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Existence of periodic standing wave solutions for a system describing pulse propagation in an optical fiber

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Abstract. We establish existence of periodic standing waves for a model to describe the propagation of a light pulse inside an optical fiber taking into account the Kerr effect. To this end, we apply the Lyapunov Center Theorem taking advantage that the corresponding standing wave equations can be rewritten as a Hamiltonian system. Furthermore, some of these solutions are approximated by using a Newton-type iteration, combined with a collocation-spectral strategy to discretize the system of standing wave equations. Our numerical simulations are found to be in accordance with our analytical results.

Key words and phrases. Schrödinger equations, standing wave solutions, nonlinear optics, spectral scheme.

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Resumen. Establecemos existencia de soluciones estacionarias periódicas para un modelo que describe la propagación de un pulso de luz en el interior de una fibra óptica teniendo en cuenta el efecto Kerr. Para este fin, aplicamos el Teorema Central de Lyapunov tomando ventaja de que las correspondientes ecuaciones de onda estacionaria pueden escribirse como un sistema Hamiltoniano. Además, algunas de estas soluciones son aproximadas usando una iteración de tipo Newton, combinada con un estrategia colocación-espectral para discretizar el sistema de ecuaciones de onda estacionaria. Las simulaciones numéricas presentadas se encuentran de acuerdo con nuestros resultados analíticos.

Palabras y frases clave. Ecuaciones Schrödinger, soluciones de onda estacionaria, óptica no lineal, esquema espectral.

1. Introduction

In this paper, we consider from the theoretical and numerical point of view, periodic solutions to the system of two coupled nonlinear Schrödinger equations (henceforth called CNLS system)

$$i\frac{\partial u}{\partial \xi} + K\frac{\partial^2 u}{\partial x^2} + \sigma_1 u + a|u|^2 u + g|v|^2 u + ev^2 u^* = 0, \tag{1}$$

$$i\frac{\partial v}{\partial \xi} + K\frac{\partial^2 v}{\partial x^2} + \sigma_2 v + c|v|^2 v + g|u|^2 v + eu^2 v^* = 0, \tag{2}$$

$x \in \mathbb{R}$, $\xi \geq 0$, which is a model to describe one-dimensional light propagation through a linearly birefringent lossless optical fiber, taking into account the Kerr effect. Here $u(\xi, x), v(\xi, x)$ are the normalized complex slowly-varying envelopes of the two polarized eigenmodes, ξ is the normalized distance, x is the normalized time, the coefficients K, a,c,g, e, σ_1, σ_2 are positive constants, and u^* denotes the complex conjugate of the function u. The constant K is the so-called dispersion coefficient, $\sigma_1 - \sigma_2$ is the wavenumber difference, the coefficients a, c describe the self-modulation of the eigenmodes, and g, e are the coupling parameters of the cross-modulation between the two wave packets. The model's coefficients depend on the wavenumber of the carrier wave, the modal structure, and the birefringence effect inside the optical fiber. The derivation of system (1)-(2) in the field of optics can be found in the works by Menyuk et al. [18], [19], and Agrawal [2]. It is worth mentioning that system (1)-(2) with variable coefficients and $e = 0$ also arises in different physical scenarios, such as for instance, in wave propagation in two-component Bose-Einstein condensates with spatially inhomogeneous interactions, which has been a field of intense research activity in Physics in the last few years [24, 29, 23, 1, 9, 30, 27, 7, 22]. We refer the readers to the works [8, 4, 12, 13, 33, 34] for derivation and further applications of the CNLS system. In the case that $K = 1$, $e = 0$, $\sigma_1 = \sigma_2 = 0$, the CNLS system reduces to the celebrated Manakov system introduced originally in [16].

In this paper we study the existence of periodic standing waves (u, v) of system (1)-(2) in the form

$$u(\xi, x) = e^{i\alpha\xi}\ \widetilde{u}(x), \quad v(\xi, x) = e^{i\alpha\xi}\ \widetilde{v}(x), \tag{3}$$

where $\widetilde{u}, \widetilde{v}$ are periodic real functions and α is a real constant. It is important to point out that exact solutions to system (1)-(2) have been obtained only in particular cases. For example, when $e = 0$ the work by Tan and Boyd [28] reviews some explicit periodic cnoidal and dnoidal solutions to system (1)-(2), and the existence of periodic traveling-wave solutions by using a topological approach was recently studied by Nguyen [21]. Furthermore, some exact periodic stationary solutions to system (1)-(2) with non-trivial phase, $e = 0$ and the coefficients $\sigma_1 = \sigma_1(x), \sigma_2 = \sigma_2(x)$ taking the form of the square of the Jacobi sine function, have been computed by Deconinck et al. in [6]. A family of periodic and quasi-periodic traveling wave solutions of the CNLS system with $e = 0$ was also computed in [5] (see also references therein). The integrability of the Manakov system was proved in [16], but only for the case $a = g = c$. Explicit single-phase bounded elliptic solutions of the Manakov system were obtained in [14] in terms of the Weierstrass sigma function with a real quasiperiod. Furthermore, the Manakov system has been studied extensively in the literature [18, 17, 32, 15, 25, 31, 35, 26] and [2] (chapter 6) and references therein. However, we point out that the full system (1)-(2) is not integrable for arbitrary values of model's parameters and initial conditions, and most solutions can be computed only by using numerical methods.

In this paper our first goal is to generalize the previous results by establishing analytically existence of periodic standing waves in the form (3) of the full system (1)-(2), considering the extra cross-mode nonlinear terms preceded by the coefficient e. It is important to note that these nonlinear terms have been neglected in several previous works on propagation of light beams along optical fibers, where the birefringence effect in the fiber is assumed to be high. However, as pointed by Menyuk [18], these terms could play an important role in a fiber with very low birefringence. This fact is the physical motivation to include these extra terms in the CNLS system. Recently, Muñoz and Quiceno [11] also illustrated by using numerical simulations the effect of these cross-mode terms on the stability/instability mechanism of periodic plane wave solutions of the system. Unlike previous works where topological, inverse scattering transform or quadrature techniques have been used, we apply the Lyapunov Center Theorem [20] to demonstrate analytically existence of periodic solutions to system (1)-(2) in the form (3), taking advantage that the corresponding standing wave equations can be rewritten as a Hamiltonian system.

For our second objective, we compute numerically some solutions to system (1)-(2) in the form (3) by using a Newton's iteration, combined with a collocation-spectral strategy to discretize the corresponding standing wave equations. This strategy allows us to compute very accurate approximations to periodic standing waves of the system for a variety of model's coefficients. To the best knowledge of the authors, such numerical approach has not been performed in previous works on the full CNLS system and it is a contribution from a numerical point of view of the present paper.

The rest of this paper is organized as follows. In section 2, we present the main theoretical results employed in the paper to study existence of periodic solutions of some Hamiltonian systems. In sections 3 and 4, we establish our analytical results on regard to existence of periodic standing wave solutions in the form (3) of two coupled nonlinear systems, respectively: the system (1)-(2) and a CNLS system with generalized nonlinear terms, by using the Lyapunov center theorem. In section 5, we introduce a numerical Newton-type procedure combined with a Fourier-collocation strategy to approximate periodic standing wave solutions of the CNLS system for a variety of parameter's regimes covered by our analytical results. Finally, section 6 contains the conclusions of our work.

2. Mathematical preliminaries

In this section, we will study existence of periodic solutions to system (1)-(2) in the form

$$u(\xi, x) = e^{i\alpha\xi}\tilde{u}(x), \qquad v(\xi, x) = e^{i\alpha\xi}\tilde{v}(x),$$

where $\tilde{u}$, $\tilde{v}$ are periodic real functions. Therefore the pair $(\tilde{u}, \tilde{v})$ must satisfy the system

$$\begin{aligned} -K\tilde{u}'' + (\alpha - \sigma_1)\tilde{u} &= a\tilde{u}^3 + b\tilde{v}^2\tilde{u}, \\ -K\tilde{v}'' + (\alpha - \sigma_2)\tilde{v} &= c\tilde{v}^3 + b\tilde{u}^2\tilde{v}, \end{aligned} \tag{4}$$

where $b = g + e$. In the present work, the theoretical strategy used to establish existence of periodic solutions to this set of equations is the Lyapunov Center Theorem, which can be obtained as an application of the Hopf Bifurcation Theorem. Here α, σ_1, σ_2, K, a, b and c are positive real constants with $\alpha \neq \sigma_1$ and $\alpha \neq \sigma_2$.

Theorem 2.1 (Lyapunov's Center Theorem)**.** ([20]) *Consider the system*

$$x' = Ax + f(x), \tag{5}$$

where f is a smooth function which vanishes along with its first partial derivatives at the origin $x = 0$. Assume that system (5) admits a first integral of the form

$$\mathscr{H} = \frac{1}{2}(x, \mathscr{S}x) + g(x),$$

where $\mathscr{S}$ is a $n \times n$ real symmetric matrix with $\det \mathscr{S} \neq 0$. Let the matrix A have eigenvalues $\pm i\lambda_1$, $\lambda_3, \ldots, \lambda_n$, with $\lambda_1 \neq 0$. If $\frac{\lambda_j}{i\lambda_1} \notin \mathbb{Z}$, for $j = 3, 4, \ldots, n$, (this is called the non-resonance condition) then system (5) has a one parameter family of periodic solutions emanating from the origin with period $\frac{2\pi}{\lambda_1}$.

For simpleness, we abandon the tildes in system (4). Observe that with the change of variables $u' = w$ and $v' = z$, we obtain the system

$$\begin{aligned} u' &= w, \\ v' &= z, \\ Kw' &= (\alpha - \sigma_1)u - (au^3 + bv^2u), \\ Kz' &= (\alpha - \sigma_2)v - (cv^3 + bu^2v). \end{aligned} \tag{6}$$

Note that system (6) is a Hamiltonian system, i.e., it can be rewritten in the form

$$U' = \mathscr{J}\nabla\mathscr{H}(U),$$

with $U = (u, v, w, z)$, $\mathscr{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $\mathscr{H}$ is the Hamiltonian defined as

$$\begin{aligned}\mathscr{H}(U) &= \frac{1}{2K}\left(\frac{a}{2}u^4 + \frac{c}{2}v^4 + bu^2v^2 + K(z^2 + w^2) + (\sigma_1 - \alpha)u^2 + (\sigma_2 - \alpha)v^2\right) \\ &= \frac{1}{2}(U, \mathscr{S}U) + g(U),\end{aligned}$$

where

$$\mathscr{S} := \begin{pmatrix} \frac{\sigma_1-\alpha}{K} & 0 & 0 & 0 \\ 0 & \frac{\sigma_2-\alpha}{K} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g(U) := \frac{1}{2K}\left(\frac{a}{2}u^4 + \frac{c}{2}v^4 + bu^2v^2\right).$$

Observe that $\det \mathscr{S} \neq 0$. A direct calculation gives us the equilibrium points of system (6)

$$P_0 = (0,0,0,0),\ P_{1_\pm} = \pm\left(\left(\frac{\alpha - \sigma_1}{a}\right)^{1/2}, 0, 0, 0\right),\ P_{2_\pm} = \pm\left(0, \left(\frac{\alpha - \sigma_2}{c}\right)^{1/2}, 0, 0\right),$$

$$P_{3_\pm} = \pm\left(\gamma^{1/2}, \varphi^{1/2}, 0, 0\right) \quad \text{and} \quad P_{4_\pm} = \pm\left(\gamma^{1/2}, -\varphi^{1/2}, 0, 0\right),$$

where

$$\gamma = \frac{1}{ac - b^2}(c(\alpha - \sigma_1) - b(\alpha - \sigma_2)) \quad \text{and} \quad \varphi = \frac{1}{ac - b^2}(-b(\alpha - \sigma_1) + a(\alpha - \sigma_2)). \tag{7}$$

Finally, let us denote

$$\Gamma = \frac{\alpha - \sigma_1}{\alpha - \sigma_2}.$$

3. Existence of periodic solutions of the CNLS system

In order to applicate the Lyapunov Center Theorem 2.1 to the Hamiltonian system (6), we require the analysis of the eigenvalues of the corresponding Jacobian matrices evaluated at the equilibrium points P_0, $P_{1\pm}$, $P_{2\pm}$, $P_{3\pm}$ and $P_{4\pm}$.

3.1. Spectral Analysis for P_0, $P_{1_\pm}$ and $P_{2_\pm}$.

In the first place, observe that the Jacobian matrix A of the vector field of system (6) evaluated at the equilibrium points P_0, $P_{1_\pm}$ and $P_{2_\pm}$ has the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{pmatrix}, \tag{8}$$

where

$$\eta = \begin{cases} \frac{1}{K}(\alpha - \sigma_1) & \text{for the point } P_0, \\ \frac{-2}{K}(\alpha - \sigma_1) & \text{for the points } P_{1_\pm}, \\ \frac{1}{K}\left((\alpha - \sigma_1) - \frac{b}{c}(\alpha - \sigma_2)\right) & \text{for the points } P_{2_\pm} \end{cases}$$

and

$$\kappa = \begin{cases} \frac{1}{K}(\alpha - \sigma_2) & \text{for the point } P_0, \\ \frac{1}{K}\left((\alpha - \sigma_2) - \frac{b}{a}(\alpha - \sigma_1)\right) & \text{for the points } P_{1_\pm}, \\ \frac{-2}{K}(\alpha - \sigma_2) & \text{for the points } P_{2_\pm}. \end{cases}$$

Therefore, the characteristic equation of the Jacobian matrix A is

$$\lambda^4 - (\eta + \kappa)\lambda^2 + \eta\kappa = 0, \tag{9}$$

and the characteristic exponents of the equilibrium points are

$$\lambda_1 = \sqrt{\eta} = -\lambda_2, \quad \text{and} \quad \lambda_3 = \sqrt{\kappa} = -\lambda_4. \tag{10}$$

In case of the points P_0 and $P_{1_\pm}$ note that, if $\eta < 0$ and $\kappa > 0$, then $\lambda_1, \lambda_2 \in i\mathbb{R}$, $\lambda_3, \lambda_4 \in \mathbb{R}$ and $\frac{\lambda_j}{\lambda_1} \notin \mathbb{Z}$ for $j = 3, 4$. Similarly for the points P_0 and $P_{2_\pm}$, we have that $\eta > 0$ and $\kappa < 0$ imply that $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_3, \lambda_4 \in i\mathbb{R}$ and $\frac{\lambda_j}{\lambda_3} \notin \mathbb{Z}$ for $j = 1, 2$. Therefore, we have verified the hypotheses of Lyapunov's Center Theorem for system (6). More concretely,

Theorem 3.1. *Assume that $\alpha < \sigma_1$ and $\alpha > \sigma_2$ or $\alpha > \sigma_1$ and $\alpha < \sigma_2$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from the point P_0 with period L_0 defined as*

$$L_0^2 = \frac{4\pi^2 K}{\sigma_1 - \alpha} \qquad \text{or} \qquad L_0^2 = \frac{4\pi^2 K}{\sigma_2 - \alpha},$$

respectively.

Theorem 3.2. *Assume that $\alpha > \sigma_1$ and $(a - b)\alpha > a\sigma_2 - b\sigma_1$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from each point of $P_{1_\pm}$ with period L_0 defined as*

$$L_0^2 = \frac{2\pi^2 K}{\alpha - \sigma_1}.$$

Theorem 3.3. *Assume that $(c - b)\alpha > c\sigma_1 - b\sigma_2$ and $\alpha > \sigma_2$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from each point of $P_{2_\pm}$ with period L_0 defined as*

$$L_0^2 = \frac{2\pi^2 K}{\alpha - \sigma_2}.$$

Furthermore for all equilibrium points, in case of $\eta, \kappa < 0$, we have $\lambda_j \in i\mathbb{R}$ for $j = 1, 2, 3, 4$. If $\frac{\lambda_1}{\lambda_3} \neq 1$, then $\frac{\lambda_1}{\lambda_3}$ or $\frac{\lambda_3}{\lambda_1}$ is not an integer. It follows that the Lyapunov Center Theorem can be applied. With this in mind, we have established the following theorems:

Theorem 3.4. *Assume that $\alpha < \sigma_1$, $\alpha < \sigma_2$ and $\Gamma \neq 1$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from P_0 with period L_0 given by*

$$L_0^2 = \frac{4\pi^2 K}{\sigma_2 - \alpha}, \qquad \text{or} \qquad L_0^2 = \frac{4\pi^2 K}{\sigma_1 - \alpha},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ or $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$, respectively.

Theorem 3.5. *Suppose that $\alpha > \sigma_1$, $(a-b)\alpha < a\sigma_2 - b\sigma_1$ (particularly $\alpha < \sigma_2$) and $\Gamma \neq \frac{a}{b-2a}$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from each point of $P_{1_\pm}$ with period L_0 given by*

$$L_0^2 = \frac{4a\pi^2 K}{b(\alpha - \sigma_1) - a(\alpha - \sigma_2)} \quad \text{or} \quad L_0^2 = \frac{2\pi^2 K}{\alpha - \sigma_1},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ or $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$, respectively.

Theorem 3.6. *Suppose that $(c-b)\alpha < c\sigma_1 - b\sigma_2$ (particularly $\alpha < \sigma_1$), $\alpha > \sigma_2$ and $\Gamma \neq \frac{b-2c}{c}$. Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from each point of $P_{2_\pm}$ with period L_0 given by*

$$L_0^2 = \frac{4c\pi^2 K}{b(\alpha - \sigma_2) - c(\alpha - \sigma_1)} \quad \text{or} \quad L_0^2 = \frac{2\pi^2 K}{\alpha - \sigma_2},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ or $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$, respectively.

3.2. Spectral Analysis for $P_{3_\pm}$ and $P_{4_\pm}$.

Now, our focus is on the analysis of the equilibrium points $P_{3_\pm}$ and $P_{4_\pm}$. The matrix A corresponding to the Jacobian of the vector field of system (6) has the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta & \mp\tau & 0 & 0 \\ \mp\tau & \kappa & 0 & 0 \end{pmatrix},$$

around $P_{3_\pm}$ and $P_{4_\pm}$, respectively, with

$$\eta = \frac{-2a}{K}\gamma, \qquad \kappa = \frac{-2c}{K}\varphi \quad \text{and} \quad \tau = -\frac{2b}{K}(\gamma\varphi)^{1/2}, \tag{11}$$

where γ and φ are as in (7) and observe that $\tau^2 = \frac{b^2}{ac}\eta\kappa$. A direct calculation shows us that the eigenvalues λ of A must satisfy the equation

$$\lambda^4 - (\eta+\kappa)\lambda^2 + (\eta\kappa - \tau^2) = 0,$$

whose roots are given by

$$\lambda_1 = \frac{\sqrt{2}}{2}\left(\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa\right)^{1/2} = -\lambda_2,$$

$$\lambda_3 = \frac{\sqrt{2}}{2}\left(-\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa\right)^{1/2} = -\lambda_4.$$

We remark that $\eta > 0$ or $\kappa > 0$ imply that $\gamma < 0$ or $\varphi < 0$ and therefore the points $P_{3_\pm}$ and $P_{4_\pm}$ do not exist in $\mathbb{R}^2$. For this reason, we consider the case $\eta < 0$ and $\kappa < 0$. Observe that $\eta, \kappa < 0$ and $ac - b^2 < 0$ imply that the quantities

$$\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa \quad \text{and} \quad -\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa,$$

are positive and negative, respectively. It follows that $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_3, \lambda_4 \in i\mathbb{R}$ and $\frac{\lambda_j}{\lambda_3} \notin \mathbb{Z}$ for $j = 1, 2$. Moreover, observe that if the inequality

$$\frac{a}{b} < \frac{\alpha - \sigma_1}{\alpha - \sigma_2} < \frac{b}{c}$$

holds, then $\eta, \kappa < 0$ and $ac - b^2 < 0$. Similarly $\eta, \kappa < 0$ and $ac - b^2 > 0$ imply that

$$\pm\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa < 0.$$

It follows that $\lambda_j \in i\mathbb{R}$ for $i = 1, 2, 3, 4$. Furthermore we have that

$$0 < \frac{\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa}{-\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} + \eta + \kappa} < 1.$$

In consequence $\frac{\lambda_j}{\lambda_3} \notin \mathbb{Z}$ for $i = 1, 2$. Note that $\alpha > \sigma_1$, $\alpha > \sigma_2$ and

$$\frac{b}{c} < \frac{\alpha - \sigma_1}{\alpha - \sigma_2} < \frac{a}{b},$$

imply that $\eta\kappa < 0$ and $ac - b^2 > 0$. Again, we have verified the hypotheses of Lyapunov's Center Theorem for system (6). Therefore, we can state the following theorem:

Theorem 3.7. *Suppose that the positive constants a, b and c are such that $\alpha > \sigma_1$, $\alpha > \sigma_2$ and*

$$\frac{a}{b} < \Gamma < \frac{b}{c} \qquad \text{or} \qquad \frac{b}{c} < \Gamma < \frac{a}{b}.$$

Then system (6) has a one parameter family of periodic solutions (u, v, w, z) emanating from each point of $P_{3_\pm}$ and $P_{4_\pm}$ with period L_0 defined as

$$L_0^2 = \frac{8\pi^2}{\left((\eta-\kappa)^2 + \frac{4b^2}{ac}\eta\kappa\right)^{1/2} - \eta - \kappa},$$

where η and κ are as in (11).

3.3. Explicit periodic solutions in some particular cases of the CNLS system and some comments on orbital stability

We point out that nontrivial solutions (i.e. when $\tilde{u}$ or $\tilde{v}$ is not constant) of the form (3) with $\tilde{u} = \tilde{v}$ to system (1)-(2) exist only when $a = c$, $\sigma_1 = \sigma_2 = \sigma$. In fact, in that case the standing wave equations (4) reduce to a unique scalar equation

$$\tilde{u}'' + \frac{\sigma - \alpha}{K}\tilde{u} + \frac{a+b}{K}\tilde{u}^3 = 0. \tag{12}$$

Observe that using the change of variables $\eta = \sqrt{\frac{a+b}{K}}\xi$, $\tilde{u}(\xi) = U(\eta)$, we get that $U(\eta)$ satisfies the equation

$$U'' - wU + U^3 = 0, \tag{13}$$

where $w = \frac{\alpha-\sigma}{a+b}$, and the second derivative U'' is taken with respect to the variable η. In [3] is constructed a family of exact periodic solutions (of dnoidal type) to equation (13). We refer the interested reader to [3] for details. Note that if U is a solution to equation (13), then $u(\xi, x) = e^{iw\xi}U(x)$ is a standing wave solution of the cubic Schrödinger equation

$$i\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0. \tag{14}$$

In consequence, since equation (12) admits exact periodic solutions, we obtain exact periodic solutions corresponding to the CNLS system (1)-(2) in the particular form $u(\xi, x) = v(\xi, x) = e^{i\alpha\xi}\tilde{u}(\xi)$.

A very important concept connected to traveling wave solutions of a dispersive system, such as the Schrödinger equation, is called *orbital stability*. In a few words, we say that the solution

$$u(\xi, x) = e^{iw\xi}U(x) \tag{15}$$

to equation (14) is orbitally stable if any solution with the initial data at $\xi = 0$ sufficiently close to $U(x)$, forever remains (modulo phase and translation symmetries) in a given small neighborhood of the trajectory of $e^{iw\xi}U(x)$. A complete analysis of orbital stability of solutions of the scalar Schrödinger equation (14) was developed by Angulo in [3]. He proved that the standing waves in the form (15) are nonlinearly stable in the energy space $H^1_{per}([0,L])$ with regard to the L-periodic flow of the Schrödinger equation (14), and unstable by perturbations with period $2L$. However, we found that there are still some technical details in order to adapt the methodology used in [3] to the study of orbital stability of periodic solutions to the CNLS system (1)-(2), even for the simpler case $\tilde{u} = \tilde{v}$, $\sigma_1 = \sigma_2$, $a = c$, mentioned above, due to the structure of the conserved quantities available for this system. We are working on this very interesting problem and we expect to present these results in a future paper.

4. Periodic standing wave solutions for a generalized CNLS system

In this section, we will establish existence of periodic solutions of the form (3) for the generalized CNLS system

$$i\frac{\partial u}{\partial \xi} + K\frac{\partial^2 u}{\partial x^2} + \sigma_1 u + a|u|^{2p}u + gv(v^*)^{2p-1}u^{2p-1} + ev^{2p}(u^*)^{2p-1} = 0, \quad (16)$$

$$i\frac{\partial v}{\partial \xi} + K\frac{\partial^2 v}{\partial x^2} + \sigma_2 v + c|v|^{2p}v + gu(u^*)^{2p-1}v^{2p-1} + eu^{2p}(v^*)^{2p-1} = 0, \quad (17)$$

where $x \in \mathbb{R}$, $\xi \geq 0$, $p \geq 1$ is a natural number, the constants K, a, c, g, e, σ_1 and σ_2 are as in system (1)-(2), and the notation u^* denotes the complex conjugate of the function u. We remark that the system above reduces to system (1)-(2) in the case that $p = 1$.

By replacing (3) into system (16)-(17) and ignoring the tildes, we get the system

$$\begin{aligned} -Ku'' + (\alpha - \sigma_1)u &= au^{2p+1} + bv^{2p}u^{2p-1}, \\ -Kv'' + (\alpha - \sigma_2)v &= cv^{2p+1} + bu^{2p}v^{2p-1}, \end{aligned}$$

where $b = g + e$. Observe that by using the change of variables $u' = w$ and $v' = z$, we get the Hamiltonian system

$$\begin{aligned} u' &= w, \\ v' &= z, \\ Kw' &= (\alpha - \sigma_1)u - (au^{2p+1} + bv^{2p}u^{2p-1}), \\ Kz' &= (\alpha - \sigma_2)v - (cv^{2p+1} + bu^{2p}v^{2p-1}), \end{aligned} \quad (18)$$

where the Hamiltonian is defined as

$$\mathscr{H}(U) = \frac{1}{2K}\left(\frac{1}{p+1}\left(au^{2p+2}+cv^{2p+2}\right)+\frac{b}{p}u^{2p}v^{2p}+K(z^2+w^2)\right.$$
$$\left.+(\sigma_1-\alpha)u^2+(\sigma_2-\alpha)v^2\right).$$

A simple calculation gives us the following equilibrium points:

$$P_0=(0,0,0,0),\ P_{1_\pm}{=}\pm\left(\left(\frac{\alpha-\sigma_1}{a}\right)^{1/2p},0,0,0\right),\ P_{2_\pm}{=}\pm\left(0,\left(\frac{\alpha-\sigma_2}{c}\right)^{1/2p},0,0\right).$$

In order to show existence of periodic solutions to system (16)-(17), we need again to perform a spectral analysis for the system's equilibrium states. The Jacobian matrix A of the vector field of system (18) around the equilibrium points P_0, $P_{1\pm}$ and $P_{2\pm}$ has the form (8), with

$$\eta = \begin{cases} \frac{1}{K}(\alpha-\sigma_1) & \text{for the point } P_0, \\ \frac{-2p}{K}(\alpha-\sigma_1) & \text{for the points } P_{1_\pm}, \\ \frac{1}{K}(\alpha-\sigma_1) & \text{for the points } P_{2_\pm} \end{cases}$$

and

$$\kappa = \begin{cases} \frac{1}{K}(\alpha-\sigma_2) & \text{for the point } P_0, \\ \frac{1}{K}(\alpha-\sigma_2) & \text{for the points } P_{1_\pm}, \\ \frac{-2p}{K}(\alpha-\sigma_2) & \text{for the points } P_{2_\pm}. \end{cases}$$

The characteristic equation is the same given in (9) and the characteristic exponents $\lambda_i, i=1,2,3,4$ are as in (10). For the equilibrium point P_0, a similar analysis as in the previous section allows us to conclude that theorems 3.1 and 3.4 are still valid for system (18). In case of the points $P_{1_\pm}$ and $P_{2_\pm}$, we can state the following results:

Theorem 4.1. *Suppose that $\alpha>\sigma_1$ and $\alpha>\sigma_2$. Then system (18) has a one parameter family of periodic solutions (u,v,w,z) emanating from each point of $P_{1_\pm}$ with period L_0 defined as*

$$L_0^2=\frac{2\pi^2K}{p(\alpha-\sigma_1)}.$$

Theorem 4.2. *Let be $\alpha>\sigma_1$, $\alpha<\sigma_2$ and $\Gamma\neq\frac{-1}{2p}$ (wich implies that $\frac{\lambda_1}{\lambda_3}$ or $\frac{\lambda_3}{\lambda_1}$ is not an integer). Then system (18) has a one parameter family of periodic solutions (u,v,w,z) emanating from each point of $P_{1_\pm}$ with period L_0 given by*

$$L_0^2=\frac{4\pi^2K}{\sigma_2-\alpha},\quad or\quad L_0^2=\frac{2\pi^2K}{p(\alpha-\sigma_1)},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ *or* $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$*, respectively.*

Theorem 4.3. *Assume that* $\alpha > \sigma_1$ *and* $\alpha > \sigma_2$*. Then system (18) has a one parameter family of periodic solutions* (u, v, w, z) *emanating from each point of* $P_{2_\pm}$ *with period* L_0 *defined as*

$$L_0^2 = \frac{2\pi^2 K}{p(\alpha - \sigma_2)}.$$

Theorem 4.4. *Suppose that* $\alpha < \sigma_1$*,* $\alpha > \sigma_2$ *and* $\Gamma \neq -2p$ *(wich implies that* $\frac{\lambda_1}{\lambda_3}$ *or* $\frac{\lambda_3}{\lambda_1}$ *is not an integer). Then system (18) has a one parameter family of periodic solutions* (u, v, w, z) *emanating from each point of* $P_{2_\pm}$ *with period* L_0 *given by*

$$L_0^2 = \frac{4\pi^2 K}{\sigma_1 - \alpha}, \quad \text{or} \quad L_0^2 = \frac{2\pi^2 K}{p(\alpha - \sigma_2)},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ *or* $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$*, respectively.*

We point out that in case that $p > 1$, the system (18) has additional equilibrium points of the form $(u, v, 0, 0)$, where u, v satisfy the equations

$$\begin{aligned} au^{2p} + bv^{2p}u^{2p-2} &= (\alpha - \sigma_1), \\ cv^{2p} + bu^{2p}v^{2p-2} &= (\alpha - \sigma_2). \end{aligned}$$

Unfortunately, no explicit solutions of the system above are available for arbitrary values of the constants a, g, c, e and the exponent p. Therefore, here we only consider the particular case $b = g + e = 0$ (in which case, the theorems 4.1, 4.2, 4.3 and 4.4 are still valid), and obtain the following four equilibrium points:

$$P_{4\pm} = \left(\pm \left(\frac{\alpha - \sigma_1}{a} \right)^{1/2p}, \pm \left(\frac{\alpha - \sigma_2}{c} \right)^{1/2p}, 0, 0 \right).$$

For these equilibrium points, the corresponding Jacobian matrices have the form of (8) with

$$\eta = \frac{-2p}{K}(\alpha - \sigma_1), \qquad \kappa = \frac{-2p}{K}(\alpha - \sigma_2)$$

and the characteristic exponents are as in (10). In this case, we obtain the following existence theorem of periodic solutions to system (18):

Theorem 4.5. *Assume that that* $\alpha > \sigma_1$*,* $\alpha > \sigma_2$*,* $p > 1$*, and* $\Gamma \neq 1$*. Then system (18) with the parameter* $b = g + e = 0$*, has a one parameter family of periodic solutions* (u, v, w, z) *emanating from each point of* $P_{4\pm}$ *with period* L_0 *given by*

$$L_0^2 = \frac{2\pi^2 K}{p(\alpha - \sigma_2)} \quad \text{or} \quad L_0^2 = \frac{2\pi^2 K}{p(\alpha - \sigma_1)},$$

depending on whether $\frac{\lambda_1}{\lambda_3} \notin \mathbb{Z}$ *or* $\frac{\lambda_3}{\lambda_1} \notin \mathbb{Z}$*, respectively.*

5. Numerical results

In this section, we compute approximations of some periodic standing wave solutions to system (1)-(2), whose existence was established in the previous section. As showed above, this is reduced to compute periodic solutions $(\tilde{u}, \tilde{v})$ to system (4). To do this, we use a Newton's procedure combined with a collocation-spectral technique introduced by the author in [10] for approximating periodic solutions of a weakly-dispersive, weakly nonlinear Boussinesq system, related to the propagation of water waves on the surface of a shallow channel.

In the first place, we set $K = 1$, $\sigma_1 = 1.5$, $\sigma_2 = 0.5$, $\alpha = 1$, $a = 1$, $e = 2$, $g = 2$, $b = e + g = 4$, $c = 1$. Thus, the expected period of the standing wave is

$$L_0 = \sqrt{\frac{4\pi^2 K}{\sigma_1 - \alpha}} = 8.8858,$$

in accordance with Theorem 2.2. Therefore the computational domain for the variable x is the interval $[0, 2l]$, with $l = L_0/2$, and we set $N = 2^9$ FFT points in all computations with Newton's method. The starting profiles for the iterative procedure are

$$u_0(x) = \cos\left(\frac{4\pi x}{2l}\right), \quad v_0(x) = \cos\left(\frac{2\pi x}{2l}\right),$$

where $l = 4.4429$. The resulting profiles $(\tilde{u}, \tilde{v})$ after 10 iterations are displayed in Figure 1. In order to check that we have computed really a periodic standing wave solution to system (1)-(2), we run the numerical solver also introduced by the author in [10] for computing the evolution of the solution by using as initial values the profiles $(\tilde{u}, \tilde{v})$ given in Figure 1. The computational domain of the variable x is the interval $[0, L_0]$, with 2^7 FFT points, and the step size for the variable ξ is $\Delta\xi = 10^{-4}$. In Figures 2, 3, we compare the real and imaginary parts of the solution to system (1)-(2) computed with the numerical scheme in [10] at $\xi = 20$ (solid line) with the expected profiles (showed in dotted line)

$$u(\xi, x) = e^{i\alpha\xi}\widetilde{u}(x), \quad v(\xi, x) = e^{i\alpha\xi}\widetilde{v}(x).$$

We see that these profiles agree with good accuracy of approximately $2e-4$ in the supremum norm.

In second place, we set $K = 1$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\alpha = 1.5$, $a = 4$, $e = 1$, $g = 1$, $b = g + e = 2$, $c = 3$. The period in accordance with Theorem 2.3 is

$$L_0 = \sqrt{\frac{2\pi^2 K}{\alpha - \sigma_1}} = 6.2832.$$

The starting profiles for Newton's iteration are

$$u_0(x) = 0.5\cos\left(\frac{2\pi x}{2l}\right), \quad v_0(x) = \cos\left(\frac{4\pi x}{2l}\right),$$

where $l = 3.1416$. The results after 15 iterations are displayed in Figure 4.

Next, we set $K = 1$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $\alpha = 3$, $a = 5$, $e = 1$, $g = 1$, $b = g + e = 2$, $c = 4$. The period of the periodic wave obtained in accordance with Theorem 2.4 is given by

$$L_0 = \sqrt{\frac{2\pi^2 K}{\alpha - \sigma_2}} = 2.8099.$$

The starting profiles for Newton's iteration are

$$u_0(x) = 0.5\cos\left(\frac{2\pi x}{2l}\right), \quad v_0(x) = \cos\left(\frac{4\pi x}{2l}\right),$$

where $l = 1.4050$. The results after 9 iterations are displayed in Figure 5.

Finally, we set $K = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\alpha = 1$, $a = 2$, $e = 1$, $g = 1$, $b = g + e = 2$, $c = 0.1$. Thus $\Gamma = 1.1429$ and the period of the periodic wave obtained from Theorem 2.8 (case $a/b < \Gamma < b/c$) becomes

$$L_0 = \sqrt{\frac{8\pi^2}{((\eta - \kappa)^2 + \frac{4b^2}{ac}\eta\kappa)^{1/2} - \eta - \kappa}} = 3.5387.$$

The starting profiles for Newton's iteration are

$$u_0(x) = 2.5\cos\left(\frac{4\pi x}{2l}\right), \quad v_0(x) = 2.5\cos\left(\frac{2\pi x}{2l}\right),$$

where $l = 1.7694$. The results after 8 iterations are displayed in Figure 6.

In all numerical experiments performed, we corroborated the existence of an approximate standing wave solution whose period coincides with the one expected theoretically.

6. Conclusions

In this paper, we considered the existence of periodic standing wave solutions to the CNLS system (1)-(2), which is a model for several physical scenarios. In particular, it describes the propagation of a pulse along an optical fiber in the presence of nonlinearity (Kerr effect) and anomalous dispersion. We further consider a type of CNLS system with generalized nonlinear terms. In each case, the problem was reduced to that of finding periodic solutions of a system of two coupled second-order ordinary differential equations, which has a Hamiltonian structure. Then by using the Lyapunov Center Theorem, we established analytically the existence of periodic solutions to the original system (1)-(2) and for the generalized CNLS system considered for several parameter's regimes. Furthermore, in order to illustrate the geometric properties of such solutions, we computed some approximations by employing a Newton's iteration together with a collocation-spectral scheme for discretization of the variable x. All of the numerical results are in perfect agreement with the theory presented. In particular, the period of each numerical approximation coincides with the one expected theoretically. Thus, we found that the numerical scheme introduced in the paper is a valuable tool to compute further periodic standing wave solutions for a variety of parameter regimes of the CNLS system. An important problem we would wish to consider in a future work is the instability/stability under small initial disturbances of the standing wave solutions obtained theoretically and numerically in the present paper. This study could be initiated for instance by using the numerical schemes introduced here.

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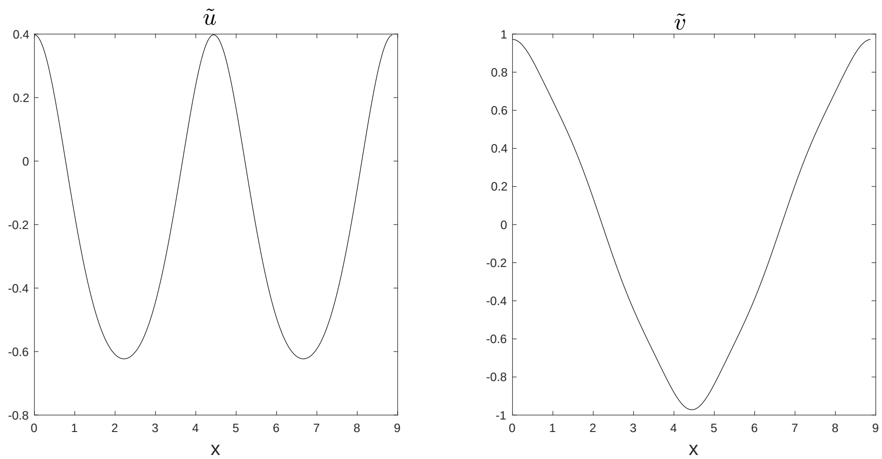

FIGURE 1. Approximation of functions $\widetilde{u}, \widetilde{v}$ in the periodic solution (3) to system (4).

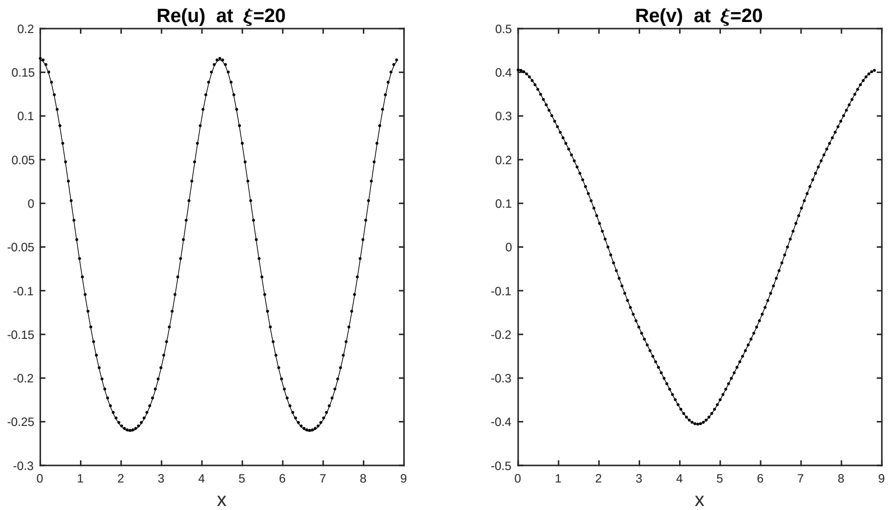

FIGURE 2. Checking the approximation of the periodic solution to system (1)-(2) given by $u(\xi, x) = e^{i\alpha\xi}\tilde{u}(x)$, $v(\xi, x) = e^{i\alpha\xi}\tilde{v}(x)$, where $\tilde{u}, \tilde{v}$ are the profiles displayed in Figure 1.

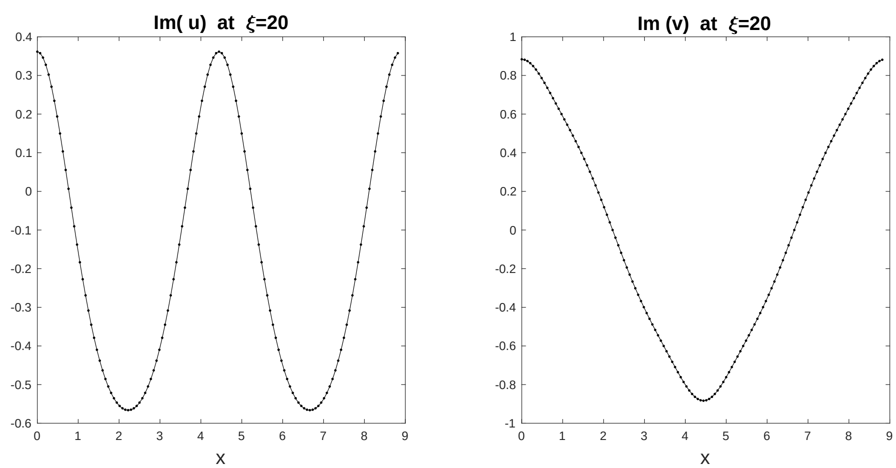

FIGURE 3. Checking the approximation of the periodic solution to system (1)-(2) given by $u(\xi, x) = e^{i\alpha\xi}\tilde{u}(x)$, $v(\xi, x) = e^{i\alpha\xi}\tilde{v}(x)$, where $\tilde{u}, \tilde{v}$ are the profiles displayed in Figure 1.

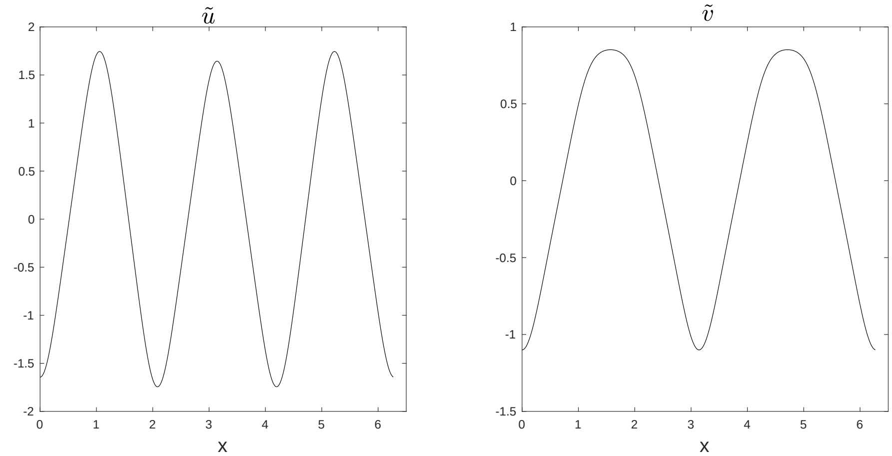

FIGURE 4. Approximation of functions $\widetilde{u}, \widetilde{v}$ in the periodic solution (3) to system (4).

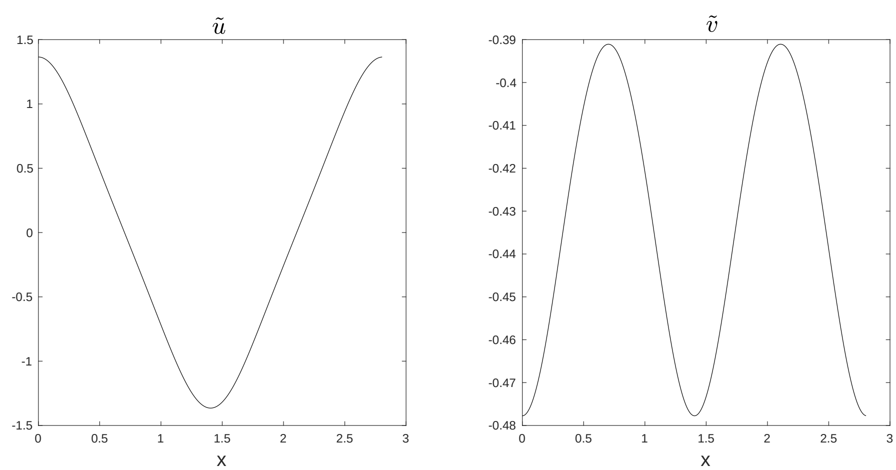

FIGURE 5. Approximation of functions $\widetilde{u}, \widetilde{v}$ in the periodic solution (3) to system (4).

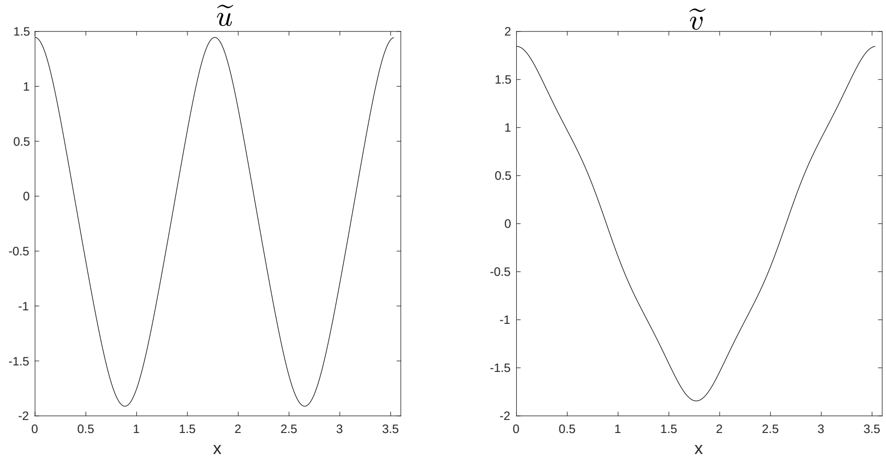

Figure 6. Approximation of functions $\widetilde{u}, \widetilde{v}$ in the periodic solution (3) to system (4).

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