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A short survey on observability

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Abstract. The exploration of the notion of observability exhibits transparently the rich interplay between algebraic and geometric ideas in *geometric invariant theory*. The concept of *observable subgroup* was introduced in the early 1960s with the purpose of studying extensions of representations from an affine algebraic subgroup to the whole group. The extent of its importance in *representation and invariant theory* in particular for Hilbert's $14th$ problem was noticed almost immediately. An important strenghtening appeared in the mid 1970s when the concept of *strong observability* was introduced and it was shown that the notion of observability can be understood as an intermediate step in the notion of reductivity (or semisimplicity), when adequately generalized. More recently starting in 2010, the concept of observable subgroup was expanded to include the concept of *observable action* of an affine algebraic group on an affine variety, launching a series of new applications and opening a surge of very interesting activity. In another direction around 2006, the related concept of *observable adjunction* was introduced, and its application to module categories over tensor categories was noticed. In the current survey, we follow (approximately) the historical development of the subject introducing along the way, the definitions and some of the main results including some of the proofs. For the unproven parts, precise references are mentioned.

Key words and phrases. observability, invariants, actions.

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Resumen. El estudio de la noción de observabilidad muestra de modo transparente la rica interacción entre las ideas algebraicas y geométricas en la *teoría geométrica de invariantes*. El concepto de *subgrupo observable* fue introducido al inicio de la década de 1960 con el propósito de estudiar las extensiones de representaciones desde un subgrupo algebraico afín a todo el grupo (también algebraico afín). La importancia de la noción de subgrupo observable en la teoría de representaciones y la teoría de invariantes, en particular para el estudio del 14to problema de

Hilbert, fue observada de inmediato. En la mitad de la década de 1970 apareció un refinamiento importante de esta noción: el concepto de *observabilidad fuerte* fue introducido y se mostró que la noción de observabilidad puede entenderse como un paso intermedio hacia la noción de reductividad (o semi-simplicidad), haciendo las generalizaciones adecuadas. Recientemente, al inicio de la década de 2010, el concepto de subgrupo observable fue expandido de modo de incluir la noción de *acción observable* de un grupo algebraico afín en una varidad algebraica afín. Esta generalización inició una serie de trabajos interesantes, con varias aplicaciones novedosas. En otra dirección, cerca de 2006 fue introducido el concepto de *adjunción observable*, que tuvo aplicación inmediata en el estudio de las módulo categorías sobre categorías tensoriales. En la revisión que sique, seguimos de modo aproximado el desarrollo histórico de esta temática, introduciendo a lo largo del camino las definiciones y los resultados centrales, junto con algunas de las pruebas. Para los resultados sin demostración, se mencionan referencias precisas.

Palabras y frases clave. observabilidad, invariantes, acciones.

1. Introduction

The concept of observable subgroup of an affine algebraic group G was introduced by A. Bialynicki–Birula, G. Hochschild and G.D. Mostow in 1963 in [2]: *Extension of representations of algebraic linear groups* (hereafter referred to as ERA).

Initially the notion of observability was related to the following situation.

Assume that $H \subseteq G$ is a pair of a subgroup and a group. We say that a representation (V, ρ) of G is an extension of a representation (U, σ) of H if: $U \subseteq V$ and the action $\rho : G \times V \to V$ restricts to $\sigma : H \times U \to U$[1].

The main question adressed by the authors of [2] concerns the following problem: in the case that H and G are affine algebraic groups, and the representations are finite dimensional and rational, does every representation of H admits an extension? In the situation that the answer is positive the group is said to be *observable*.

In the introduction of ERA the authors write:

[1]For the above question to make sense, the general definition has to be adapted to particular situations involving a basic field –where the representations are defined– and a precise description of the actions we are working with (i.e. maps such as ρ and σ above) that have to be adapted to the additional structure of the groups under consideration – analytic, differentiable, algebraic, etc.

> *Let G be an algebraic linear group over an arbitrary field F. If ρ is a rational representation of G by linear automorphisms of a finite–dimensional F–space U, we refer to this structure (U, ρ) by saying that U is a finite–dimensional rational G–module. A G–module that is a sum (not necessarily direct) of finite–dimensional rational G–modules is called a rational G–module. Let H be an algebraic subgroup of G. We are interested in determining when every finite–dimensional rational representation of H can be extended to a rational representation of G, i.e., when every finite–dimensional rational H–module can be imbedded as a H–submodule in a rational G–module.*

Notations and prerequisites. In this paper we assume that the reader is familiar with the basic results and notations of the theory of affine algebraic groups its actions and representations which appear –eventually with slight differences– in the initial chapters of the standard textbooks on the subject such as: A. Borel's [3], C. Chevalley's [4], G. Hochschild's [19], J. E. Humphrey's [23], T.A. Springer's [48] or the more recent monograph [46]. We work with groups and varieties defined over an algebraically closed field that will be denoted as $\Bbbk$.

If G is an affine algebraic group then the algebra $\Bbbk[G]$ of polynomial functions on G (with pointwise operations of sum and product) is in fact a Hopf algebra and its operations are defined as follows. The comultiplication $\Delta : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$ written as $\Delta(f) = \sum f_1 \otimes f_2$ –Sweedler's notation– is characterized by the fact that for all $x, y \in G$: $\sum f_1(x) f_2(y) = f(xy)$. The antipode $S : \Bbbk[G] \to \Bbbk[G]$ is defined for all $x \in G$ as $S(f)(x) = f(x^{-1})$ and the counit $\varepsilon : \Bbbk[G] \to \Bbbk$ is $\varepsilon(f) = f(e)$. In particular de left and right translations of f by an element $x \in G$ are $x \cdot f = \sum f_1 f_2(x)$; $f \cdot x = \sum f_1(x) f_2$.

A –not necessarily finite dimensional– rational (left) G–module M can be defined in terms of a (right) $\Bbbk[G]$–comodule structure $\chi_M : M \to M \otimes \Bbbk[G]$, and this structure map is written à la Sweedler as $\chi(m) = \sum m_0 \otimes m_1 \in M \otimes \Bbbk[G]$. It is related with the action of G on M by the formula ($x \in G$, $m \in M$): $x \cdot m = \sum m_0 m_1(x)$. The category of rational G–modules is denoted as ${}_G\mathcal{M}$, and by definition it coincides with the category of $\Bbbk[G]$–comodules. If $N \in {}_G\mathcal{M}$, we denote as ${}^G N := \{n \in N : x \cdot n = n \text{ for all } x \in G\}$ and it is clear that ${}^G N = \{n \in N : \chi(n) = n \otimes 1\}$ with χ the $\Bbbk[G]$–comodule structure on N. If M is a finite dimensional rational G–module and $m \in M$, $\alpha \in M^*$ we call $\alpha|m \in \Bbbk[G]$ the polynomial $\alpha|m = \sum \alpha(m_0) m_1$ or in explicit terms: $(\alpha|m)(x) = \alpha(x \cdot m)$ for $x \in G$. It is clear that $x \cdot (\alpha|m) = \alpha|(x \cdot m)$ for all $x \in G$. Also, in the case of a closed inclusion $H \subseteq G$ of affine algebraic groups, if $N \in {}_G\mathcal{M}$, $N|_H$ is the H–module obtained by result of the restriction of the G–action to an H–action. In this situation if the structure of $\Bbbk[G]$–comodule of N is $\chi(n) = \sum n_0 \otimes n_1 \in N \otimes \Bbbk[G]$, the structure of $N|_H$ as a $\Bbbk[H]$–comodule is $(\mathrm{id} \otimes \pi)\chi(n) = \sum n_0 \otimes \pi(n_1) \in N \otimes \Bbbk[H]$ where $\pi : \Bbbk[G] \to \Bbbk[H]$ is the restriction morphism.

Concerning some algebraic aspects: all algebras will be commutative –unless explicitly stated– and over a base field $\Bbbk$ that is algebraically closed. An algebra is affine if it is commutative, finitely generated and with no non–zero nilpotents.

It is well known that the category ${}_G\mathcal{M}$ for an affine algebraic group G is abelian, and has enough injectives. This guarantees that the basic machinery of homological algebra is available in the working platform of this survey. In particular, this category has the particularity that $\Bbbk[G] \in {}_G\mathcal{M}$ is an injective object and also that if $M \in {}_G\mathcal{M}$ is an arbitrary rational G–module, then $M \otimes \Bbbk[G]$ is injective. In this manner one has that the coaction map $\chi : M \to M \otimes \Bbbk[G]$ produces an imbedding of M in an injective object and this guarantees that the category has enough injectives.

Sometimes we deal with the categories of (R, G)–modules –denoted as ${}_{(R,G)}\mathcal{M}$, where R is a rational commutative G–module algebra. We say that M is an (R, G)–module, provided that it is a rational G–module, a module over the ring R and that the actions are related in the following manner if $x \in G$, $r \in R$, $m \in M$, $x \cdot (rm) = (x \cdot r)(x \cdot m)$. The morphisms are defined in the obvious way.

2. Antecedents, faithful representations of Lie groups

The concerns that led to the discovery of the concept of observability seem to derive from the persuit of the understanding and simplification of a series of results on the existence of faithful finite dimensional representations of Lie groups (due to E. Cartan, M. Goto, D. Ado, A. Malcev, K. Iwasawa, G. Hochschild and others).

Below we trace backwards the main steps of this process.

Previously to the results appearing in ERA, Hochschild and Mostow published in 1957/58 two important papers ([20, 28]) on the extension of representations of Lie groups that are cited explicitly in the aforementioned introduction of ERA:

> *In the analogous situation for Lie groups, an analysis of the the obstructions to the extendibility of representations of a subgroup has been made only for normal subgroups, [20, 28], and not much is known in the general case. The algebraic case turns out to be much more accessible.*

The differences between the algebraic case and the Lie group situation are remarkable and it is patent from the comparison between the results for Lie groups in [20, 28] and the situation of algebraic groups in [2].

For example, in the first mentioned papers and in a rather laborious way, the authors prove the following result.

Theorem 2.1. [28, Theorem 4.1] *Let $H \subseteq G$ be a closed normal inclusion in the category of (real or complex) analytic groups and denote as N the radical of the commutator subgroup G' of G. Assume that ρ is a finite dimensional representation of H and that ρ' is the semisimple representation associated to ρ. Then, ρ can be extended to G (with a finite dimensional extension) if and only if the following three conditions hold:*

(1) ρ' *is trivial in* $H \cap N$*;*

(2) The representation σ *of* HN *defined by* $\sigma(xu) = \rho'(x)$ *for* $x \in H$ *,* $u \in N$ *is continuous when* HN *is endowed with the topology induced by* G*;*

(3) Call G_f *the intersection of all the kernels of all the finite dimensional representations of* G*. Then* ρ *is trivial in* $G_f \cap H$*.*

The above theorem is the main result of [28], whereas in the first paper [20], a particular case is proved with additional topological conditions. It is interesting to compare it with the following very simple criterion for the extension of a representation in the case of affine algebraic groups without the hypothesis of normality (this subject will be treated in more detail and precision in Section 3.1).

First we need to introduce some definitions.

Definition 2.2. Let $H \subseteq G$ be a closed inclusion of affine algebraic groups.

(1) A character $\chi : H \to \Bbbk$ is said to be extendable to G if there is a polynomial function $f \in \Bbbk[G]$ such that $f(1) = 1$ and for all $x \in H$, $x \cdot f = \chi(x) f$ –or equivalently, for all $y \in G$, $f(yx) = f(y)\chi(x)$.

(2) If $M = (M, \cdot)$ is a rational H–module, and χ is a character of H, we call M_χ the rational H–module $(M, \cdot_\chi)$ where $\cdot_\chi$ is defined on M as $x \cdot_\chi m = \chi(x)(x \cdot m)$ for all $m \in M$. Clearly $M_\chi = M \otimes \Bbbk_\chi$ where $\Bbbk_\chi$ is the one dimensional H–module associated to the character χ.

Next theorem guarantees that for affine algebraic groups, every representation can be extended "up to the twist by an extendable character".

Theorem 2.3. [2, Theorem 1][46, Theorem 8.2.3] *If* $H \subseteq G$ *is a closed inclusion of affine algebraic groups for any rational finite dimensional* H*–module* M *there exists a finite dimensional rational* G*–module* N *and a character* $\chi : H \to \Bbbk$ *such that:*

(1) The character χ *is extendable;*

(2) $M_\chi \subseteq N|_H$*, where* $N|_H$ *denotes that we consider the action of* N *restricted to* H*.*

Moreover, in the case that M *is a simple* H*–module,* N *can be taken to be a simple* G*–module and even more particulary a simple* G*–submodule of* $\Bbbk[G]$*. Also given a pair* $0 \neq m \in M$ *and* $z \in G$ *there is such an injection* $M_\chi \subseteq \Bbbk[G]$ *such that* $m(z) \neq 0$*.*

Theorem 2.4. [46, Theorem 11.2.9] *In the situation above, if the character* χ^{-1} *is extendable then the finite dimensional* H *module* M *can be imbedded (as a* H

submodule) in a finite dimensional G–module N. In particular if for every extendable character χ, the character χ^{-1} is also extendable the subgrup H is observable in G. Moreover if $H \subseteq G$ is observable, then all characters of H are extendable to G.

Proof. Imbed first $M_\chi \subseteq N|_H$ and then consider the inclusion of H–modules $\Bbbk\chi^{-1} \to \Bbbk[G]$ that sends $\chi^{-1} \mapsto f$ where f is the polynomial guaranteeing the extendibility of χ^{-1}. Clearly the tensor products of the corresponding maps gives an inclusion of H–modules from $M := M_\chi \otimes \Bbbk\chi^{-1} \to (N \otimes \Bbbk[G])|_H$. As the image of M inside of $N \otimes \Bbbk[G]$ lies in $N \otimes \Bbbk f$, that is finite dimensional rational G–module, the proof of the first assertion is finished. The second assertion follows directly from the first. It only remains to prove that if χ is an arbitrary character of an observable H, then χ is exendible. Given χ, an arbitrary character of H, we can find a finite dimensional G–module N and an H–inclusion of $\Bbbk_\chi \to N$. If we call $n \in N$ the image of χ, we have that for all $x \in H$, $x \cdot n = \chi(x)n$. If $\alpha \in N^\vee$ is a linear functional such that $\alpha(n) = 1$ and take the polynomial $\alpha|n \in \Bbbk[G]$ (recall that $(\alpha|n)(y) = \alpha(y \cdot n)$ for all $y \in G$). It is clear that if $x \in H$ then $(x \cdot (\alpha|n))(y) = (\alpha|n)(yx) = \alpha((yx) \cdot n) = \alpha(y \cdot (x \cdot n)) = \alpha(y \cdot (\chi(x)n)) = \chi(x)\alpha(y \cdot n) = \chi(x)(\alpha|n)(y)$. Moreover, $(\alpha|n)(1) = \alpha(n) = 1$. ☑

It seems that the main motivation of the authors of [20, 28] to study the extension of representations from normal Lie subgroups to the whole group, was the search for the simplification and unification of some of the proofs of the standard results on faithful representations of Lie groups. In this respect, in the introduction to [28] and after describing the main results of [20] the author writes:

> *From the extension [results of [20]...] one deduces quickly all the standard results on faithful representations of Lie groups.*

Indeed, in [20, Section 3], short new proofs of the following three classical and important theorems are presented: E. Cartan's theorem on the existence of a faithful representation of a simply connected solvable Lie group, that is unipotent in a maximal normal nilpotent subgroup; Goto's theorem on the existence of a faithful represention of a connected Lie group G provided we know the existence of a representation for a maximal semisimple subgroup together with additional topological conditions on the radical of the commutator subgroup of G, and Malcev theorem that guarantees the existence of a faithful representation of a connected Lie group once we know that such a representation exists for the radical of G and for a maximal semisimple analytic subgroup of G.

3. Observability and geometry, observability and invariant theory

3.1. Observability and geometry

One of the more interesting results of ERA is the discovery of the relationship between the extension of the representations from H to G and the geometric structure of the homogeneous space G/H.

It is substantially harder to study homogeneous spaces in the category of algebraic groups than for example in the closely related category of Lie groups. The basic general results concerning the existence of a natural structure of algebraic variety on G/H are due to M. Rosenlicht and A. Weil in the mid 1950s (see [42] and [51]). The proof that G/H is quasi–projective is due to W. Chow and appeared in 1957 (see [5]).

The proof that the quotient of an affine group by a normal closed subgroup is also an *affine* algebraic group seemed to have appeared for the first time in 1951[2], in Chevalley's very important foundational book, [4].

In ERA the following theorem –that provides a very precise characterization of observability in geometric terms– is proved.

Theorem 3.1. [2, Theorem 4] *Let $H \subseteq G$ be a closed inclusion of affine algebraic groups. Then H is observable in G if and only if the homogeneous space G/H is a quasi–affine variety.*

In particular, the above theorem guarantees that a normal subgroup is always observable and hence, that the normality hypothesis unavoidable for the situation of Lie groups as presented in [20, 28], is unnecesary in the category of algebraic groups.

For the proof of Theorem 3.1 we need some preparation.

Lemma 3.2. *Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups, if $0 \neq I \subseteq \Bbbk[G]$ is an H stable ideal, there is a non zero element $f \in I$ and an extendable character χ of H such that $x \cdot f = \chi(x)f$ for all $x \in H$ and that $f(1) \neq 0$.*

Proof. Take V a simple rational H–submodule of I. Choose a basis $\{e_1, \cdots, e_n\}$ with the property that $e_1(1) = 1, e_i(1) = 0$ for $i = 2, \cdots, n$. Take V^* the linear dual of V that is a rational H–module, and apply Theorem 2.3 to obtain an extendable character χ of H and inclusion $\iota : V^* \to \Bbbk[G]_{\chi^{-1}}$ with the property that $\iota(e_1^*)(1) \neq 0$. The equivariance property of ι reads as: $\iota(x \cdot \alpha) = x \cdot_{\chi^{-1}} \iota(\alpha) = \chi^{-1}(x)x \cdot \iota(\alpha)$. It is clear that $\sum e_i \otimes e_i^*$ is H–stable, i.e. for all $x \in H$ we have that $x \cdot e_i \otimes x \cdot e_i^* = \sum e_i \otimes e_i^*$, if we apply $\mathrm{id} \otimes \iota$ to this equality, we deduce that $\sum e_i \otimes \iota(e_i^*) = \sum x \cdot e_i \otimes \iota(x \cdot e_i^*) = \chi^{-1}(x) \sum x \cdot e_i \otimes x \cdot \iota(e_i^*)$. Then, the element $f = \sum e_i\iota(e_i^*) \in I$ satisfies the following equivariance property $f = \sum e_i\iota(e_i^*) =$

[2]Probably this result was known to specialists since the beginning of the theory.

$\chi^{-1}(x)\sum(x\cdot e_i)(x\cdot\iota(e_i^*)) = \chi^{-1}(x)x\cdot\sum e_i\iota(e_i^*) = \chi^{-1}(x)x\cdot f$. Moreover, $f\neq 0$ as $f(1)=\sum e_i(1)\iota(e_i^*)(1) = e_1(1)\iota(e_1^*)(1) = \iota(e_1^*)(1)\neq 0$. ☑

The following characterization of observability seems to have appeared for the first time in [46, Chapter 11, Section 5].

Theorem 3.3. [46, Theorem 11.5.1] *Assume that $H\subseteq G$ is a closed inclusion of affine algebraic groups. Then H is observable in G if and only if, for every H–stable ideal $I\subseteq \Bbbk[G]$, there is a non zero element $f\in I$ such that $x\cdot f = f$ for all $x\in H$. Also, $H\subseteq G$ is observable if and only if for every closed proper subset of the quotient space $C\subseteq G/H$ there is a non zero invariant polynomial such that $f(C)=0$.*

Proof. In accordance with the lemma just proved, we can find $h\in I$ with the property that $x\cdot h = \chi(x)h$ for some extendable character χ and with $h(1)=1$. If H is observable, the character χ^{-1} is also extendable and then there is an element $g\in\Bbbk[G]$ such that $g(1)\neq 0$ and $x\cdot g = \chi^{-1}(x)g$ for all $x\in H$. Hence $hg\in I$ and is H–invariant and not zero. For the converse, if we have an extendable character χ, we have an element $0\neq f\in\Bbbk[G]$ that is χ-semi invariant and the associated principal ideal $I=\Bbbk[G]f$ is not zero and H–stable. By hypothesis, we can find $hf = g\in I$ with the property that $fg = h = x\cdot h = (x\cdot f)(x\cdot g) = (\chi(x)f)(x\cdot g)$. If G is connected we can cancel $f\neq 0$ and we have that $x\cdot g = \chi^{-1}(x)g$. So that χ^{-1} is extendable and in accordance with Theorem 2.4, the group H is observable. The case that G is not connected can be proved following the same methods. The second assertion is basically a reformulation of the first (ideal–theoretical) characterization of observability in geometric terms. ☑

Observation 3.4. If H is observable in G –with G connected– then ${}^H[\Bbbk[G]] = [{}^H\Bbbk[G]]$ (compare with [46, Lemma 11.5.4] where this result is proved and also the converse). It is clear that in general $[{}^H\Bbbk[G]]\subseteq {}^H[\Bbbk[G]]$. Conversely, take $0\neq g\in {}^H[\Bbbk[G]]$ and consider the H–stable ideal $I_g = \Bbbk[G]g\cap\Bbbk[G]$. In accordance with Theorem 3.3 we can find a non zero polynomial f_1 in I_g that is also H–fixed. It we write $f_1 = f_2 g$ for g as above, using the fact that f_1 and g are fixed by H, we conclude that f_2 is also fixed.

Next we prove Theorem 3.1.

Proof of Theorem 3.1:

From the following general fact (see [46, Theorem 1.4.48]): if $C\subseteq X$ is a closed subset of a quasi–affine variety, then there is a global section $0\neq f\in\mathcal{O}_X(X)$ such that $f|_C = 0$, and the second assertion of Theorem 3.3, it follows directly that if G/H is quasi affine then H is observable in G.

We sketch the proof of the converse assertion and we work in the case that G is irreducible (it is easy to show that it is enough to treat this particular situation).

Assume that $H \subseteq G$ is observable, and using the fact that ${}^H[\Bbbk[G]] = [{}^H\Bbbk[G]]$ (see Observation 3.4) we can take a family of field generators of the invariant rational functions $\{f_1, \cdots, f_n\} \subseteq {}^H[\Bbbk[G]] = [{}^H\Bbbk[G]]$ that are of the form $f_i = u_i/u_0$ with $\{u_0, \cdots, u_n\} \subseteq {}^H\Bbbk[G]$ for $i = 1, \cdots, n$. Let N the finite dimensional rational G–module generated by $\{u_0, \cdots, u_n\}$, $M = \bigoplus_{i=0}^n N$ and take $m_0 = (u_0, \cdots, u_n) \in M$. It is a standard result in the theory of affine algebraic groups that $H = \{x \in G : x \cdot f = f \text{ for all } f \in {}^H[\Bbbk[G]]\}$ (see for example [46, Corollary 8.3.4]) and then in our case we have that $H = G_{m_0}$ the stabilizer of m_0. It can be proved that G/H is isomorphic to the G–orbit of m_0 in M (result that is obvious in the case of zero characteristic, but that in general a proof of the separabililty of the action in this situation is needed) and as such it is a quasi–affine variety (for more details see [46, Section 8.3]). ☑

3.2. Observability and Hilbert's 14th problem

About ten years after the introduction of the concept of observability, an important relation with the so called Hilbert's 14th problem was discovered by G. Grosshans in [10]. As such, the concept of observability became another important element in the toolkit of invariant theory.

We describe briefly some parts of the contents of the important paper mentioned above, wherein the author distinguishes three situations –that he names as "the main problems"[3].

`Problem 1. Galois characterization of the observable subgroups.`

The author presents an interesting new perspective of the concept of observable subgroup.

Definition 3.5. If G is an affine connected algebraic group, define the sets $\mathfrak{H} = \{H : H \subseteq G \text{ is a closed inclusion}\}$ and $\mathfrak{R} = \{R \subseteq \Bbbk[G] : R \text{ is a } \Bbbk\text{–subalgebra of } \Bbbk[G]\}$ and the maps:

(1) $\mathbb{F} : \mathfrak{H} \to \mathfrak{R}$, $\mathbb{F}(H) := {}^H\Bbbk[G] = \{f \in \Bbbk[G] : x \cdot f = f, \forall x \in H\}$;

(2) $\mathbb{S} : \mathfrak{R} \to \mathfrak{H}$, $\mathbb{S}(R) = \operatorname{Stab}(R) := \{x \in \Bbbk[G] : x \cdot r = r, \forall r \in R\}$.

In the above situation it is usual to write $\mathbb{F}(H) = H'$ and similarly, $\mathbb{S}(R) = R'$.

Theorem 3.6. *In the above context, if we endow the sets $\mathfrak{H}$, $\mathfrak{R}$ with the order given by inclusion, the maps $\mathbb{F}, \mathbb{S}$ form an (order inverting) Galois connection. Moreover, the fixed subgroups for this connection, i.e. $\{H \in \mathfrak{H} : H'' = H\}$ are the observable subgroups.*

Corollary 3.7. *In the above situation one has that:*

[3] We set the problems –specially the second and third– in a slightly more general context than the original one due to Grosshans. For this we follow basicaly the presentation of [46, Chapter 11,13].

(1) For any $H \subseteq G$, H'' is observable in G;

(2) $H'' = \bigcap\{K : H \subseteq K \subseteq G\,,\, K \text{ observable}\}$;

(3) If A is a commutative rational G–module algebra, and $H \subseteq G$ is a closed inclusion, then ${}^{H}A = {}^{H''}A$.

Proof. (1) For a Galois connection $H''' = H'$ hence (1);

(2) In the above situation $H \subseteq K$ implies that $H'' \subseteq K'' = K$ and then $H'' \subseteq \bigcap\{K : H \subseteq K \subseteq G\,,\, K \text{ observable}\}$ and being H'' observable, the proof of this part is finished;

(3) It is clear that ${}^{H''}A \subseteq {}^{H}A$. Take now, $a \in {}^{H}A$ and let V be a rational finite dimensional G–module that contains a. If we call $K = \{x \in G : x \cdot a = a\}$, then Theorem 3.8 guarantees that K is observable. As a is fixed by H we deduce that $H \subseteq K$ and then $H'' \subseteq K$ and that means that a is fixed by all the elements of H''. ☑

Theorem 3.8. [2, Theorem 8],[10] *Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups and that there is a finite dimensional rational $M \in {}_G\mathcal{M}$ with the property that there exists $m_0 \in M$ such that $H = G_{m_0}$. Then H is observable in G.*

Proof. Take an arbitray $\alpha \in M^*$. The element $\alpha|m_0$ satisfies the following equivariance condition for all $x \in G$ $x\cdot(\alpha|m_0) = \alpha|(x\cdot m_0)$. Then, $\alpha|m_0 \in {}^{H}\Bbbk[G]$. Assume now that $z \in G$ is such that $z \cdot f = f$ for all $f \in {}^{H}\Bbbk[G]$. Then for all α we have that $z \cdot (\alpha|m_0) = \alpha|m_0$ and this implies that $\alpha(z \cdot m_0) = \alpha(m_0)$ for all $\alpha \in M^*$. Then $z \cdot m_0 = m_0$ and then $z \in H$. Hence $H = \{z \in G : z \cdot f = f \text{ for all } f \in {}^{H}\Bbbk[G]\}$. It is well known from the general theory of affine algebraic groups, that the above characterization of H as the stabilizer of ${}^{H}\Bbbk[G]$ guarantees that $[{}^{H}\Bbbk[G]] = {}^{H}[\Bbbk[G]]$. Along the proof of Theorem 3.1 we proved that the condition $[{}^{H}\Bbbk[G]] = {}^{H}[\Bbbk[G]]$ guarantees the observability of H on G. ☑

`Problem 2. Descent of the finite generation condition.`

Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups and that A is a rational G–module algebra we say that the *the finite generation condition descends from G to H* if for all A as above, in the inclusion ${}^{G}A \subseteq {}^{H}A$ the finite generation of the smallest $\Bbbk$–algebra implies the finite generation of the larger.

It is natural to search for conditions for G and H for which the finite generation of invariants descends from G to H.

The first thing to notice is that having H'' the same invariants than H we can assume without loss of generality that H is observable in G as ${}^{G}A \subseteq {}^{H''}A = {}^{H}A$. This is a crucial observation that reduces some problems in invariant theory to the case of observable subgroups.

Definition 3.9. Let $H \subseteq G$ be a closed inclusion we say that "the pair (H, G) satisfies the codimension two condition" if there exists a finite dimensional rational G–module V and an element $v \in V$ such that: (i) $H = \{x \in V : x \cdot v = v\}$ and $G/H \cong G \cdot v$; (ii) for each irreducible component C of $\overline{(G \cdot v)} \setminus (G \cdot v)$, we have that $\operatorname{codim}_{\overline{G \cdot v}} C \geq 2$.

In that context, the following theorem is proved in [10]:

Theorem 3.10. [12, Theorem 4.3] *For the situation above if H is observable in G, the conditions:*

(1) The $\Bbbk$–algebra $H' \subseteq \Bbbk[G]$ is finitely generated;

(2) The pair (H, G) satisfies the codimension two condition;

(3) For any finitely generated rational G–module; algebra A, ${}^{H}A$ is a finitely generated $\Bbbk$–algebra,

are related as follows. Conditions (1) and (2) are equivalent and condition (3) implies both of them. In the case that the action of G on V is separable and G is reductive, the three conditions are equivalent.

For a proof of this theorem we refer the reader to [12] or to a more recent exposition appearing in [46, Section 13.5, 13.6].

The so called "codimension two condition" is used in order to apply the following theorem on the extension of regular functions. "Let X be an irreducible normal variety and $f \in \mathcal{O}_X(U)$ be a function defined in an open subset U such that $\operatorname{codim}_X(X \setminus U) \geq 2$, then f can be extended to a function defined in X". See [10, Lemma 1] or [46, Theorem 2.6.14] for (similar) proofs of this general result.

It is worth noticing that in case of the special hypothesis on the separability of the action, the ring H' "behaves like a universal object as far as finite generation is concerned" (see [10, page 231]).

`Problem 3. Hilbert's` 14^{th} `problem.`

The original Hilbert's 14^{th} problem examines the answers to the following question (see [15]).

`Hilbert's problem.` *Let $A = \Bbbk[X_1, \cdots, X_n]$ be the polynomial algebra in n variables, let H be a subgroup $H \subseteq \mathrm{GL}_n(\Bbbk)$ and consider the action of H on A given by the restriction of the natural action of $\mathrm{GL}_n(\Bbbk)$. Is the subalgebra of H–invariants of A finitely generated?*

This problem can be generalized to the following context.

`Generalized Hilbert's problem.` *Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups, and that A is a finitely generated commutative $\Bbbk$–algebra. Assume that G acts rationally in the affine algebra A. Find conditions for the pair (H, G) that guarantee that if A^G is finitely generated so is A^H.*

It is clear that Theorem 3.10 guarantees that if G is reductive then, the generalized Hilbert's 14^{th} problem has a positive answer if H is observable in G.

3.3. The perspective of observability in Hilbert's 14th problem

The original formulation by D. Hilbert of his famous 14th problem reads as follows (as it appeared translated into English in [15]):

> "By a finite field of integrality I mean a system of functions from which a finite number of functions can be chosen, in which all other functions of the system are rationally and integrally expressible. Our problem amounts to this: to show that all relatively integral functions of any given domain of rationalityalways constitute a finite field of integrality".

In modern language this problem can be formulated as follows –see [30]– : "Let $\Bbbk$ be a field [$\{x_1, \dots, x_n\}$ a family of indeterminates] and let K be a subfield of $\Bbbk(x_1, \dots, x_n)$: $\Bbbk \subset K \subset \Bbbk(x_1, \dots, x_n)$. Is the ring $K \cap \Bbbk[x_1, \dots, x_n]$ finitely generated over $\Bbbk$?".

This problem of the *finite generation of special subalgebras of the polynomial algebra* $\Bbbk[x_1, \dots, x_n]$ is known as *Hilbert's 14^{th} problem* because it appeared with that number in the list of 23 problems presented by Hilbert in the International Congress of Mathematicians celebrated in Paris in 1900 ([15]).

A particularly important case is the following:

Lt $G \subset \mathrm{GL}_n$ be a subgroup, consider the induced action of G on $\Bbbk[x_1, \dots, x_n]$ and call $K = {}^G\Bbbk(x_1, \dots, x_n)$. As ${}^G\Bbbk[x_1, \dots, x_n] = K \cap \Bbbk[x_1, \dots, x_n]$, the finite generation of rings of invariants could –in principle– be deduced from an affirmative answer to Hilbert's problem.

In 1900, when Hilbert formulated his 14^{th} problem, a few particular cases were already solved. Classical invariant theorists were concerned with the invariants of "quantics" (invariants for certain actions of $\mathrm{SL}_m(\mathbb{C})$). In this situation the finite generation was proved by Gordan in 1868 for $m = 2$ and by Hilbert in 1890 for arbitrary m. Hilbert mentioned as motivation for his 14^{th} problem work by Hurwitz and also by Maurer–that turned out to be partially incorrect–.

Maurer's work contains some partial relevant results that were later rediscovered by Weitzenböck and guaranteed a positive answer for the case of the invariants of $(\mathbb{C}, +)$ and $(\mathbb{C}^*, \times)$. Later Weyl and Schiffer gave a complete positive answer for semisimple groups over $\mathbb{C}$. More recently –based on the platform established by Mumford in [29]–, Nagata's school contributions (see [31] and [32]) together with Haboush's results ([13]) settled the question affirmatively for reductive groups over fields of arbitrary characteristic.

In the case of non reductive groups, positive answers are more scarce. It is worth mentioning –besides the contributions by Maurer and Weitzenböck for the case of the additive group of the field of complex numbers– a result by Hochschild and Mostow (valid in characteristic zero): if U is the unipotent radical of a subgroup H of G that contains a maximal unipotent subgroup of G, then the U–invariants of a finitely generated commutative G–module algebra are finitely generated ([21]).

Around the same time of the publication of the paper just mentioned, Grosshans' published the above mentioned papers that provide more general insights into the problem of the finite generation of invariants for a non reductive group in arbitrary characteristic. For example the results of [21] can be understood as of Grosshans' pairs and the same with the classical result of Maurer's results on the invariants of the additive group. The so called Popov–Pommerening conjecture concerning the finite generation of the U–invariants of a finitely generated G–module algebra when G is a reductive group and U is a unipotent subgroup normalized by a maximal torus of G can also be formulated within that framework. The reader interested in these and many other topics in invariant theory should read the survey [35].

It took almost 60 years to discover that, in general, the answer to Hilbert's 14^{th} question is negative. The first counterexample was devised by M. Nagata and presented at the International Congress of Mathematicians in 1958 ([31]). Nagata's counterexample consisted of a commutative unipotent algebraic group U acting linearly and by automorphisms on a polynomial algebra, with a non finitely generated algebra of invariants.

4. Observability, Integrals and reductivity

In 1977 in the article *Induced modules and affine quotients* (referred as IMAQ), Cline, Parshall and Scott introduced a new viewpoint in the subject of observability (see [6]) by relating it with homological concepts, such as the exactness of the induction functor and injectivity conditions. With hindsight we could say that in a non–explicit way, the idea of observability was related to a generalization of the concept of reductivity (see [44, 45]).

The authors summarize –rather succinctly– the results of their paper as follows:

> *Let G be an affine algebraic group over an algebraically closed field $\Bbbk$. A closed subgroup H of G is exact if induction of rational H–modules to rational G-modules preserves short exact sequences. The main result of this paper is that H is exact iff the quotient variety G/H is affine. (In case G is reductive this means that H is reductive.) Also, we obtain a characterization of exactness in terms of a strong observability criterion, in this respect our theorem generalizes a result of Bialynicki-Birula [2] on reductive groups in characteristic zero.*

In the definition of strong observability, besides the existence of an extension of an H–module M by a G–module N, the authors demand a condition that controls the relation between the H–invariants of the submodule and the G–invariants of the module.

The concept of exactness will be treated in detail in Section 5. Below we give the basic operative definition in order to proceed as fast as possible to the main results.

Definition 4.1. Suppose that $H \subseteq G$ is a closed inclusion of affine algebraic groups. We say that a rational H–module M is strongly extendable, if there is a rational G–module N such that $M \subseteq N|_H$ and ${}^HM \subseteq {}^GN$. If the pair $H \subseteq G$ is such that all rational H–modules are strongly extendable to G we say that H is strongly observable in G.

Observation 4.2. In the paper we are currently considering the authors write down a stronger condition for the fixed parts of the modules N and M in the above definition, they ask that ${}^HM = {}^GN$, but later in [6, Remark 4.4,(c)] they comment that it can be weakened as above.

Definition 4.3. Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups. We say that H is exact in G if for an arbitrary short exact sequence $0 \to P \to Q \to R \to 0$ of $(\Bbbk[G], H)$– modules, the sequence $0 \to {}^HP \to {}^HQ \to {}^HR \to 0$ is exact.

Generalizing the relationship discovered in [2], between the geometry of G/H and the observability of H in G, the authors of [6] show that this more precise concept of "strong observability", has relevant connections with: a. the geometric structure of the homogeneous space G/H (strengthenig the results known for the observability situation); b. the exactness properties of the induction functor from H–modules to G–modules; c. the descent of the injectivity condition by restriction of the action.

Indeed, in [6, Theorem 4.3, Proposition 2.1] the following neat and comprehensive result is proved.

Theorem 4.4. *For a closed inclusion of affine algebraic groups $H \subseteq G$, the following conditions are equivalent:*

(1) The subgroup H is strongly observable in G.

(2) The rational G–module $\Bbbk[G]$ is injective when considered as an H–module. More generally for every injective rational G–module I, then $I|_H$ is also injective.

(3) The subgroup H is exact in G[4].

(4) The homogeneous space G/H is affine.

[4]It is also usual to define this exactness in terms of the induction functor: H is exact in G if the induction functor $\mathrm{Ind}_H^G : {}_H\mathcal{M} \to {}_G\mathcal{M}$ is exact (see Definition 5.1)).

The fact that (4) implies (3) was proved (as it is mentioned in the paper) – almost at the same time but using different methods– in Haboush's paper [14]. Also another proof appeared around the same time in [41][5]. Moreover, in the introduction of [6], it is mentioned that the equivalence of (3) and (4) had been conjectured by J.A. Green before.

4.1. Strong observability, injetivity and integrals

We deal next with the first two conditions of Theorem 4.4 leaving the third and fourth for later consideration. Our proofs will be different from the orginals as we use "integral tools". Given an affine algebraic group H we define the notion of integral in H (or $\Bbbk[H]$) with values in an H–algebra R and show the relation of integrals with strong observability. This relation is implict in [6, Theorem 3.1] where the authors consider the strong observability for the situation that H unipotent. Therein the authors mention [16, Proposition 2.2] as an antecedent where the integrals appear as cross–sections –in the same manner than in IMAQ–.

Definition 4.5. (1) An (scalar) integral for an affine group H is a linear map $\sigma : \Bbbk[H] \to \Bbbk$ that is invariant –i.e. $\sigma(x \cdot f) = \sigma(f)$ for $x \in H$ and $f \in \Bbbk[H]$–. It is said to be total if $\sigma(1) = 1$.

(2) An integral with values in a rational H–module algebra R is a linear map $\sigma : \Bbbk[H] \to R$ that is H–equivariant –i.e. $\sigma(x \cdot f) = x \cdot \sigma(f)$)–. We say that it is total if $\sigma(1) = 1$.

The relation of integrals with strong observability is deployed explicitly in [46, Theorems 11.4.8, 11.4.10]. In the theorem that follows (Thm. 4.6) we prove the equivalence of conditions (1) and (2) of Theorem 4.4 by proving that the two conditions are equivalent to the existence of a normalized integral.

Theorem 4.6. *Given the closed inclusion $H \subseteq G$, H is strongly observable in G if and only if H admits a total integral with values in $\Bbbk[G]$ and this happens if and only if $\Bbbk[G]$ is injective as an H–module.*

Proof. First we prove the equivalence of the injectivity condition with the existence of a total integral. If $\Bbbk[G]$ is injective in ${}_H\mathcal{M}$, we can complete the diagram

$$\begin{array}{ccc} \Bbbk & \longrightarrow & \Bbbk[H] \\ \downarrow & \swarrow_{\sigma} & \\ \Bbbk[G] & & \end{array}$$

and produce a morphism of H–modules $\sigma : \Bbbk[H] \to \Bbbk[G]$, sending 1 into 1.

[5]See also the discussion later in the paper in Section 4.4.

Conversely, assume that $\sigma : \Bbbk[H] \to \Bbbk[G]$ is a total integral and define the map $\Lambda : \Bbbk[G] \otimes \Bbbk[H] \to \Bbbk[G]$ by the formula for $r \in \Bbbk[G]$ and $f \in \Bbbk[H]$, $\Lambda(r \otimes f) = \sum r_1 \sigma\big(S(\pi(r_2))f\big)$ where $\Delta(r) = \sum r_1 \otimes r_2 \in \Bbbk[G] \otimes \Bbbk[G]$. If $\chi(r) = \sum r_1 \otimes \pi(r_2)$ is the H–comodule structure map for $\Bbbk[G]$, then $(\Lambda\chi)(r) = \sum r_1 \sigma\big(S(\pi(r_2))\pi(r_3)\big) = r\sigma(1) = r$. Also, if $r \in \Bbbk[G]$ and $x \in H$, then $\sum x \cdot r_1 \otimes \pi(r_2) \cdot x^{-1} = \sum r_1 \otimes \pi(r_2)$, equality that can be proved directly by evaluation of both sides at an element $(y, z) \in G \times H$ (the left and right side yield the value $r(yz)$ after evaluation). Then for all $x \in H$,

$$\begin{aligned}\Lambda(r \otimes x \cdot f) = & \sum r_1 \sigma\big(S(\pi(r_2))(x \cdot f)\big) = \sum (x \cdot r_1) \sigma\big(S(\pi(r_2) \cdot x^{-1})(x \cdot f)\big) = \\ & \sum (x \cdot r_1) \sigma\big(x \cdot \big(S(\pi(r_2))f\big)\big) = x \cdot \sum r_1 \sigma\big(S(\pi(r_2))f\big) = \\ & x \cdot \Lambda(r \otimes f)\end{aligned}$$

If we write as $\Bbbk[G]_0 \otimes \Bbbk[H]$ the rational H–module with trivial H–action in the first tensor factor and the regular action on the second, the above considerations show that $\chi : \Bbbk[G] \to \Bbbk[G]_0 \otimes \Bbbk[H]$ splits the H–morphism $\Lambda : \Bbbk[G]_0 \otimes \Bbbk[H] \to \Bbbk[G]$. Hence, $\Bbbk[G]$ is a direct H–module summand of $\Bbbk[G]_0 \otimes \Bbbk[H]$ and hence (as it is well known that $\Bbbk[H]$ is injective as a rational H–module) the polynomial algebra $\Bbbk[G]$ is also injective as a rational H–module.

Next we show how to produce a total integral if we know that the inclusion $H \subseteq G$ is strongly observable.

Assume that H is strongly observable and consider the H–module $\Bbbk[H]$. By the hypothesis of strong observability, one can find an inclusion $\Bbbk[H] \subset N$ where N is a rational G–module and ${}^H\Bbbk[H] = \Bbbk = N^G$. Take a linear functional α on N such that $\alpha(1) = 1$ and define $f \mapsto \sigma(f) : \Bbbk[H] \to \Bbbk[G]$ as: $\sigma(f)(x) = \alpha(x \cdot f)$ for $x \in G$. The integral is total as $\sigma(1)(x) = \alpha(x \cdot 1) = \alpha(1) = 1$.

For the proof of the H–equivariance of σ we compute $\sigma(y \cdot f)(x) = \alpha(x \cdot y \cdot f) = \alpha(xy \cdot f) = \sigma(f)(xy) = (y \cdot \sigma(f))(x)$.

We finish the proof of the theorem by showing that the existence of a total integral implies the strong observability of H in G.

Assume that σ is a total integral. First show that H is observable in G. Assume that γ is a rational character of H and fix an $f \in \Bbbk[G]$ with the property that $\pi(f) = f|_H = \gamma$. Define the following element of $\Bbbk[G]$: $g = \sum \sigma\left(S(\pi(f_2))\gamma\right) f_1 \in \Bbbk[G]$. A direct computation shows that for all $x \in H$ we have that $x \cdot g = \gamma(x)g$. As $g(1) = 1$ we conclude that g extends γ and being γ an arbitrary character we deduce the observability of H in G.

In order to prove that the observability is strong we proceed as follows. Given $M \in {}_H\mathcal{M}$ we take $S = \bigoplus S_i$ the socle of M, S_i a simple object in ${}_H\mathcal{M}$. Using the fact that H is observable, and S_i simple it is easy to show that we can find H–equivariant inclusions $\eta_i : S_i \to T_i$ with T_i a G–module, and $\eta_i({}^H S_i) \subseteq {}^G T_i$. Then we have a map $\eta : S \to \bigoplus T_i$ with the required property for the strong observability. In other words, we have proved that if H is observable in G, an arbitrary rational H–module has its socle strongly extendable to a G–module. We go one step further and prove that in our case, this G–module (that we call L) can be taken to be injective. This is done by imbeding the G–module thus obtained, using the structure map $\chi : L \to L \otimes \Bbbk[G]$. This map is equivariant when G acts trivially in the first tensor component, and using the fact that we have a total integral, we see that $L \otimes \Bbbk[G]$ is injective as an H–module. All in all, we have proved that the original H–module M has its socle S strongly extended to a G–module M that is injective as an H–module. The injectivity of M guarantees the extension of the map from S to M and this extension does the job without increasing the H–invariants as ${}^H S = {}^H M$. ☑

4.2. Integrals, observability and invariants

Here we describe briefly some aspects on the development of the ideas concerning total integrals mainly in the context of algebraic groups.

It was realized around 1961 that the concept of "integral" taking values in an arbitrary $\Bbbk[H]$–comodule algebra (or rational H–module algebras) instead of in the base field $\Bbbk$ could be a relevant tool to control the representations and the geometry of the actions of the group H. A particularly interesting case is when the $\Bbbk[H]$–comodule algebra is $\Bbbk[G]$ for G an affine algebraic group and H a given subgroup.

An important motivation was the following. In [16] and [17], Hochschild set the basis of the cohomology theory of affine algebraic groups –rational cohomology. It was soon observed that if G is an affine algebraic group and $H \subseteq G$ a normal closed subgroup, then it was necessary to prove that $\Bbbk[G]$ is injective as an H–module in order to guarantee the convergence of the Lyndon-Hochschild-Serre spectral sequence –that relates the cohomology of G, H and G/H–.

The necessary injectivity result is a direct consequence of the equivalence of (2) and (4) in Theorem 4.4 and it was treated and proved in certain cases in the mentioned papers [16] and [17]. For example, the injectivity of $\Bbbk[G]$ as a rational H–comodule and the cohomological consequences, were established in [17, Prop. 2.2] but only for the case that the integrals are multiplicative –strong restriction that rarely occurs except in the case of unipotent subgroups. As far as we are aware, the injectivity of $\Bbbk[G]$ as an H–module, for H normal in G was proved in full generality only much later in [6], [14] and [33] (the three articles appeared in 1977). Non multiplicative general integrals appeared around 1977, even though at first they were used in a subordinate way to produce multiplicative ones.

Concerning this fact, we mention the following two results from [6]. In Proposition 1.10 (attributed to Hochschild: [16, Prop. 2.2]) the authors prove that if the closed inclusion $H \subseteq G$ of affine algebraic groups admits an *equivariant cross–section*, then $\Bbbk[G]$ is injective as an H–module. Such a cross section is a closed subvariety $S \subseteq G$ such that the map given by multiplication $(s, x) \mapsto sx : S \times H \to G$ is an isomorphism of varieties. The proof of the injectivity result follows from the fact that the H–module algebras $\Bbbk[S]\otimes\Bbbk[H]$ and $\Bbbk[G]$ are equivariantly isomorphic with respect to the natural actions on each tensor factor and endowing $\Bbbk[S]$ with the trivial H–action.

The use of integrals (without mentioning the name) appears in the following theorem where the authors deal with the relationship between the existence of a total integral with values in $\Bbbk[X]$ and the existence of affine quotients of X –at least for the case of a unipotent group–. This situation can be generalized for non unipotent groups, but one needs to restrict the variety X to be an affine algebraic group as in Theorem 4.4.

Theorem 4.7. [6, Thm. 3.1] *Let U be a connected unipotent group acting on an affine variety X. The following are equivalent.*

(1) $\Bbbk[X]$ is a rationally injective U–module.

(2) There is a U–equivariant morphism of varieties $\rho : X \to U$, (i.e., there is a U–equivariant algebra homorphism $\Bbbk[U] \to \Bbbk[X]$).

(3) There is a U–module homomorphism $\alpha : \Bbbk[U] \to \Bbbk[X]$ with $\alpha(1) = 1$.

When these conditions are satisfied, the quotient X/U exists and is affine.

Proof. (1) ⇒ (2) As U is unipotent one can write $\Bbbk[U]$ as $\Bbbk[U] = \Bbbk[X_1, \cdots , X_n]$ with the property that if $P_i = \Bbbk[X_1, \cdots , X_i]$ then, for all $u \in U$, $u \cdot X_i \cong X_i (\mod P_{i-1})$. Then we start with $P_0 = \Bbbk$ for which we take the inclusion $\Bbbk \to \Bbbk[X]$ and construct by induction a U–equivariant algebra homomorphism $\alpha_i : P_i \to \Bbbk[X]$. Given $\alpha_{i-1} : P_{i-1} \to \Bbbk[X]$ we extend it as a U–equivariant morphism of U–modules $\beta_i : P_i \to \Bbbk[X]$ using the injectivity of $\Bbbk[X]$. Then, define α_i as the morphism of algebras that on the generators take values $\alpha_i(X_j) = \beta_i(X_j)$ for $1 \leq j \leq i$. It is easy to see that α_i is U–equivariant.

(3) ⇒ (1) This is the content of Theorem 4.6 item (2). See also the comment that follows after the proof.

It is clear that the quotient variety X/U will be the cross–section associated to ρ, i.e. $\rho^{-1}(1_U)$. ☑

Nowadays, all these considerations have been proved to be valid in a more general framework. In particular the theory Hopf–Galois extensions is well established –see for example [27] for an exposition of the original results of [47]–. From today's perspective one can say that [6, Thm. 3.1] is a predecessor of the theory that relates the existence of integrals with the Galois theory of Hopf algebras as in [9] –see [27] for a comprehensive exposition and a complete bibliography–.

In a parallel development, Sweedler collected in his classical book [50] (1969) the basic properties of the (scalar) integrals in the set up of general Hopf algebras. Therein he also proved, a generalization for arbitrary Hopf algebras of Hochschild's result guaranteeing that the existence of an (scalar) total integral for the Hopf algebra of an affine algebraic group is equivalent to the complete reducibility of the representations of the group ([18]). The general situation of the existence of total H–integrals with values in $\Bbbk[G]$ for $H \subseteq G$ and its relation with semisimplicity, appeared first in [44].

These developments culminate beautifully in a series of articles by Y. Doi and later by Y. Doi and M. Takeuchi starting in 1983. The authors define the general notion of total integral from a Hopf algebra H in an H–comodule algebra A and prove the corresponding injectivity result as well as many other interesting properties of the category of the (A, H)–comodules (see [7], [8] and [9]).

4.3. Observability, exactness and quotients

In this section we complete the proof of Theorem 4.4 (i.e. the main result of IMAQ). We show the relation of strong observability with the exactness of the induction functor and also with the affineness of the associated homogeneous space. The induction functor also plays an important, but different role in the characterization of observability through the surjectivity of the associated counit natural transformation –this is shown in the next section.

We need first a proof of the fact that the exactness condition implies the observability.

Proposition 4.8. *Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups. If H is exact in G then, H is observable in G.*

Proof. Take $M \in {}_G\mathcal{M}$ and consider the morphism $\pi \otimes id : \Bbbk[G] \otimes M \to \Bbbk[H] \otimes M$, that is clearly a morphism of $(\Bbbk[G], H)$–modules (see Definition 4.3) provided that we endow $\Bbbk[H]$ with the structure of $\Bbbk[G]$ module given by π. The associated morphism obtained by restriction to the H–fixed part is the map ${}^H(\Bbbk[G] \otimes M) \to {}^H(\Bbbk[H] \otimes M) = M$, $\sum f_i \otimes m_i \mapsto \sum f_i(1)m_i$. Thanks to the exactness hypothesis we deduce that this morphism E_M –that is the counit of the adjunction between induction and restriction– is surjective. This is one of the possible characterizations of observability and hence the result is proved (see also [6, Lemma 4.2] for another line of reasoning). ☑

The relation of observability and the induction functor is treated below in Section 5: Definition 5.1 and Lemma 5.2.

We will need for the proof the following easy and handy Lemma that appears for example in [46, Theorem 1.4.49], and that guarantees that within the class of quasi–affine varieties, the validity of the Nullstellensatz characterizes the affine ones.

Lemma 4.9. *Assume that X is a quasi–affine variety with the property that if J is an arbitrary proper ideal $J \subsetneq \mathcal{O}_X(X)$, then $\mathcal{Z}(J) \neq \varnothing$; then X is affine. In* *particular if H is an observable subgroup of G, if for all $J \subsetneq {}^H\Bbbk[G]$ we also have that $J\Bbbk[G] \neq \Bbbk[G]$, then G/H is affine.*

Theorem 4.10. *Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups. The following three conditions are equivalent:*

(1) The subgroup H is exact in G;

(2) The homogeneous space G/H is affine;

(3) There is a total integral $\sigma : \Bbbk[H] \to \Bbbk[G]$.

Proof. We prove that (2) $\Rightarrow$ (1) folowing Haboush's argument in [14]. For a rational G–module M and $U \subseteq G/H$ open in G/H, we consider the usual diagonal action of H on $\mathcal{O}_G(\pi^{-1}(U)) \otimes M$ and define $\mathcal{I}_M$, the sheaf on G/H such that $\mathcal{I}_M(U) = {}^H(\mathcal{O}_G(\pi^{-1}(U)) \otimes M)$. It is clear that the global sections of this sheaf is the induced module ${}^H(\Bbbk[G] \otimes M)$ and a direct computation shows that the stalk of the sheaf $\mathcal{I}_M$ at $eH \in G/H$ is M. Hence, it is clear that for an exact sequence $0 \to P \to Q \to R \to 0 \in {}_H\mathcal{M}$, the sequence $0 \to \mathcal{I}_P \to \mathcal{I}_Q \to \mathcal{I}_R \to 0$ is also exact. In the situation that G/H is affine, Serre's cohomological characterization of affineness guarantees that the sequence of global sections of the above sheaf is also exact. This means that the induction functor is exact and it follows easily that this implies that H is exact in G.

The proof that (1) $\Rightarrow$ (2) is as follows, from the exactness hypothesis we deduce that G/H is quasi affine. In order to apply Theorem 4.9 take $J \subsetneq {}^H\Bbbk[G]$ a proper ideal. In the case that $J\Bbbk[G] = \Bbbk[G]$, we can find $\{j_1, \cdots, j_n\} \subseteq J$ such that the morphism of $(\Bbbk[G], H)$ modules $\Phi : \bigoplus_{i=1}^n \Bbbk[G] \to \bigoplus_{i=1}^n \Bbbk[G]$, $\Phi(g_1, \cdots, g_n) = \sum g_i j_i$ is surjective. Then, the morphism $\Phi : \bigoplus_{i=1}^n {}^H\Bbbk[G] \to \bigoplus_{i=1}^n {}^H\Bbbk[G]$ is also surjective and that means that $J = {}^H\Bbbk[G]$.

Next we prove that (1) $\Rightarrow$ (3).

Let $\iota : M \hookrightarrow N$ be an inclusion of finite dimensional rational H–modules and consider the diagram in ${}_H\mathcal{M}$

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & N \\ {\scriptstyle\phi}\downarrow & \swarrow {\scriptstyle\widehat{\phi}} & \\ \Bbbk[G] & & \end{array}$$

Consider $X = \operatorname{Hom}_\Bbbk\big(M, \Bbbk[G]\big)$ and $Y = \operatorname{Hom}_\Bbbk\big(N, \Bbbk[G]\big)$ endowed with the standard rational $(\Bbbk[G], H)$–module structure. The inclusion ι induces a surjective morphism of $(\Bbbk[G], H))$–modules. From the exactness of H in G, we conclude that $\iota^*\big({}^HY\big) = {}^HX$. Any element $\widehat{\phi} \in {}^HY$ mapped into $\phi \in {}^HX$ is the extension of ϕ we are looking for. For the case of infinite dimensional H–modules a Zorn's Lemma type of argument does the job to extend the morphism in the above diagram. We have thus proved that $\Bbbk[G]$ is injective as an H–module. And this implies condition (3).

The proof that (3) $\Rightarrow$ (1) goes as follows. Take an arbitrary $(\Bbbk[G], H)$–module M and consider the map $\mathcal{R}_M : M \to M : \mathcal{R}_M(m) = \sum \sigma(S(m_1))m_0$. It is easy to show that $\mathcal{R}_M(M) = {}^H M$ and that for a morphism $f : M \to N$ of $(\Bbbk[G], H)$–modules, $f \circ \mathcal{R}_M = \mathcal{R}_n \circ f$. From the commutativity of the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\mathcal{R}_M} & {}^H M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f|_{{}^H M}} \\ N & \xrightarrow{\mathcal{R}_N} & {}^H N, \end{array}$$

we deduce that if f is surjective, so is the restriction $f|_{{}^H M}$. Hence H is exact in G. ☑

4.4. Strong observability and reductivity

In IMAQ, for example in Corollary 4.5 or in Remark 4.4, the notion of strong observabity (viewed as an injetivity condition) is studied for a closed inclusion $H \subset G$ in the case that G is reductive. This sort of considerations are also present in the mentioned work of Haboush where (using different methods), similar results are proved. For example in [13, Proposition 3.2], the author proves that *if $H \subseteq G$ is a closed inclusion of affine algebraic groups with G reductive, then G/H is affine if and only if H is reductive*[6]. This assertion is also known as *Matsushima's criterion* and appeared for the first time in [26], and later proofs appeared in work by Borel and Harish–Chandra, Bialynicki-Birula, Richardson, Haboush, Cline Parshall and Scott (IMAQ), etc. The last three works, are valid in arbitrary characteristic and were published more or less simultaneously. In the introduction to Richardson's paper [41] appears the following citation of a letter from Borel to the author (1977):

[6] The difficult part is the conclusion of the reductivity of H from the geometric hypothesis about the quotient space G/H.

> *... The fact that G/H affine implies that H is reductive, has been know for almost 15 years, although not formally published. But this was only because of the difficulty to give references for some necessary foundational material on étale cohomology. In fact, using the Chevalley groups schemes over $\mathbb{Z}$ it can be seen that the étale cohomology* $\mod \mathbb{Z}/\ell\mathbb{Z}$ *, ℓ prime* $\neq \operatorname{char} \Bbbk$ *of a reductive $\Bbbk$–group, is the same as the ordinary cohomology of the corresponding complex group. If one takes for granted the existence of a spectral sequence for the fibration of a group by a closed subgroup, then it is clear that the proof given in my joint paper with Harish–Chandra goes over verbatim for arbitrary characteristic, using étale cohomology. This was pointed out to me by Grothendieck (in 1961 as I remember it) as soon as I outlined this proof to him. I have always found mildly amusing that the so called 'algebraic proof' of Bialynicki–Birula is restricted to characteristic zero, while the 'trascendental' one is not. The fact mentioned above about the cohomology of reductive groups is proved by M. Raynaud (Inv. Mat. 6 (1968)) but, apart from that, it seems difficult even now to give clear-cut references to the basic facts on étale cohomology needed here, so a more direct proof such as yours is still useful.*

Nowadays it is clear that the mentioned criterion admits for arbitrary characteristic, proofs that are much more elementary than the one suggested by Grothendieck using étale cohomology. In [44, 45] the authors propose a different perspective that yields an easy proof for the above result and many others. For that, one has to reinterpret the condition of the exactness of K in H as an assertion on the linear reductivity of the action of K on H –or on $\Bbbk[H]$. In this case if we look at the trivial action of H on $\Bbbk$ we obtain the concept of linear reductivity. Using this viewpoint, Matsushima's criterion can be read as follows: in the hypothesis that the action of H on $\Bbbk$ is linearly reductive we have that the action of K on H is linearly reductive, if and only if the action of K on $\Bbbk$ is linearly reductive[7].

Definition 4.11.

(1) Let H be an affine algebraic group and R a rational H–module algebra. We say that the action of H on R is *linearly reductive* if for every triple (M, J, λ) where $M \in (R, H) - \mathrm{mod}$, $J \subseteq R$ is an H–stable ideal and $\lambda : M \to R/J$ is a surjective morphism of (R, H)–modules; there exists an element $m \in {}^H M$, such that $\lambda(m) = 1 + J \in R/J$. In the context above, if the action of H on R is given, we say that (R, H) is a linearly reductive pair.

(2) In the case that $R = \Bbbk[X]$ and the action of H on R is linearly reductive we say that the action of H on X is linearly reductive and also that the pair (H, X) is linearly reductive.

[7]In order to simplify the assertions we concentrate in this survey in the situation of linearly reductive actions (see Observation 4.12)

Observation 4.12. In [44] besides considering the notion of linearly reductive action of H on R (or in an affine variety X) the concept of geometrically reductive action was defined in the same context. Some of the results of this section can be generalized to the context of geometric reductivity.

The proof of the theorem that follows, that appeared [46, Thm.2.4], is similar to others presented before and we omit it (compare with the results in Section 4).

Theorem 4.13. *Let H an affine algebraic group and R a rational H–module algebra. Then, the following conditions are equivalent:*

(1) The action of H on R is linearly reductive.

(2) If $\varphi : M \to N$ is a surjective morphism in the category ${}_{(R,H)}\mathcal{M}$, then $\varphi\big({}^HM\big) = {}^HN$.

(3) There exists a total integral $\sigma : \Bbbk[H] \to R$.

(4) The H–module algebra R is an injective object in the category ${}_H\mathcal{M}$.

(5) Every object $M \in {}_{(R,H)}\mathcal{M}$ is injective in ${}_H\mathcal{M}$.

Morever, in the case that $H = U$ is unipotent, the action of U on R is linearly reductive, if and only if there is a multiplicative normal integral from $\Bbbk[U]$ into R.

It is clear that the trivial action of H on $\Bbbk$ is linearly reductive, if and only if H is a lineraly reductive affine algebraic group.

Once we free the notion of obervability of the restriction to the group/subgroup situation, we acquiere a degree of flexibility that seems to provide a better understandig of the main issues of this area. In that sense we mention below (without proofs) a few other results from [45].

(1) Let $K \subseteq H$ be a closed inclusion of affine algebraic groups. The following two conditions are equivalent:
   (a) The action of K in H and the action of H in H/K are linearly reductive
   (b) K is linearly reductive.

(2) Let $K \subseteq H$ be as above and R a rational K–module algebra and consider $R_H = \operatorname{Ind}_K^H(R)$ the induced H–module algebra. Assume moreover that the action of K on H is lineraly reductive. Then if the action of H on R_H is linearly reductive, so is the action of K on R. For the definition of the functor Ind_K^H see Section 5.

(3) (*Generalized Matsushima's criterion.*) Suppose that we have $K \subset H$ a pair given by a group and a subgroup, and that R is an H–module algebra with the property that the action of H on R is linearly reductive. Then if the action of K on H is linearly reductive, then the action of K on R is linearly reductive.

5. Observable adjunctions

The concept of *observable adjunction* and of *observable module category* appeared in 2006 (see [1]) as a direct product of the following observations based in the consideration of the monoidal categories ${}_G\mathcal{M}$ and ${}_H\mathcal{M}$ instead of the groups G and H.

Let $H \subseteq G$ be a closed inclusion of affine algebraic groups and let $\mathcal{D} = {}_H\mathcal{M}$ and $\mathcal{C} = {}_G\mathcal{M}$ be the corresponding categories of rational representations. Call $\mathbb{L} : \mathcal{C} \to \mathcal{D}$ the restriction functor, usually denoted as Res_G^H, from rational G–modules to H–modules.

It is well known that the monoidal functor $\mathbb{L}$ (see Definition 5.3) has a right adjoint that is usually named as the induction functor, denoted as Ind_H^G and herein abbreviated as $\mathbb{R}$.

Definition 5.1. If $H \subseteq G$ is a closed inclusion of affine algebraic groups and $M \in {}_H\mathcal{M}$, we endow $\Bbbk[G] \otimes M$ with a structure of H–module acting on the left, and with a left structure of G–module where $x \in G$ acts as x^{-1} on the right in the first tensor factor, and define $\mathbb{R}(M)$ as the G–module $\mathbb{R}(M) := \operatorname{Ind}_H^G(M) := {}^H(\Bbbk[G] \otimes M)$. If $f : M \to M'$ is a morphism of rational H–modules, we define $\operatorname{Ind}_H^G(f) := (\operatorname{id} \otimes f)|_{{}^H(\Bbbk[G]\otimes M)}$.

It is well known (see for example [46, Corollary 7.7.12]) that $\mathbb{L} \dashv \mathbb{R}$ (i.e. $\mathbb{L}$ is the left adjoint of $\mathbb{R}$) or in explict terms that: for all $M \in {}_H\mathcal{M}$ and $N \in {}_G\mathcal{M}$ there is a natural isomorphism (in the category of $\Bbbk$–spaces) $\operatorname{Hom}_H(\operatorname{Res}_G^H(N), M) \cong \operatorname{Hom}_G(N, \operatorname{Ind}_H^G(M))$. In the classical literature the above isomorphism was called the *Reciprocity law.*

The counit of the adjunction is the following family of maps:

$$\varepsilon_M : {}^H(\Bbbk[G]\otimes M) \to M,\ \varepsilon_M\left(\sum f_i \otimes m_i\right) = \sum f_i(1)m_i \text{ for } \sum f_i \otimes m_i \in {}^H(\Bbbk[G]\otimes M). \tag{1}$$

The observability can be characterized in terms of the natural transformation ε.

Lemma 5.2. *In the above situation $H \subseteq G$ is observable if and only if for all $M \in {}_H\mathcal{M}$, $\varepsilon_M : {}^H(\Bbbk[G] \otimes M) \to M$ is surjective.*

Proof. We prove that if for all $M \in {}_H\mathcal{M}$, the counit $\varepsilon_M : {}^H(\Bbbk[G] \otimes M) \to M$ is surjective, then H is observable in G.

We use the characterization in terms of extendable characters (see the beginning of Section 6.1 and the footnote therein). Let χ a character of H, consider the character χ^{-1} and write as $\Bbbk_{\chi^{-1}}$ the one dimensional H–module defined by χ^{-1}.

It is not hard to see that

$$\operatorname{Ind}_H^G(\Bbbk_{\chi^{-1}}) = \{f \in \Bbbk[G] : x \cdot f = \chi(x)f, \forall x \in H\} = \Bbbk[G]_\chi$$

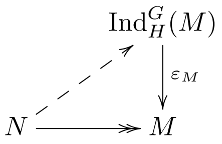
and that $\varepsilon : \Bbbk[G]_\chi \to \Bbbk$ is the evaluation at the identity element of G.

Using the surjectivity of ε we can guarantee the existence of $f \in \Bbbk[G]_\chi$ such that $f(1) = 1$ and then f is a non zero χ–semi invariant.

Next we show that if $H \subseteq G$ es observable then ε is surjective for all $M \in {}_G\mathcal{M}$.

First observe that if every H–representation M can be extended to a G–representation N, $M \subseteq N$, by dualization every H–representation can be obtained as the projection of a G–representation. Hence it is clear that $H \subseteq G$ is observable, if and only if for an arbitrary H–module M there is a G–module N and a surjective morphism of H–modules such that $N \twoheadrightarrow M$.

In this situation the universal property of the adjunction guarantees the existence of a map as in the diagram.



The surjectivity of the horizontal map implies the surjectivity of the vertical map ε_M. ☑

The above result is the justification for the following definition of *observable adjunction*. First we introduce some nomenclature.

Definition 5.3. A monoidal category is a sextuple $\mathcal{C} = (\mathcal{C}, \otimes, \Bbbk, \Phi, \ell, r)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, $\Bbbk$ is a fixed object, the unit; Φ is a natural isomorphism: the associativity constraint with components $\Phi_{c,d,e} : (c \otimes d) \otimes e \to c \otimes (d \otimes e)$, ℓ and r are the unit constraints, that are natural isomorphisms with components $r_c : c \otimes \Bbbk \to c$ and $\ell_c : \Bbbk \otimes c \to c$. Moreover, all these data satisfy certain coherence conditions –commutative diagrams (see MacLane's classic book: *Categories for the working mathematician*: [25]).

If $\mathcal{C}$ and $\mathcal{D}$ monoidal categories and $T : \mathcal{C} \to \mathcal{D}$ is a functor a (strong) monoidal structure in T is a natural isomorphism $T(c) \otimes T(d) \to T(c \otimes d)$ and an isomorphism $\Bbbk \to T(\Bbbk)$ with certain coherence conditions (see Joyal and Street: *Braided tensor categories.* [24]). A monoidal functor is a functor together with a monoidal structure.

Given a monoidal category, a $\mathcal{C}$–module category is a category $\mathcal{M}$ together with a functor $\boxtimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\mu_{x,y,m} : (x \otimes y) \boxtimes m \to x \boxtimes (y \boxtimes m)\,,\ \lambda_m : \Bbbk \boxtimes m \to m,$$

with compatibility conditions that we omit and involve the associativity constraint Φ and also the left and right unit constrains ℓ, r.

From now on we assume that all categories are $\Bbbk$–linear and that the tensor structures and associated natural transformation are compatible with the linear structure.

Definition 5.4. A non–trival module category over a tensor category $\mathcal{C}$ is said to be simple if any proper submodule category is trivial. The trivial module category is the category $\mathcal{M} = 0$.

Definition 5.5. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories and $\mathbb{L} : \mathcal{C} \to \mathcal{D}$ a monoidal functor. Suppose that $\mathbb{L}$ admits a right adjoint functor $\mathbb{R} : \mathcal{D} \to \mathcal{C}$. and call $\varepsilon_d : \mathbb{L}\mathbb{R}d \Rightarrow d$ the counit. If $\varepsilon : \mathbb{L}\mathbb{R} \Rightarrow \mathrm{id} : \mathcal{D} \to \mathcal{D}$ is a surjective natural transformation, we say that $\mathcal{D}$ is observable in $\mathcal{C}$ and that the pair $(\mathbb{L}, \mathbb{R})$ observes $\mathcal{D}$ in $\mathcal{C}$.

Definition 5.6. In the above context we endow $\mathcal{D}$ with a structure of $\mathcal{C}$ module category by the following rule: $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ is $c \boxtimes d := \mathbb{L}(c) \otimes_{\mathcal{D}} d$.

The following theorem illustrates the use of this concept in the theory of module categories.

Theorem 5.7. [1, Theorem 2.3] *Given an observable adjunction $\mathbb{L} \dashv \mathbb{R}$, $\mathbb{L} : \mathcal{C} \to \mathcal{D}$, $\mathbb{R} : \mathcal{D} \to \mathcal{C}$, if $\mathcal{D}$ is ind–rigid and the adjunction is observable then $\mathcal{D}$ is simple as a $\mathcal{C}$–module category.*

The concept of ind–rigid category is a categorical generalization of the idea of locally finite linear action. A category is ind–rigid if every object is the colimit of rigid objects, i.e. every object behaves as a rational module in the sense that it is the sum of finite dimensional (i.e. rigid) subobjects.

In the mentioned paper, the above considerations are used to study in some concrete cases the ideas related to the general definition of observability in particular, it is treated the case of Hopf algebra quotients $\pi : A \to B$ and the situation of the category of the linearized sheaves of a G–variety.

6. Observable actions of groups on varieties

6.1. Brief description of the major results

To illustrate the basic ideas of the current section we revisit some of the relevant results around the concept of observable subgroup H of a connected group G. Consider the following four equivalent properties of a closed inclusion $H \subseteq G$ that summarize the results of ERA, [2].

(1) For every H–stable and closed subset $Y \subset G$ there is a non zero H–invariant polynomial function that is zero on Y.

(2) The homogeneous space G/H is a quasi–affine variety.

(3) $^H\left[\Bbbk[G]\right] = \left[^H\Bbbk[G]\right]$.

(4) For every character $\rho \in \mathcal{X}(H)$ there is a non zero polynomial $f \in \Bbbk[G]$, with the property that for all $x \in H$, $x \cdot f = \rho(x)f$, i.e. every character is extendable[8].

Around 2010 it was observed by Renner and Rittatore in the paper: *Observable actions of algebraic groups*, see [36] (abbreviated as OAAG), that if (1) is taken as the definition of observable subgroup, it can be easily and profitably generalized, by taking an arbitrary action of a group on a variety rather than the action of a subgroup in a larger group.

Regarding this idea the following definition appeared in the mentioned paper:

Definition 6.1. Assume that H is an affine algebraic group and that X is an affine H–variety. The action of H on X is said to be *observable*, if every H–stable and closed subvariety $Y \subset X$ admits an H–invariant polynomial function that is zero on Y.

In this more general situation, some adaptations are needed in order to obtain results similar to the ones listed above. Here we just give a succint description and more details appear later.

For example, concerning the equivalence of conditions (1) and (3), in this general case one needs to consider also the set $\Omega(X) = \{x \in X : O(x)$ is closed and of maximal dimension$\}$ in which case the valid result guarantees that the following two conditions (a) and (b) taken together, are equivalent to the observability of the action: (a) $\left[^H\Bbbk[X]\right] = {}^H\left[\Bbbk[X]\right]$; (b) $\Omega(X)$ has non–empty interior.

This general result is consistent with the case of group–subgroup, because in the case that $H \subset G$, one has that $\Omega(G) = G$.

The characterization of observability in terms of the quasi–affineness of the homogeneous space G/H, also has a version in the generalized context guaranteeing the existence of a geometric quotient X/H in a principal H–invariant open subset of X.

For the above characterization of the observability of subgroups in terms of the extension of characters, one has also some partial results when generalizing: if the group H acting on the affine variety X is solvable (or if the variety is factorial), the action is observable if and only if the set of extendable characters is a group (the concept of extendable character can be defined in exactly the same manner as before). We consider this subject again in Section 7 when commwnting about L. Renner's later work.

Definition 6.2. If H is an affine algebraic group acting regularly on the affine variety X. A character $\chi : H \to \Bbbk$ is said to be extendable, if there is a non zero

[8] The equivalence of (4) with the other conditions is not fully proved in this survey, its complete proof can be found in [46, Thm. 11.2.9]. Also in Section 7 this topic is further discussed.

polynomial $f \in \Bbbk[X]$ with the property that $x \cdot f = \chi(x)f$, for all $x \in H$. We denote as $\mathrm{E}_H[X]$ the set of extendable characters –that is in fact a unital monoid with respect to the pointwise product.

It is interesting to notice that there is a close relation between the concepts of observable action and unipotency: indeed it can be shown that a group is *universally observable* (i.e. its action is observable in any variety where it acts rationally) if and only if it is unipotent.

The study –in the rather "opposite" direction– of *observable actions of reductive groups* is also interesting. For example, in OAAG it is shown that the action is observable if and only if the set of closed orbits of maximal dimension is not empty. Moreover, it can be proved that there is a maximal H–stable closed subset of the original variety, such that the restricted action is observable. In othere words, for reductive groups all the actions are generically observable.

Even though, the study by the mentioned authors of this generalized concept of observability has many other interesting results, in what follows we limit ourselves in this short survey to the three areas of results described above.

6.2. A characterization of observable actions

The result that follows is a first approximation to a geometric perspective of the concept of observable actions. Given a regular action of an affine algebraic group H on an affine variety X, if the algebra of invariants ${}^H\Bbbk[X]$ is finitely generated we say the the affinized quotient of X by H exists. In that situation we call $X/_{\mathrm{aff}}\, H$ the variety with the aforementioned algebra of invariants as polynomial algebra and call $\pi : X \to X/_{\mathrm{aff}}\, H$ the map associated to the natural inclusion ${}^H\Bbbk[X] \subseteq \Bbbk[X]$.

Theorem 6.3. *Assume that H is an affine group acting regularly on an irreducible affine variety X and suppose that the affinized quotient $\pi : X \to X/_{\mathrm{aff}}\, H$ exists. If all the fibers of π are (closed) orbits, then the action is observable.*

Proof. If $Y \subset X$ is a H–stable closed subset with dense image in $X/_{\mathrm{aff}}\, H$, then $\pi(Y)$ contains an open subset of $X/_{\mathrm{aff}}\, H$. Hence, using our hypothesis concerning the relationship between the fibers and the orbits, it follows that $Y = \pi^{-1}\big(\pi(Y)\big)$, and as $\pi^{-1}\big(\pi(Y)\big)$ contains an open subset of X we conclude that $Y = X$.

It follows that if $Y \subsetneq X$ is an H–stable closed subset strictly contained in X it cannot have dense image; therefore there exists $z \in (X/_{\mathrm{aff}}\, H) \setminus \overline{\pi(Y)}$. Let $f \in \Bbbk\big[X/_{\mathrm{aff}}\, H\big] = {}^H\Bbbk[X]$ be such that $f(z) = 1$ and $f\big(\overline{\pi(Y)}\big) = 0$. Then f is a non-zero invariant polynomial that is zero when restricted to Y. ☑

The theorem below characterizes the observability in terms of conditions for the invariant rational functions and a geometric condition on the orbits. The theorem just proved helps in the proof of one of the implications.

Theorem 6.4. *Let H be an affine group acting regularly on an irreducible affine variety X. Then the following conditions are equivalent:*

(1) The action of H on X is observable.

(2) The following two conditions are satisfied:

(a) Every invariant rational function on X is the quotient of two polynomials $\big[{}^H\Bbbk[X]\big] = {}^H\big[\Bbbk[X]\big]$.

(b) The set $\Omega(X)$ has nonempty interior.

Proof. We prove first that (1) $\Rightarrow$ (2). It follows from the definition of observability that there is an invariant function $f \in \Bbbk[X]$ with the property that $\varnothing \neq X_f \subset X^{\max}$, and then [46, Theorem 7.3.5] guarantees that $X_f \subseteq \Omega(X)$. This proves (b). Clearly $\big[{}^H\Bbbk[X]\big] \subseteq {}^H\big[\Bbbk[X]\big]$. Let $g \in {}^H\big[\Bbbk[X]\big]$, and consider the ideal $I = \big\{f \in \Bbbk[X] : fg \in \Bbbk[X]\big\}$. Then I is H–invariant, and hence there exists $0 \neq f \in {}^H\Bbbk[X]$ such that $fg \in {}^H\Bbbk[X]$, which proves (a).

In order to prove the converse, i.e. (2) $\Rightarrow$ (1), take $f \in {}^H\Bbbk[X]$ such that ${}^H\Bbbk[X_f]$ is finitely generated (the existence of such an element f is due to Grosshans in [11] and a proof appears also in [46, Theorem 7.5.6]). It is not hard to see that the action of H on X is observable if and only if the action on X_f is so. Thus, we can assume without loss of generality that ${}^H\Bbbk[X]$ is finitely generated. Let $\pi : X \to X/_{\mathrm{aff}}\, H$ be the affinized quotient, i.e. $X/_{\mathrm{aff}}\, H$ is the affine variety whose algebra of polynomial functions is ${}^H\Bbbk[X]$. By general results on affinized quotients (e.g. [46, Theorem 14.7.1]) there exists $f \in {}^H\Bbbk[X]$ such that $\pi^{-1}(y) = \overline{H \cdot x}$ for all $y \in V = (X/_{\mathrm{aff}}\, H)_f \cong X_f/_{\mathrm{aff}}\, H$. Moreover, for a certain $(X_f)_0$, an H–stable open subset of X_f, we have the following commutative diagram:

$$
\begin{array}{ccccc}
(X_f)_0 & \hookrightarrow & X_f & \hookrightarrow & X \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
(X_f)_0/H & \underset{\varphi}{\hookrightarrow} & X_f/_{\mathrm{aff}}\, H & \hookrightarrow & X/_{\mathrm{aff}}\, H
\end{array}
$$

where $\big(\rho,\, (X_f)_0/H\big)$ is a geometric quotient. Since $\Bbbk\big((X_f)_0/H\big) = {}^H\Bbbk\big((X_f)_0\big) = \Bbbk(X_f)^H$, it follows by hypothesis that $\Bbbk\big((X_f)_0/H\big) = \Bbbk(X_f/_{\mathrm{aff}}\, H)$. Since ρ and π separate closed orbits, it follows that φ is an open immersion.

Since $\Omega(X)$ contains a nonempty open subset, it follows that $\Omega(X) \cap (X_f)_0 \neq \varnothing$. Let $g \in {}^H\Bbbk[X]$ be such that $(X_f/_{\mathrm{aff}}\, H)_g \subset (X_f)_0/H$. If $y \in (X_f/_{\mathrm{aff}}\, H)_g$, then $\pi^{-1}(y) = \overline{O(x)}$, where $x \in \Omega(X) \cap (X_f)_0$, hence $\pi^{-1}(y)$ is a closed orbit of maximal dimension. Therefore, $\pi\big|_{X_{fg}} : X_{fg} \to (X_f/_{\mathrm{aff}}\, H)_g \cong X_{fg}/_{\mathrm{aff}}\, H$ is such that all its fibers are closed orbits. Replacing X by X_{fg}, we can hence assume that all the fibers of the affinized quotient are closed orbits. Therefore, the proof of the observability of the action now follows directly from Theorem 6.3. ☑

6.3. Observable actions and unipotency

By the very definitions, both the unipotency of a group as well as the observability of an action are conditions that can be formulated in terms of the existence of enough invariants for certain actions of the group in question. Therefore, it is natural to expect some close connection between both concepts. This is illustrated below by showing that an affine algebraic group that is "universally" observable has to be unipotent — compare also with the notion of *unipotent action* as defined in [44] or [45, Section 7].

To implement the proof we use a result appearing in [43], that guarantees that an affine algebraic group H is unipotent *if and only if* for all affine H–variety X the H–orbits on X are *closed*.

Theorem 6.5. *Let H be an irreducible affine algebraic group such that every action of H on an affine algebraic variety is observable. Then H is a unipotent group.*

Proof. We first prove that every H–orbit on an affine H–variety X is closed. Indeed, if $O \subset X$ is an orbit, then the action of H on the affine variety $\overline{O}$ is observable. Hence, changing X by $\overline{O}$, we may assume that X has an open (and dense) orbit O. If we call $I \subset \Bbbk[X]$ the H–stable ideal of $X \setminus O$, if this algebraic set is not empty, the ideal I is not zero. If $f \in \Bbbk[X]$ is a H–fixed not zero function in I, it is clear that f is constant on the orbit and hence on X. Thus, this constant function taking the value zero on a non empty set, has to be zero everywhere and this is a contradiction. Using the fact that we mentioned above, as all the orbits are closed we conclude that the group H is unipotent. ☑

6.4. Observable actions of reductive groups

In this section, following [36], we study the properties of observable actions when the acting group is reductive. It can be proved that given an action of H on an affine variety X there is a maximal closed H–subvariety of X such that the restricted action is observable.

Definition 6.6. Recall that if H is an affine group acting in the variety X, we define the socle of X –denoted as X_{soc} as:

$$X_{\mathrm{soc}} := \bigcup_x \{O(x) : \overline{O(x)} = O(x)\}.$$

Theorem 6.7. *Let H be reductive group acting on an affine algebraic variety X. Then the action is observable if and only if $\Omega(X) \neq \varnothing$. In particular, X_{soc} is the largest H–stable closed subset $Z \subset X$ such that the restricted action $H \times Z \to Z$ is observable.*

Proof. If the action is observable, it follows from Theorem 6.4 that $\Omega(X) \neq \varnothing$. Assume now that $\Omega(X) \neq \varnothing$ and let $Z \subsetneq X$ be a H–stable closed subset and call I the ideal associated to Z; we want to show that ${}^H I \neq \{0\}$. If $\Omega(X) \subset Z$ it follows that $Z = X$; hence $\Omega(X) \setminus Z \neq \varnothing$. Recall that the semi–geometric quotient $\pi : X \to X/H = \operatorname{Spm}\big({}^H\Bbbk[X]\big)$ separates closed orbits –Spm is the maximal spectrum functor. It follows that $\Omega(X) \setminus \pi^{-1}\big(\pi(Z)\big) \neq \varnothing$, since the closed orbits belonging to Z and $\pi^{-1}\big(\pi(Z)\big)$ are the same. Let $O \subset \Omega(X) \setminus Z$ be a closed orbit. Then $\pi^{-1}\big(\pi(O)\big) = O$, again because π separates closed orbits. Since π also separates H–stable closed subsets, it follows that there exists $f \in {}^H\Bbbk[X]$ such that $f \in I' \subset I$ where I' is the ideal of $\pi^{-1}\big(\pi(Z)\big)$ and $f(O) = 1$; in particular, $f \in {}^H I \setminus \{0\}$ and the action is observable.

It follows by the very definition of X_{soc} that $\Omega(X_{\mathrm{soc}}) \neq \varnothing$. Let Z be an H–stable irreducible closed subset such that the restricted action is observable; then $\Omega(Z)$ is a nonempty open subset of Z, consisting of closed orbits in Z, and hence in X. It follows that $Z = \overline{\Omega(Z)} \subset X_{\mathrm{soc}}$. If Y is any H–stable closed subset, it can be proved that the restriction of the action to any irreducible component Z is observable, and hence $Y \subset X_{\mathrm{soc}}$. ☑

Theorem 6.8. *Let H be a reductive group acting on an affine variety X and call I_0–the ideal associated to X_{soc}–. Then I_0 is the largest H–stable ideal such that ${}^H I = (0)$.*

Proof. Let $I = \sum\{J : {}^H J = (0)\}$ be the sum of all H–stable ideals such that ${}^H J = (0)$, and consider the canonical H–morphism $\varphi : \bigoplus\{J : {}^H J = (0)\} \to I$. Since φ is surjective, it follows from the reductivity of H that for every $f \in {}^H I$ there exist $n \geq 0$ and $h \in {}^H \bigoplus\{J : {}^H J = (0)\} = (0)$ such that $\varphi(h) = f^{p^n}$, where $\operatorname{char}\Bbbk = p$, then as our algebras are free of nilpotents, we deduce that ${}^H I = (0)$.

Let $O \subset X$ be a closed orbit, call Z the set of zeros of I and assume that $O \cap Z = \varnothing$. Since ${}^H\Bbbk[X]$ separates H–stable closed subsets, if follows that there exists $f \in {}^H\Bbbk[X]$ such that $f\big|_O = 1$ and $f\big|_Z = 0$, hence ${}^H I \neq (0)$ and we get a contradiction. Therefore, $X_{\mathrm{soc}} \subset Z$.

Observe that if $f \in {}^H\big(\sqrt{I}\big)$ is such that $f^n \in I$, it follows that for any $a \in H$, then $a \cdot (f^n) = f^n \in I$, and hence $f = 0$. Thus, ${}^H\big(\sqrt{I}\big) = (0)$ and by maximality then $I = \sqrt{I}$. By Theorem 6.7, if we prove that the action $H \times Z \to Z$ is observable (Z is the set of zeros of I), then $X_{\mathrm{soc}} = Z$. But $\Bbbk[Z] \cong \Bbbk[X]/I$, and hence the H–stable ideals of $\Bbbk[Z]$ are of the form J/I, were $J \subset \Bbbk[X]$ is an H–stable ideal containing I. Then if $J/I \neq (0)$ it follows that $I \subsetneq J$ and hence, by maximality of I, ${}^H J \neq (0)$. Thus, ${}^H(J/I) \neq (0)$, since ${}^H\Bbbk[X]$ injects in $\Bbbk[X]/I$. ☑

7. A glimpse into some recent contributions

In this section, even more sketchly than in the preceeding ones, we look at certain important recent results around the concept of observability. Our description centers on several papers by Lex Renner [37], [38], [39] and [40] (2012/2015), that are a natural continuation of previous work of its author with A. Rittatore (see Section 6 in this paper and also [36] (2010)) where a wide generalization of the concept of observability is presented. By expanding the platform of observability from the case of an action a subgroup acting by translations in a larger group, to the situation of actions of a group in a general affine variety, a vast range of new questions and problems were opened. Many of these are addressed in the papers mentioned above.

Along this section we assume that X is an irreducible variety.

7.1. Visible and stable actions, extendable characters, semi–observability

In [37] and [38], L. Renner introduces the concepts that appear in the title of this section (the concept of stability was introduced previously by Popov), the importance of its introduction can be appreciated in the light of the conditions displayed in Section 6.1 –particulary the equivalent conditions (1), (3) and (4). In the case of a general action the conditions are not equivalent as in the context of a subgroup action by translation, and hence, a detailed study of the role played by each, seems necessary and hence it is natural that they are given specific names. The concept of semi–observability[9] is also invisible in the group/subgroup situation as it is remarked in the comments after Definition 7.6.

Definition 7.1. A regular action $X \times H \to X$ of an affine algebraic group on an affine variety, is said to be *visible*, if every affine invariant rational function is the quotient of two invariant polynomials.

Concerning the above definition notice that the inclusion $[{}^H\Bbbk[X]] \subseteq {}^H[\Bbbk[X]]$ is always true.

We recall the concept of stable action that has been studied for quite a long time and plays along very well with the above considerations. It seems to be due V. Popov (in 1972) and an english translation apperead in [34].

Definition 7.2. Let $X \times H \to X$ be a regular action of the affine algebraic group on an affine variety X. The action is *stable*, if there exists an open non–empty subset $U \subset X$ with the property that the orbits of its points are closed.

Using the new nomenclature it is clear that we can reformulate Theorem 6.4 as follows.

[9]Notice that we use the name of *semi–observability* for the concept that in the mentioned papers is named as χ*–observability*. The new name seems to be more consistent with the use of the expression *semi–invariant* as a weakening of the notion of invariant by adding a character. Also, whereas the expression "observability" is related to the existence of invariant polynomials, in the present context what is guaranteed by the semi–observability is the existence of a semi–invariant polynomial.

Theorem 7.3. *In the situation of Definition 7.1, the action of H on X is observable if and only if it is visible and stable.*

Considering the situation of actions by translations of subgrups in larger groups and using the new nomenclature, the theorem of ERA mentioned in Section 6.1 reads as follows: the action is visible if and only if the subgroup is observable. In the case of general actions, the example of the group $\mathrm{GL}_n(\Bbbk)$ acting on $\mathrm{M}_n(\Bbbk)$ by translations (see [36, Example 3.11]), shows that the converse is not true.

In [37, Thm.2.8], besides writing down this simplified statement of the main theorem of [36], the author provides a simplified proof. Therein the concept of visibility is also studied in relation with many of the standard themes in invariant theory. We only mention the following: assume that H is a connected affine algebraic group that admits a Levi decomposition of the form $G = L \times U$ with L reductive and U unipotent, then any stable action is visible. The proof proceeds by reducing first to the situation of a reductive group and then applying standard results of the invariant theory of reductive groups.

The concept of semi–invariant has been an important tool since the very origins of invariant theory in the middle of the 19th. century, specially in order to construct new invariants from old ones. Its relationship with observability was first observed in the seminal work by Bialinicki–Birula, Hochschild and Mostow (ERA, [2, Thm.1, Thm.3, Thm. 9]).

The concept of extendable character appeared associated to any semi–invariant polynomial in the group/subgroup situation. For a general action of the form $X \times H \to X$ the monoid of extendable characters is defined and used in the theory for the first time in [36], see Definition 6.2.

Already in ERA was observed that the algebraic structure of the monoid of extendable characters is very relevant for the understanding of observability, but the precise formulation of this relationship appeared later in the literature (see [12], [6] and [46]): H is observable in G if and only if all the characters of H are extendable (see the beginning of Section 6.1). In the case of a general action of an affine group H in an affine variety X the situation is more complex and it is described below.

Lemma 7.4. *If H is an affine algebraic group acting regularly in an affine variety X.*

(1) If the action is observable, then $\mathrm{E}_H[X]$ is a group.

(2) Assume that $\mathrm{E}_H[X]$ is a group and that every H invariant rational function in X is the quotient of two χ semi–invariant polynomials for some $\chi \in \mathrm{E}_H[X]$, then the action of H on X is visible.

Proof. (1) Assume that $\chi \in \mathrm{E}_H[X]$ and call f its associated semi–invariant polynomial. Consider the ideal generated by f, using the observability hypothesis we produce a polynomial g with the property that fg is invariant. Hence g is semi–invariant with character χ^{-1}.

(2) If $F \in {}^H[\Bbbk[X]]$ is written as $F = f/g$ for f, g two non zero χ invariants for some $\chi \in \mathrm{E}_H[X]$. Our hypothesis guarantees the existence of $0 \neq h \in \Bbbk[X]$ such that is a χ^{-1}–semi–invariant. Then $F = fh/gh$ belongs to $[{}^H\Bbbk[X]]$.

☑

From the above we deduce the following result that appears in [36]. The proof is based in the well known fact that in a factorial affine variety, every invariant rational function is the quotient of two χ–invariant polynomials for some character χ.

Theorem 7.5. *If H is an affine algebraic group acting regularly in a affine factorial variety X. Then, the following are equivalent.*

(1) The action of H on X is observable;

(2) The action is stable and $\mathrm{E}_H[X]$ is a group.

Another important concept introduced in the set of papers under consideration is the concept of semi–observability of an action of an affine algebraic group H in a variety X (where it is named as χ–observability).

Definition 7.6. We say that the action of an affine algebraic group H on an affine variety X is semi–observable if for every H–stable prime ideal $I \subseteq \Bbbk[X]$ there is a $\chi \in \mathrm{E}_H[X]$ such that $I_\chi \neq 0$, where $I_\chi = \{f \in I : \forall x \in H, x \cdot f = \chi(x)f\} \subseteq \Bbbk[X]_\chi \subseteq \Bbbk[X]$

Observation 7.7. Assume that $H \subseteq G$ is a closed inclusion of affine algebraic groups. Then the action by translations of H on G is semi–observable. This is the content of [46, Cor. 8.2.5] and it follows more or less directly from classical results of Chevalley on the definition of the subgroup H by semi–invariants. In that sense this result is already implicit in [2, §2].

This observation and the Theorem that follows, provides an explanation that closes the gap between the theory of actions of a subgroup on a group and the theory of general actions, by showing that the main obstruction for the validity of the neat equivalence results listed in Section 6.1, lies in the fact that in the general situation the semi–observability is not guaranteed. The proof of the theorem that follows is an adaptation to the current context of the procedures of [46, Thm.11.2.9]. Compare also with Theorem 2.4.

Theorem 7.8. [38, Thm.3.5] *Assume that the action $X \times H \to X$ is semi–observable. Then the following are equivalent.*

(1) The action $X \times H \to X$ is observable;

(2) The action $X \times H \to X$ is observable in codimension one (see Definition 7.10);

(3) $\mathrm{E}_H[X]$ *is a group.*

Proof. The proof that (1) $\Rightarrow$ (2) is clear. Assuming (2) if we take $\chi \in \mathrm{E}_H[X]$ and the corresponding non zero χ–invariant polynomial, the principal ideal generated by f is a non–zero of pure height one H–stable ideal of $\Bbbk[X]$. Hence, we can find a polynomial $g \in \Bbbk[X]$ such that fg is H–fixed. Hence, g is a non–zero χ^{-1} semi–invariant. Assuming (3) if we take a non–zero H stable ideal of $\Bbbk[X]$, using they basic hypothesis of semi–observability, we can find a character $\chi \in \mathrm{E}_H[X]$ with an associated polynomial $f \in \Bbbk[X]$ that is χ–semi–invariant. As $\mathrm{E}_H[X]$ is a group, there is a non zero polynomial g associated to the character χ^{-1} and then $fg \in {}^H I$ is a non zero invariant polynomial. ☑

A very interesting consequence of the semi–observability of the action is the following.

Corollary 7.9. [38, Cor.3.7] *Assume that the action $X \times H \to X$ is semi–observable. Then the orbits in general position are affine.*

Proof. From the semi–observability we deduce the existence of a semi–invariant $f \in \Bbbk[X]$ such that the action on X_f is observable (use the above theorem), From Theorem 7.3 we deduce that a typical H–orbit in X_f is closed in X_f and hence it is affine. ☑

7.2. Observability in codimension one

Heretofore we described the neat manner in which –in the three articles we have been considering ([36, 37, 38])– the generalized concept of observability has been characterized in terms of the conditions of visibility and stability of the action –with the ingredient of the semi–stability playing along. Next we consider other characterization that has a strong geometrical content and that involves the notion of *observability in codimension one.* The author states that "the purpose of this paper is to identify the study of such actions as an important part of invariant theory" (c.f. [38]).

First notice that the definition of observable action (Definition 6.1) can be reformulated algebraically as follows. Let $X \times H \to X$ is a regular action of the affine algebraic group on the affine variety X. We say that the action is observable, if for every H–stable prime ideal $I \subseteq \Bbbk[X]$ the set of invariants ${}^H I = I \cap {}^H\Bbbk[X] \neq \{0\}$.

The following definition –that weakens the one above– is explored in [38].

Definition 7.10. Assume that $X \times H \to X$ is a regular action of the affine algebraic group on the affine variety X. We say that the action is observable in codimension one, if for every H–stable prime ideal $J \subseteq \Bbbk[X]$ of height one, the set of invariants ${}^HJ = J \cap {}^H\Bbbk[X] \neq \{0\}$

Below (using the notations of Definitions 7.1,7.2, 7.6,7.10) we describe part of the results obtained relating general observability with observability in codimension one (and the other concepts considered above) that appear with complete proofs in [38] (the labels of the items correspond to the numeration of the results in the mentioned article).

Prop.2.2 If X is normal, and observable in codimension one, the action of H is visible.

Thm.2.5 If the action is visible, the following conditions are equivalent:

(1) The action is observable in codimension one;
(2) If $Y \subset X$ is closed, H–stable and of pure codimension one, then there is a non–zero invariant polynomial that is zero in Y.
(3) There is a dense open set of orbits U such that for all $x \in U$ $\operatorname{codim}_{\overline{H\cdot x}}(\overline{H \cdot x} \setminus (H \cdot x)) \geq 2$.

Thm.2.9 The following conditions are equivalent:

(1) The action is observable;
(2) (a) The action is visible;
    (b) The action is observable in codimension one;
    (c) The typical orbit is affine.

We only say some words about the proof of the last item Thm.2.9. The results mentioned above Theorem 7.8 and Corollary 7.9, guarantee that (1) $\Rightarrow$ (2). Take $x \in X$, from the affiness we deduce that of Hx is closed or $\dim(\overline{Hx} \setminus Hx) = \dim(Hx) - 1$ but from Thm.2.5 (3) mentioned above, the generic situation for an orbit is its closedness, hence the action is stable. As it is visible by hypothesis, we deduce from the characterization of observability following Definition 7.2 that the proof is finished.

The results mentioned above, are retaken in [39] and the author obtains some strengthenings. For example, the following is proved in Theorem 3.4: assuming that X and H are irreducible, the ensuing properties for an action $X \times H \to X$, are equivalent: (1) the field extension $[{}^H\Bbbk[X]] \subseteq {}^H[\Bbbk[X]]$ is finite and there is an $0 \neq f \in {}^H\Bbbk[X]$ such that $\operatorname{codim}_{X_f}(X_f \setminus (X_f)_{\max}) \geq 2$; (2) the action is observable in codimension one. In the same vein, the author proves in the second paper a version of Thm.2.9. without condition (a).

7.3. The adjoint action

One of the more studied actions in group theory is the action by conjugation $(x, h) \mapsto h^{-1}xh : \underline{G} \times G \to \underline{G}$. For example, in the case of affine algebraic groups, the subject of the geometric structure of the orbits by this action, has reached a high degree of development –see for example the exposition in [22] or the more classical [49]. It is natural to apply the general observability techniques developped for actions of groups on varieties in the particular context of the action by conjugation. In [40, Theorem 3.11] the author proves that the adjoint action is observable. As the author says in the abstract of the paper: *A major step in our proof is to show that the adjoint action is induced generically from the conjugating action of $N_G(T)$ (the normalizer of a maximal torus) on a certain open subset of $C_G(T)$ (the centralizer of the torus).*

8. Final remarks

Arising in the late 1950s and early 1960s from questions about the existence of faithful representations of Lie groups, the concept of observability in its development along almost sixty years, reached out in a profitable interaction with most of the crucial themes of –geometric and algebraic– invariant theory. Even though the generalizations and the tools that have been introduced, have attained a high degree of sophistication, we have tried to show along the current survey that many of the new developments can be seen as the natural evolution of concepts, ideas and results already contained in the initial paper [2], written in 1963. Today the original concept, together with its generalizations, have grown into an important component of the toolkit of modern invariant theory.

References

[1] N. Andruskiewitsch and W. Ferrer Santos, *On observable module categories*, in Groups, Rings and Group Rings, eds. A. Gianbruno, C. Polcino Milies, K.S. Sehgal. Lecture Notes in Pure and Appl. Math. No 248, Chapman & Hall/CRC, Boca Raton (2006), 11–23.

[2] A. Bialynicki-Birula, G. Hochschild, and D. Mostow, *Extension of representations of algebraic linear groups*, Amer. J. Math. **85** (1963), 131–144.

[3] A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics, Vol 126, 2d. enlarged ed., Springer-Verlag, New York, 1991.

[4] C. Chevalley, *Théorie des groupes de Lie. Tome II. Groupes algébriques*, Actualités Sci. Ind. no. 1152, Hermann & Cie., Paris, 1951.

[5] W.-L. Chow, *On the projective embedding of homogeneous varieties*, in: Algebraic geometry and topology. A symposium in honor of S.Lefschetz, pp 122–128, Princeton University Press, Princeton, N. J., 1957.

[6] E. Cline, B. Parshall, and L. Scott, *Induced modules and affine quotients*, Math. Ann. **230** (1977), no. 1, 1–14.

[7] Y. Doi, *On the structure of relative Hopf modules*, Comm. Algebra **11** (1983), no. 3, 243–253.

[8] ______, *Algebras with total integrals*, Comm. Algebra **13** (1985), no. 10, 2137–2159.

[9] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Comm. Algebra **14** (1986), no. 5, 801–818.

[10] F. D. Grosshans, *Observable groups and Hilbert's fourteenth problem*, Amer. J. Math. **95** (1973), 229–253.

[11] ______, *Localization and invariant theory*, Adv. in Math. **21** (1976), no. 1, 50–60.

[12] ______, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Mathematics, Vol. 1673, Springer–Verlag, Berlin,, 1997.

[13] W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. **102** (1975), no. 2, 67–83.

[14] ______, *Homogeneous vector bundles and reductive subgroups of reductive algebraic groups*, Amer. J. Math. **100** (1978), 1123–1137.

[15] D. Hilbert, *Problèmes futures de mathématiques*, Comptes Rendus du $2^{ème}$ Congrès International des Mathématiciens, Gauthier-Villars, Paris (1902), 58–111.

[16] G. Hochschild, *Cohomology of algebraic linear groups*, Illinois J. Math. **5** (1961), 492–519.

[17] ______, *Rationally injective modules for algebraic linear groups*, Proc. Amer. Math. Soc. **14** (1963), 880–883.

[18] ______, *The structure of lie groups*, Holden-Day, San Francisco, 1965.

[19] ______, *Basic theory of algebraic groups and Lie algebras*, Graduate Texts in Mathematics, Vol 75, Springer-Verlag, New York, 1981.

[20] G. Hochschild and G. D. Mostow, *Extension of representations of Lie groups and Lie algebras, I.*, Amer. J. of Math. **79** (1957), 924–942.

[21] ______, *Unipotent groups in invariant theory*, Proc. Nat. Acad. Sci. U.S.A. **70** (1973), 646–648.

[22] J. Humphreys, *Conjugacy classes in semisimple algebraic groups*, AMS Mathematical Surveys and Monographs **43** (1995), 196.

[23] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, Vol 21, Springer-Verlag, New York, 1975.

[24] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78.

[25] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate texts in mathematics, no. 5, Springer–Verlag, New York, 1998.

[26] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. Jour. **16** (1960), 205–218.

[27] S. Montgomery, *Hopf algebras and their action on rings*, CMBS Regional Conference Series in Mathematics, vol. 82, Published by the AMS, Providence, Rhode Island, 1993.

[28] G. D. Mostow, *Extensions of representations of Lie groups, II*, Amer. J. of Math. **80** (1958), 331–347.

[29] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol 34, Springer-Verlag, Berlin, 1965.

[30] ______, *Hilbert'os fourteenth problem - the finite generation of subrings such as rings of invariants*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), Amer. Math. Soc.., Providence, R. I., 1976.

[31] M. Nagata, *On the fourteenth problem of Hilbert*, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York (1960), 459–462.

[32] ______, *Lectures on the fourteenth problem of hilbert*, Tata Institute of Fundamental Research, Bombay, 1965.

[33] U. Oberst, *Affine Quotientenschemata nach affinen, algebraischen Gruppen und induzierte Darstellungen*, J. Algebra **44** (1977), no. 2, 503–538.

[34] V. Popov, *On the stability of the action of an algebraic group on an algebraic variety*, Math. USSR Izv. **6** (1973), 367–379.

[35] V. L. Popov and E. B. Vinberg, *Invariant theory, algebraic geometry. iv*, Enciclopae- dia of Mathematical Sciences, vol. 55, Springer-Verlag, Berlin, 1994, Linear algebraic groups. Invariant theory, A translation of Algebraic geometry. 4 (Russian), Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation edited by A. N. Parshin and I. R. Shafarevich.

[36] L. Renner and A. Rittatore, *Observable actions of algebraic groups*, Transform. Groups **14** (2010), 985–999, arXiv:0902.0137v2[math.AG].

[37] L. E. Renner, *Orbits and invariants of visible group actions*, Transform. Groups **14** (2012), no. 4, 1191–1208.

[38] ______, *Observability in invariant theory*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 205–216.

[39] ______, *Observability in invariant theory II: Divisors and rational invariants*, Proc. Amer. Math. Soc. **143** (2015), no. 10, 4113–4121.

[40] ______, *The adjoint action is observable*, J. Algebra **441** (2015), 159–165.

[41] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), no. 1, 38–41.

[42] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

[43] W. Ferrer Santos, *Closed conjugacy classes, closed orbits and structure of algebraic groups*, Comm. Algebra **19** (1991), no. 12, 3241–3248.

[44] ______, *A generalization of the concept of linearly and geometrically reductive group*, Topics in Algebraic Geometry. Proceedings of the workshop on algebraic geometry. Eds. L. Brambila, X. Gómez-Mont. Guanajuato, Mexico, 1989. Aportaciones Matemáticas, Notas de Investigación , vol 5. Sociedad Matemática Mexicana, 1992.

[45] W. Ferrer Santos and A. Rittatore, *Linearly reductive and unipotent actions of affine groups*, International Journal of Mathematics **26** (2015), no. 5, 1–31.

[46] ______, *Actions and Invariants of Algebraic Groups. 2d. Edition*, Series: Monographs and Research Notes in Mathematics, CRC Press, Florida, 2017.

[47] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel, J. of Mat. **72** (1990), no. 1-2, 167–195.

[48] T. A. Springer, *Linear algebraic groups, 2 ed*, Modern Birkhäuser classics, Birkhäuser, Basel.

[49] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Mathematics, Vol. 366, Springer-Verlag, Berlin, 1974.

[50] M. Sweedler, *Hopf algebras*, Mathematics Lecture Notes Series, W.A. Benjamin, inc.,New York, 1969.

[51] A. Weil, *On algebraic groups and homogeneous spaces*, Amer. J. Math. **77** (1955), 493–512.

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