)(\beta-1)}\|\varphi_0\|_1\left(\int_D w(s,x)\frac{\varphi_0^2(x)}{\|\varphi_0\|_1}dx\right)^{\beta}$$
$$= \left(\frac{e^{-\lambda_0 K(s,0)}}{\|\varphi_0\|_1}\right)^{\beta-1} E\left[w(s,\cdot)\right]^{\beta},$$

where we have used Jensen's inequality with respect to the probability measure $\frac{\varphi_0(x)dx}{\|\varphi_0\|_1}$.

Let $y(t) := E\left[w(t,\cdot)\right]$. Plugging (19) into (18), and afterward multiplying the resulting inequality by ϵ^{-1} with $\epsilon \to 0$, we obtain that

$$y'(t) \geq \left(\|\varphi_0\|_1^{-1}e^{-\lambda_0 K(t,0)}\right)^{\beta-1} h(t)y^{\beta}(t),$$
$$y(0) = \int_D f(x)\varphi_0(x)dx.$$

Let

$$c(t) = \|\varphi_0\|^{1-\beta}e^{-\lambda_0(\beta-1)K(t,0)}h(t) \qquad \text{and} \qquad N = \int_D f(x)\varphi_0(x)dx > 0,$$

and consider the ordinary differential equation

$$p'(t) = c(t)p^{\beta}(t), \qquad p(0) = N.$$

Notice that

$$p^{-\beta}(t)p'(t) = c(t).$$

Thus, integrating both sides of the above equality from 0 to t yields

$$\frac{1}{1-\beta}\left[p^{1-\beta}(t) - N^{1-\beta}\right] = \int_0^t c(s)ds.$$

Therefore

$$p(t) = \left[\frac{1}{N^{1-\beta} - (\beta-1)\int_0^t c(s)ds}\right]^{\frac{1}{\beta-1}}. \qquad (20)$$

Since the function $\int_0^t c(s)ds$ is continuous and increases to $\int_0^\infty c(s)ds$, we have that p blows up for some $0 < T_e < \infty$ if

$$N^{1-\beta} - (\beta-1)\int_0^\infty c(s)ds < 0,$$

which hols if and only if (17) is satisfied. Thus, by comparison, we have that

$$\lim_{t\uparrow T_e} \|w(t,\cdot)\|_\infty \geq \lim_{t\uparrow T_e} E\left[w(t,\cdot)\right] \geq \lim_{t\uparrow T_e} p(t) = \infty.$$

☑

Lemma 5.3. *Let $f \in C_0(D)$ be a nonnegative and not identically zero function. There exists $g \in C_b(D)$ nonnegative and not identically zero such that $g\varphi_0 \leq f$ on D.*

Proof. Since by assumption f is not identically zero, there exists $x \in D$ such that $f(x) > 0$. Using the continuity of f we get $r > 0$ such that $f(x) > 0$ on $V_r(x) \subset D$. By Urysohn's lemma there exists a continuous function $q : \mathbb{R}^d \to [0,1]$ such that $q = 1$ on the closed ball $\overline{V_{\frac{r}{3}}(x)}$, and $q = 0$ on $\left(V_{\frac{2r}{3}}(x)\right)^c$. Hence the support of q is contained in $V_r(x)$. Putting $\zeta = \frac{1}{2}(f \wedge q)$ we get a continuous function which is not identically zero, and whose support $\widetilde{C}$ is compact, has positive Lebesgue measure and is contained in D. Moreover, $0 \leq \zeta < f$ on $\widetilde{C}$.

Let $\{t_n\}$ be any given sequence of positive numbers with $t_n \downarrow 0$. It follows from the strong continuity of $\left\{U^D_{t,s}\right\}_{t\geq s\geq 0}$, that

$$U^D_{t_n,0}\zeta \to \zeta \qquad \text{in} \ \ L^2(D),$$

and therefore

$$U^D_{t_n,0}\zeta \to \zeta \qquad \text{in} \ \ L^2(\tilde{C}).$$

Using Egoroff's theorem, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, and a set $C \subset \widetilde{C}$ of positive Lebesgue measure such that

$$U^D_{t_{n_k},0}\zeta \to \zeta \qquad \text{uniformly in} \ \ C.$$

Hence, there exists $t_0 > 0$ such that

$$U^D_{t_0,0}\zeta(x) < f(x) \qquad \text{for all} \ \ x \in C.$$

Let us define $\xi = 1_C U^D_{t_0,0}\zeta$. The intrinsic ultracontractivity (5) implies that

$$\rho := \frac{U^D_{t_0,0}\zeta}{\varphi_0} = \frac{S^D_{K(t_0,0)}\zeta}{\varphi_0} \in C_b(D).$$

Then, we can write $\xi = 1_C\rho\varphi_0$, and thus, any nonnegative continuous function g with support contained in C, such that $g \leq 1_C\rho$ satisfies the assertion of the lemma. ☑

Theorem 5.4. *Let $f \in C_0(D)$ be a nonnegative and not identically zero function, and let g as in Lemma 5.3. If condition* (17) *holds for $\widetilde{f} = g\varphi_0$, then the mild solution of* (2) *blows up in finite time.*

Proof. Using that $0 \leq \widetilde{f} \leq f$, we get

$$u(t,x) = U^D_{t,0}f(x) + \int_0^t h(s)U^D_{t,s}u^\beta(s,x)ds$$
$$\geq U^D_{t,0}\widetilde{f}(x) + \int_0^t h(s)U^D_{t,s}u^\beta(s,x)ds.$$

Let v be the mild solution of (2) with initial value $\widetilde{f}$, which is given by

$$v(t,x) = U^D_{t,0}\widetilde{f}(x) + \int_0^t h(s)U^D_{t,s}v^\beta(s,x)ds.$$

We define the operator $\widetilde{\Psi}$ by

$$\widetilde{\Psi}v(t,x) = U^D_{t,0}\widetilde{f}(x) + \int_0^t h(s)U^D_{t,s}v^\beta(s,x)ds.$$

Then

$$\widetilde{\Psi}u(t,x) \leq \Psi u(t,x),$$

where Ψ is defined in (11). Now, we define the sequences $\{v_n\}_{n=0}^\infty$ and $\{u_n\}_{n=0}^\infty$ by

$$v_n(t,x) = \begin{cases} U^D_{t,0}\widetilde{f}(x), & n = 0, \\ \widetilde{\Psi}v_{n-1}(t,x), & n \in \mathbb{N}, \end{cases}$$

and

$$u_n(t,x) = \begin{cases} U^D_{t,0}f(x), & n = 0, \\ \Psi u_{n-1}(t,x), & n \in \mathbb{N}. \end{cases}$$

If $v_{n-1}(t,x) \leq u_{n-1}(t,x)$, then

$$v_n(t,x) = \widetilde{\Psi}v_{n-1}(t,x) \leq \widetilde{\Psi}u_{n-1}(t,x) \leq \Psi u_{n-1}(t,x) = u_n(t,x).$$

The contraction mapping property in a Banach space implies that the sequence $\{v_n\}_{n=0}^\infty$ converges in the norm $|||\cdot|||$ to the unique fixed point v of $\widetilde{\Psi}$, namely

$$v_n \to v \qquad \text{and} \qquad \widetilde{\Psi}v_n \to v$$

in the norm $|||\cdot|||$ as $n \to \infty$. Similarly, the sequence $\{u_n\}_{n=0}^\infty$ converges to the unique fixed point u of Ψ, that is

$$u_n \to u \qquad \text{and} \qquad \Psi u_n \to u$$

in the norm $|||\cdot|||$ as $n \to \infty$. Then, we have demonstrated that $v \leq u$. Since $\widetilde{f}$ satisfies the conditions of Proposition 5.2, we conclude that the mild solution of (2) blows up in finite time. ☑

***Remark* 5.5.** Note that the above theorem is consistent with the corresponding result obtained in [22], which establishes that for the case $\mathcal{A} = \Delta$, $k = h \equiv 1$, $\beta > 1$ and a nonnegative initial condition $f \in C_0(D)$, where D is a bounded regular domain, the positive mild solution blows up in finite time if

$$\int_D f(x)\varphi(x)dx > \lambda_0^{\frac{1}{\beta-1}} \|\varphi_0\|_1.$$

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