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Generalized Bilateral Eighth Order Mock Theta Functions and Continued Fractions

Funciones Theta Mock generalizadas en dos variables de orden ocho

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Abstract. We give a two independent variable generalization of bilateral eighth order mock theta functions and expressed them as infinite product. On specializing parameters, we have given a continued fraction representation for the generalized function, which I think is a new representation.

Key words and phrases. q-Hypergeometric series, Mock theta functions, Continued fraction.

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Resumen. En esta contribución se obtienen funciones Theta Mock generalizadas de orden ocho en dos variables que se expresan mediante un producto infinito. Para valores particulares de los parámetros se deducen representaciones de dichas funciones mediante fracciones continuas.

Palabras y frases clave. funciones Theta Mock, fracción continuada, orden ocho en dos variables.

1. Introduction

Ramanujan in his last letter to Hardy [4, pp 354-355] defined 17 functions and called them mock theta functions. In 2000, Gordon and Mclntosh [2] gave eight new mock theta functions and called them of eighth order. Most of the work done on mock theta functions was on unilateral mock theta functions that is, for $(n > 0)$ or $(n < 0)$, where n denotes the summation index.

In the present paper, we first generalize these eighth order mock theta functions of Gordon and Mclntosh by introducing two independent variables and then consider their bilateral form. We express the bilateral mock theta functions as a sum of two unilateral mock theta functions and then express them as infinite products. The advantage of two variable generalization is that we have two extra parameters and by specializing them we get some known functions. In the last section we have represented generalized functions as Gollnitz-Gordon continued fraction, which I think is a new representation. We also prove some identities. We shall be considering only four of the eighth order mock theta functions as the rest give analogous results.

The eighth order mock theta functions of Gordon and Mclntosh [2] are

$$S_0(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(-q^2;q^2)_n},\qquad S_1(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(-q^2;q^2)_n}, \tag{1}$$

$$T_0(q)=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+3n+2}}{(-q;q^2)_{n+1}},\qquad T_1(q)=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n^2+n}}{(-q;q^2)_{n+1}}, \tag{2}$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},\qquad U_1=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \tag{3}$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}=-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}}, \tag{4}$$

$$V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}, \tag{5}$$

where $(a;q^k)_n=\prod_{j=0}^{n-1}(1-aq^{kj}),(a;q^k)_\infty=\prod_{j=0}^{\infty}(1-aq^{kj})$ and $(a;q^k)_0=1$.

2. Two Variable Generalized Eighth Order Mock Theta Functions

We define generalized eighth order mock theta functions by introducing two independent variables :

$$S_0(\alpha,z;q)=\sum_{n=0}^{\infty}\frac{(-zq;q^2)_n q^{n^2}\alpha^n}{(-\alpha q^2;q^2)_n}, \tag{6}$$

$$S_1(\alpha,z;q)=\sum_{n=0}^{\infty}\frac{(-zq;q^2)_n q^{n^2+2n}\alpha^n}{(-\alpha q^2;q^2)_n}, \tag{7}$$

$$T_0(\alpha, z; q) = \sum_{n=0}^{\infty} \frac{(-q^2/\alpha; q^2)_n q^{n^2+3n+2}}{(-q/z; q^2)_{n+1} z^{n+1}}, \tag{8}$$

$$T_1(\alpha, z; q) = \sum_{n=0}^{\infty} \frac{(-q^2/\alpha; q^2)_n q^{n^2+n}}{(-q/z; q^2)_{n+1} z^{n+1}}. \tag{9}$$

For $\alpha = 1$ and $z = 1$ they reduce to eighth order mock theta functions of Gordon and Mclntosh given in (1)-(2).

We now give their bilateral form

$$S_{0,c}(\alpha, z; q) = \sum_{n=-\infty}^{\infty} \frac{(-zq; q^2)_n q^{n^2} \alpha^n}{(-\alpha q^2; q^2)_n}, \tag{10}$$

$$S_{1,c}(\alpha, z; q) = \sum_{n=-\infty}^{\infty} \frac{(-zq; q^2)_n q^{n^2+2n} \alpha^n}{(-\alpha q^2; q^2)_n}, \tag{11}$$

$$T_{0,c}(\alpha, z; q) = \sum_{n=-\infty}^{\infty} \frac{(-q^2/\alpha; q^2)_n q^{n^2+3n+2}}{(-q/z; q^2)_{n+1} z^{n+1}}, \tag{12}$$

$$T_{1,c}(\alpha, z; q) = \sum_{n=-\infty}^{\infty} \frac{(-q^2/\alpha; q^2)_n q^{n^2+n}}{(-q/z; q^2)_{n+1} z^{n+1}}. \tag{13}$$

We have put c in the suffix following Watson, who called the bilateral sum as complete. Now

$$\begin{aligned} S_{0,c}(\alpha, z; q) &= \sum_{n=0}^{\infty} \frac{(-zq; q^2)_n q^{n^2} \alpha^n}{(-\alpha q^2; q^2)_n} + \left(1 + \frac{1}{\alpha}\right) \sum_{n=0}^{\infty} \frac{(-q^2/\alpha; q^2)_n q^{n^2+3n+2}}{(-q/z; q^2)_{n+1} z^{n+1}}, \\ &= \sum_{n=-\infty}^{\infty} \frac{(-zq; q^2)_n q^{n^2} \alpha^n}{(-\alpha q^2; q^2)_n}. \end{aligned} \tag{14}$$

So

$$\sum_{n=-\infty}^{\infty} \frac{(-zq; q^2)_n q^{n^2} \alpha^n}{(-\alpha q^2; q^2)_n} = S_0(\alpha, z; q) + \left(1 + \frac{1}{\alpha}\right) T_0(\alpha, z; q), \tag{15}$$

and similarly

$$\sum_{n=-\infty}^{\infty} \frac{(-zq; q^2)_n q^{n^2+2n} \alpha^n}{(-\alpha q^2; q^2)_n} = S_1(\alpha, z; q) + \left(1 + \frac{1}{\alpha}\right) T_1(\alpha, z; q). \tag{16}$$

Here we have used

$$(a;q)_{-n}=\frac{(-q/a)^n q^{n(n-1)/2}}{(q/a;q)_n},$$

to get the second sum in the right side of (14).

3. Bilateral Sum as Infinite Product

We now prove the following two theorems which express the bilateral sum as infinite product.

Theorem 3.1.

$$\begin{aligned}S_0(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_0(\alpha,z;q^2)&=\frac{(q^2;q^2)_\infty}{2(-\alpha q^4,-q^2/z;q^4)_\infty}\\&\times\Big[(\sqrt{\alpha z}q,\sqrt{\alpha/z}q,q/\sqrt{\alpha z};q^2)_\infty\\&+(-\sqrt{\alpha z}q,-\sqrt{\alpha/z}q,-q/\sqrt{\alpha z};q^2)_\infty\Big].\end{aligned}\tag{17}$$

Theorem 3.2.

$$\begin{aligned}S_1(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_1(\alpha,z;q^2)&=\frac{(q^2;q^2)_\infty}{2\sqrt{\alpha z}q(-\alpha q^4,-q^2/z;q^4)_\infty}\\&\times\Big[(-\sqrt{\alpha z}q,-\sqrt{\alpha/z}q,-q/\sqrt{\alpha z};q^2)_\infty\\&-(\sqrt{\alpha z}q,\sqrt{\alpha/z}q,q/\sqrt{\alpha z};q^2)_\infty\Big]\end{aligned}\tag{18}$$

Proof of Theorem 3.1. To prove Theorem 3.1, we use the following transformation formula of Bailey [1, eq.(11.33), p.239]

$$\begin{aligned}&{}_6\psi_6\left[\begin{matrix}qa^{1/2},-qa^{1/2},b,c,d,e\\a^{1/2},-a^{1/2},aq/b,aq/c,aq/d,aq/e\end{matrix};q;\frac{qa^2}{bcde}\right]\\&=\frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty}.\end{aligned}\tag{19}$$

Replacing $q\to q^2$ and putting $b=i\sqrt{z}q,\ c=-i\sqrt{z}q,\ a^2=z\alpha q^2$ and letting $d,e\to\infty$ in the above ${}_6\psi_6$ summation formula, the left-hand side of (19)

$$\begin{aligned}&=\sum_{n=-\infty}^{\infty}\frac{(-zq^2;q^4)_n q^{2n^2}\alpha^n}{(-\alpha q^4;q^4)_n}\left[\frac{1-\sqrt{z\alpha}q^{4n+1}}{(1-\sqrt{z\alpha}q)}\right]\\&=\frac{1}{(1-\sqrt{z\alpha}q)}\sum_{-\infty}^{\infty}\frac{(-zq^2;q^4)_n q^{2n^2}\alpha^n}{(-\alpha q^4;q^4)_n}-\frac{\sqrt{\alpha z}}{(1-\sqrt{z\alpha}q)}q\sum_{-\infty}^{\infty}\frac{(-zq^2;q^4)_n q^{2n^2+4n}\alpha^n}{(-\alpha q^4;q^4)_n}.\end{aligned}\tag{20}$$

By equation (15) and (16) the above

$$= \frac{S_0(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_0(\alpha,z;q^2)}{(1-\sqrt{z\alpha}q)} - \sqrt{\alpha z}q\frac{S_1(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_1(\alpha,z;q^2)}{(1-\sqrt{z\alpha}q)}.$$

The right side of (19) is

$$\begin{aligned}
&= \frac{(q^2;q^2)_\infty(\sqrt{\alpha z}q^3;q^2)_\infty(\sqrt{\alpha/z}q;q^2)_\infty(q/\sqrt{\alpha z};q^2)_\infty}{(-q^2/z;q^4)_\infty(-\alpha q^4;q^4)_\infty}\\
&= \frac{1}{(1-\sqrt{z\alpha}q)}\frac{(q^2;q^2)_\infty(\sqrt{\alpha z}q;q^2)_\infty(\sqrt{\alpha/z}q;q^2)_\infty(q/\sqrt{\alpha z};q^2)_\infty}{(-q^2/z;q^4)_\infty(-\alpha q^4;q^4)_\infty}\\
&= \frac{1}{(1-\sqrt{z\alpha}q)}\frac{(q^2;q^2)_\infty(\sqrt{\alpha z}q;q^2)_\infty(\sqrt{\alpha/z}q;q^2)_\infty(q/\sqrt{\alpha z};q^2)_\infty}{(-q^2/z;q^4)_\infty(-\alpha q^4;q^4)_\infty}.
\end{aligned}$$

Finally we have

$$\begin{aligned}
&\left[S_0(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_0(\alpha,z;q^2)\right]-\sqrt{\alpha z}q\left[S_1(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_1(\alpha,z;q^2)\right]\\
&= \frac{(q^2;q^2)_\infty(\sqrt{\alpha z}q,\sqrt{\alpha/z}q,q/\sqrt{\alpha z};q^2)_\infty}{(-q^2/z;q^4)_\infty(-\alpha q^4;q^4)_\infty}. \qquad (21)
\end{aligned}$$

Writing $-q$ for q in (21), we have

$$\begin{aligned}
&\left[S_0(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_0(\alpha,z;q^2)\right]+\sqrt{\alpha z}q\left[S_1(\alpha,z;q^2)+\left(1+\frac{1}{\alpha}\right)T_1(\alpha,z;q^2)\right]\\
&= \frac{(q^2;q^2)_\infty(-\sqrt{\alpha z}q,-\sqrt{\alpha/z}q,-q/\sqrt{\alpha z};q^2)_\infty}{(-q^2/z;q^4)_\infty(-\alpha q^4;q^4)_\infty}. \qquad (22)
\end{aligned}$$

On adding and subtracting equations (21) and (22) we get Theorem 3.1 and Theorem 3.2. ☑

4. Application of Theorem 3.1 and Theorem 3.2

We now prove the following theorems which are application of Theorem 3.1 and Theorem 3.2.

Theorem 4.1.

$$\begin{aligned}
\sum_{r=-\infty}^{\infty}\frac{q^{4r^2}/(\alpha z)^r}{(-q^2/z;q^2)_{2r}} &= \frac{(q^2;q^2)_\infty}{2(\alpha q^2/z,-q^4/z,-q^2/z;q^4)_\infty}\\
&\times\Big[(\sqrt{\alpha z}q,\sqrt{\alpha/z}q,q/\sqrt{\alpha z};q^2)_\infty\\
&\qquad + (-\sqrt{\alpha z}q,-\sqrt{\alpha/z}q,-q/\sqrt{\alpha z};q^2)_\infty\Big].
\end{aligned}$$

Theorem 4.2.

$$\sum_{r=-\infty}^{\infty} \frac{q^{4r^2+4r}/(\alpha z)^{r+1}}{(-q^2/z;q^2)_{2r+1}} = \frac{(1+z)(q^2;q^2)_\infty}{2z\sqrt{\alpha z}q(\alpha q^2/z,-1/z,-q^2/z;q^4)_\infty} \times \Big[(-\sqrt{\alpha z}q,-\sqrt{\alpha/z}q,-q/\sqrt{\alpha z};q^2)_\infty - (\sqrt{\alpha z}q,\sqrt{\alpha/z}q,q/\sqrt{\alpha z};q^2)_\infty\Big].$$

Proof of Theorem 4.1. To prove Theorem 4.1 we require the following transformation formula [3, eq.(6.8), p.527];

If $|q|, |b/a| < 1$, then

$$\sum_{r=-\infty}^{\infty} \frac{(a;q^2)_r t^r q^{r^2}}{(b;q^2)_r} = \frac{(b/a,-qb/at;q^2)_\infty}{(b;q^2)_\infty} \times \sum_{r=-\infty}^{\infty} (-atq/b,a;q^2)_r \left(\frac{b}{a}\right)^r. \quad (23)$$

Putting $a=-zq,\ b=-\alpha q^2,\ t=\alpha$ in above transformation, we have

$$\sum_{r=-\infty}^{\infty} \frac{(-zq;q^2)_r q^{r^2}\alpha^r}{(-\alpha q^2;q^2)_r} = \frac{(\alpha q/z,-q^2/z;q^2)_\infty}{(-\alpha q^2;q^2)_\infty} \times \sum_{r=-\infty}^{\infty} (-z,-zq;q^2)_r \left(\frac{\alpha q}{z}\right)^r. \quad (24)$$

By (10), we have

$$S_{0,c}(\alpha,z;q) = \frac{(\alpha q/z,-q^2/z;q^2)_\infty}{(-\alpha q^2;q^2)_\infty} \times \sum_{r=-\infty}^{\infty} (-z,-zq;q^2)_r \left(\frac{\alpha q}{z}\right)^r. \quad (25)$$

On changing the order of summation on right side of (25)

$$S_{0,c}(\alpha,z;q) = \frac{(\alpha q/z,-q^2/z;q^2)_\infty}{(-\alpha q^2;q^2)_\infty} \times \sum_{r=-\infty}^{\infty} \frac{q^{2r^2}/(\alpha z)^r}{(-q/z;q)_{2r}}. \quad (26)$$

Replacing q by q^2 in (26), we have

$$S_{0,c}(\alpha,z;q^2) = \frac{(\alpha q^2/z,-q^4/z;q^4)_\infty}{(-\alpha q^4;q^4)_\infty} \times \sum_{r=-\infty}^{\infty} \frac{q^{4r^2}/(\alpha z)^r}{(-q^2/z;q^2)_{2r}}. \quad (27)$$

Using equation (14), (15), & (17) we get Theorem 4.1.

Proof of Theorem 4.2. To prove Theorem 4.2 put $a=-zq,\ b=-\alpha q^2,\ t=\alpha q^2$ and then replace q by q^2 in the transformation formula (23). ☑

We now prove the following two theorems.

Theorem 4.3.

$$\sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{-8r})(-1/\alpha;q^2)_{2r} q^{8r^2+6r}}{(1-z\alpha)(-q^2/z;q^2)_{2r} z^{3r}\alpha^r}$$
$$= \frac{(q^2;q^2)_\infty (q^4/z\alpha, \alpha z q^4; q^4)_\infty}{2(-\alpha q^4, -q^2/z, -q^4/z, -\alpha q^2; q^4)_\infty}$$
$$\times \left[(\sqrt{\alpha z} q, \sqrt{\alpha/z} q, q/\sqrt{\alpha z}; q^2)_\infty + (-\sqrt{\alpha z} q, -\sqrt{\alpha/z} q, -q/\sqrt{\alpha z}; q^2)_\infty \right]. \tag{28}$$

Theorem 4.4.

$$\sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{-8r+4})(-1/q^2\alpha;q^2)_{2r} q^{8r^2-2r}}{(1-z\alpha q^4)(-1/z;q^2)_{2r} \alpha^r z^{3r}}$$
$$= \frac{(q^2;q^2)_\infty (1/\alpha z, \alpha z q^8; q^4)_\infty}{2\sqrt{\alpha z} q(-\alpha q^4, -q^2/z, -1/z, -\alpha q^6; q^4)_\infty}$$
$$\times \left[(-\sqrt{\alpha z} q, -\sqrt{\alpha/z} q, -q/\sqrt{\alpha z}; q^2)_\infty - (\sqrt{\alpha z} q, \sqrt{\alpha/z} q, q/\sqrt{\alpha z}; q^2)_\infty \right]. \tag{29}$$

Proof of Theorem 4.3. In the transformation formula [3, eq.(6.4), p.527]

For $|q|, |bd/atq| < 1$, then

$$\sum_{r=-\infty}^{\infty} \frac{(a;q^2)_r t^r q^{r^2}}{(b;q^2)_r} = \frac{(-bq/at, -qt; q^2)_\infty}{(-q^3/at, -aqt; q^2)_\infty}$$
$$\times \sum_{r=-\infty}^{\infty} \frac{(1+atq^{4r-1})(a, -aqt/b; q^2)_r (bat^2)^r q^{4r^2-4r}}{(1+at/q)(b, -tq; q^2)_r}. \tag{30}$$

we take $a = -zq$, $b = -\alpha q^2$ and $t = \alpha$, the left-hand side is

$$= \sum_{r=-\infty}^{\infty} \frac{(-zq;q^2)_r q^{r^2} \alpha^r}{(-\alpha q^2; q^2)_r} = S_{0,c}(\alpha, z; q), \tag{31}$$

and the right-hand side is

$$= \frac{(-q^2/z, -q\alpha; q^2)_\infty}{(q^2/z\alpha, zq^2\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{4r})(-zq, -z; q^2)_r q^{4r^2-r} z^r \alpha^{3r}}{(1-z\alpha)(-\alpha q^2, -\alpha q; q^2)_r} \tag{32}$$

$$= \frac{(-q^2/z, -q\alpha; q^2)_\infty}{(q^2/z\alpha, zq^2\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{4r})(-z;q)_{2r} q^{4r^2-r} z^r \alpha^{3r}}{(1-z\alpha)(-\alpha q;q)_{2r}}. \tag{33}$$

Reversing the order of summation, (33) is

$$= \frac{(-q^2/z, -q\alpha; q^2)_\infty}{(q^2/z\alpha, zq^2\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{-4r})(-1/\alpha; q)_{2r} q^{4r^2+3r}}{(1-z\alpha)(-q/z; q)_{2r} z^{3r} \alpha^r}. \tag{34}$$

Finally we get

$$S_{0,c}(\alpha, z; q) = \frac{(-q^2/z, -q\alpha; q^2)_\infty}{(q^2/z\alpha, zq^2\alpha; q^2)_\infty} \sum_{r=-\infty}^{\infty} \frac{(1-z\alpha q^{-4r})(-1/\alpha; q)_{2r} q^{4r^2+3r}}{(1-z\alpha)(-q/z; q)_{2r} z^{3r} \alpha^r}. \tag{35}$$

Replacing q by q^2 and using Theorem 3.1, we get Theorem 4.3.

Proof of Theorem 4.4. To prove Theorem 4.4 take $a = -zq$, $b = -\alpha q^2$ and $t = \alpha q^2$ and then replacing q by q^2 in the transformation formula (30). ☑

5. Generalized Functions Represented as Continued Fraction

In this section we shall represent generalized functions as continued fractions of Gollnitz-Gordon.

(i). The specialized generalized functions as continued fractions of Gollnitz-Gordon

Taking $\alpha = -1$ and $z = -1$ in (6), we have

$$S_0(-1, -1; q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(q^2; q^2)_n}.$$

By Slater [5, eq.36, p.155]

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} S_0(-1, -1; q) = (-q^3; q^8)_\infty (-q^5; q^8)_\infty (q^8; q^8)_\infty. \tag{36}$$

Similarly taking $\alpha = -1$ and $z = -1$ in (7) and using Slater [5, eq.34, p.155], we have

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} S_1(-1, -1; q) = (-q; q^8)_\infty (-q^7; q^8)_\infty (q^8; q^8)_\infty. \tag{37}$$

Dividing (36) by (37)

$$\frac{S_0(-1, -1; q)}{S_1(-1, -1; q)} = \frac{(-q^3; q^8)_\infty (-q^5; q^8)_\infty}{(-q; q^8)_\infty (-q; q^8)_\infty}.$$

$$= 1 - q + \frac{q^2}{1-q^3} {}_{+} \frac{q^4}{1-q^5} {}_{+} \frac{q^6}{1-q^7} {}_{+} \cdots. \tag{38}$$

which is Gollnitz-Gordon continued fraction.

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References

[1] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encycl. Math. Applics., Vol. 35, Cambridge University Press, Cambridge, 1990.

[2] B. Gordon and R.J. McIntosh, *Some eighth order mock theta functions*, J. London Math. Soc. **62(2)** (2000), 321–335.

[3] J. Mc Laughlin, *Mock Theta Function Identities Deriving from Bilateral Basic Hypergeometric Series*, Analytic Number Theory, Modular Forms and q-Hypergeometric Series, Springer Proc. Math. Stat., Springer, Cham. **221** (2017), 503–531.

[4] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York 1962 ; reprinted by the American Mathematical Society, Providence, RI, 2000.

[5] L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **54** (1952), 147–167.

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