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On a family of groups generated by parabolic matrices

Sobre una familia de grupos generados por matrices parabólicas

Christian Pommerenke[1], Margarita Toro[0,2,✉]

[1]Technische Universität Berlin, Berlin, Germany

[2]Universidad Nacional de Colombia-Sede Medellín, Medellín, Colombia

Abstract. We study various aspects of the family of groups generated by the parabolic matrices $A(t_1\zeta), \dots, A(t_m\zeta)$ where $A(z) = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ and by the elliptic matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The elements of the matrices W in such groups can be computed by a recursion formula. These groups are special cases of the generalized parametrized modular groups introduced in [16].

We study the sets $\{z : \operatorname{tr} W(z) \in [-2, +2]\}$ [13] and their critical points and geometry, furthermore some finite index subgroups and the discretness of subgroups.

Key words and phrases. modular group, parametrized modular group, singular set, discrete groups, Chebyshev polynomials.

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Resumen. Estudiamos algunos aspectos de la familia de grupos generados por matrices parabólicas $A(t_1\zeta), \dots, A(t_m\zeta)$ donde $A(z) = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ y por la matriz elíptica $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Los elementos de las matrices W en tales grupos se pueden calcular mediante una fórmula de recurrencia. Estos grupos son casos especiales de la generalización del grupo modular parametrizado estudiado en [16].

Estudiamos los conjuntos $\{z : \operatorname{tr} W(z) \in [-2, +2]\}$ [13] y sus puntos críticos y geometría, así como también algunos subgrupos de índice finito y la discreticidad de tales subgrupos.

Palabras y frases clave. grupo modular, grupo modular parametrizado, conjunto singular, grupos discretos, polinomios de Chebyshev.

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1. Introduction

A parabolic matrix is determined by one parameter. In this paper we study a family of groups generated by a finite number of parabolic matrices, where the parameter lies in a polynomial ring of one variable over the complex numbers. The groups that we consider have one additional generator, an elliptic element of order four. More specifically, we consider the parabolic matrix $A(\xi) = \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$ and the elliptic matrix $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For $\xi = 1$, the group generated by $A(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and B is the classical modular group. In [13] we studied a more general case when ξ runs through all complex numbers, so we introduce the parametrized modular group

$$\Pi = \left\langle A(\xi), B;\, B^4 = I \right\rangle \subset \mathrm{SL}(2, \mathbb{C}[\xi]),$$

see also the paper of P.M.Cohn [3]. A free purely parabolic subgroup of Π with index 4 had been considered by various authors, for instance J.Gilman and P.Waterman [8] and in [14].

In [16] we had considered the more general case of m parabolic matrices $A(p_k)$ where the p_k are any polynomials in $\mathbb{C}[\xi]$. All group elements can be written in a precise form, see (2.2) below.

The group defined in [16] is too general for many purposes. In the present paper however, we restrict ourselves to polynomials of the special form $p_k = t_k\xi$ where t_k are complex numbers, therefore we study the group

$$\Pi = \Pi[t_1\xi, \dots, t_m\xi] := \left\langle A(t_1\xi), \dots, A(t_m\xi), B;\, B^4 = I \right\rangle.$$

Now the matrices in Π can be computed by a recursion formula, see Section 2. In a way this paper takes up more the ideas of our first paper [13].

In Section 3 we study the critical points, i.e, the points where our group has an additional relation. We consider two methods to find critical points, namely by the Riley operator and by using Chebyshev polynomials.

Section 4 is about the singular set. For a member W of our group we form the set of all complex numbers ζ for which the trace of W lies in the interval $[-2, +2]$. The singular set S is then the union of all the sets formed. Its closure is of particular interest, see for instance [8, 14, 21].

In Section 5 we study the problem of discreteness of subgroups. This problem had not been considered in our previous papers. For instance we prove that the group $\left\langle A\left(\sqrt{p}\right), A\left(i\sqrt{q}\right), B\right\rangle$ with $p, q \in \mathbb{N}$ is discrete. This family of groups contains some very well known examples: For $p = 1, q = 2$ we have $\Pi[1, i\sqrt{2}] = \mathrm{SL}(2, O_2)$ and for $p = q = 1$ we obtain the Picard group $\Pi[1, i] = \mathrm{SL}(2, O_1)$. We also give examples of non discrete groups.

Using ideas of T. Jörgensen [9] and of A.F.Beardon [1], R.Riley [21, Th.1] proved the following beautiful theorem:

Theorem 1.1. (Riley) *Let $\Gamma(\zeta)\,(|\zeta|<1)$ be a holomorphic family of subgroups of* SL*(2,$\mathbb{C}$) which is non-elementary except possibly for countably many ζ. Then*

$$T := \{\zeta \in \mathbb{D} : \Gamma(\zeta) \text{ is discrete}\}$$

is closed and the critical points are dense in the complement of T.

The boundary of T lies in the closure of the singular set S defined in Section 4. We will not use these results but they serve as a guide line for our Theorem 4.2 and Proposition 5.1. In Section 6 some subgroups are discussed.

The motivation to introduce the parametrized modular group [13] was the study of representations of the group of 2-bridge links. This problems was solved by Riley in a beautiful collection of paper [18, 19, 20]. The problem of the representations of 3-bridge links is not solved, but Riley gave some examples in his seminal paper [19]. This paper give us the motivation to study the generalized parametrized group [16] and now to specialized this group to the particular case considered in this paper. In [17] we apply our ideas in order to develop an algorithm to compute representations of 3-bridge knot groups and continue the work in [16].

Now we introduce some notation and review some of our previous results that are the motivation of this paper.

In [13] we studied the subgroup of $\mathrm{SL}(2,\mathbb{Z}[\xi])$, $\Pi = \Pi\,[\xi] = \langle A\,(\xi)\,, B\rangle$, where $\mathbb{Z}[\xi]$ is the ring of polynomials in the variable ξ. For $\zeta \in \mathbb{C}$, let $\Pi(\zeta)$ be the subgroup of $\mathrm{SL}(2,\mathbb{C})$ obtained by substituting the indeterminate ξ by the number ζ and $W\,(\zeta)$ the matrix obtained by substituting ξ by ζ in an element W in Π. The modular group Γ, which have been amply studied, is $\Pi(1)$ in our notation. The groups $\Pi(2\cos(\pi/q))\,(q \geq 3)$ become the classical Hecke groups after projecting to $\mathrm{PSL}(2,\mathbb{C})$, and $\Pi(\zeta)$ for $\zeta \in \mathbb{R}$ become the generalized Hecke groups. The group Π_1 generated by $\left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ \xi & 1 \end{smallmatrix}\right)$ is a free subgroup of $\Pi\,[\xi]$ with index 4. The group Π_1 is conjugate to the much investigated two-parabolic group generated by $\left(\begin{smallmatrix} 1 & 2\lambda \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We studied algebraic and analytic properties of this group. First we provide the algebraic descriptions of the group $\Pi = \Pi\,[\xi]$ and its subgroup Π_1. We use combinatorial techniques to describe precisely the elements of both groups. To each element W in Π we associate a sequence of non zero integers and an inductive way to compute the polynomial that conforms the entries of the matrix W. We also describe some technical aspects of a word W with some particular type of associated sequences. We complete the algebraic aspects of Π by studying the set of words W such that for some ζ, $\pm W\,(\zeta)$ becomes a relator in the group $\Pi\,(\zeta)$, i.e, $W\,(\zeta) = \pm I$. Then we consider analytic aspects of sets related to the groups Π and Π_1. We define the singular set of Π, denoted $S\,(\Pi)$, as the set of elements $\zeta \in \mathbb{C}$ such that $W\,(\zeta)$ is not loxodromic, for some $W \in \Pi$. Notice that any relator provides elements in $S\,(\Pi)$, so it is a "natural" transition to pass from studying the relators to study $S\,(\Pi)$ and $S\,(\Pi_1)$. For the proofs in this part we rely heavily on the description of the elements in Π and Π_1. We give examples of singular sets for particular words, showing some pictures to illustrate the behavior of $\overline{S(\Pi)}$. We exhibit symmetry properties of $S(\Pi)$ and estimate the logarithmic capacity. In a much more general context, the closure of the singular set has been studied in [21] and [12]. Very little is known about the set-theoretic properties of $\partial\overline{S(\Pi)}$. Is it connected?, does it have infinite linear measure or even positive Haussdorff dimension? A computer generated picture by Wright in [8, p.11] suggests that it is well behaved, another in [12] suggests that it is chaotic. The singular set $S(\Pi)$ is the union of countably many analytic arcs. The closure of the singular set of analytic families of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ has been much studied, see e.g. [9, 21].

In [15] we studied free subgroups of index four of the parametrized modular group Π. We show that there are eight free subgroups, four of which are normal and four are non-normal. Then we studied the intersections of the normal subgroups. We give canonical presentations of these subgroups in terms of generators and relations. The derivation of our presentations relies on the results about the enumeration of the word in Π. We proved that the commutator subgroup Π' has infinite index in Π, which is quite different in other contexts, for instance, the first three commutator subgroups of the Picard group have finite index. At the end of [15] we find connections between Π and the Picard group and other Bianchi groups and to a group from relativity theory. In order to establish the connections we needed to enlarge the groups $\Pi(\zeta)$ and consider groups generated by two or more parabolics. This was a motivation to study the generalized parametrized modular group in [16]. Given a set of polynomials $p_1, \dots, p_m$ with complex coefficients and a indeterminate ξ which is the same for all μ, $p_\mu \neq 0$ $(\mu = 1, \dots, m)$, we define in [16] the generalized parametrized modular group $\Pi = \Pi[p_1, , \dots, p_m] = \langle A(p_1), \dots, A(p_m), B \rangle$. For $\zeta \in \mathbb{C}$, the notation $\Pi(\zeta) := \Pi[p_1, \dots, p_m](\zeta) \in \mathrm{SL}(2,\mathbb{C})$ $(\zeta \in \mathbb{C})$, means that the polynomials p_μ are evaluated at ζ. If $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then, for instance, $a = a(\xi)$ is a polynomial whereas $a(\zeta)$ is a complex number. We did not impose any restrictions on the polynomials $p_1, \dots, p_m$ but we were able to show the existence of a simple algorithm to obtain the elements of Π. We show a way to describe the element in the group by a set of polynomial, but in general these polynomials are not uniquely determined. However, we have uniqueness under some special conditions on the p_μ, $(\mu = 1, \dots, m)$. By imposing the restrictions on the polynomial we were able to proved similar results to the ones in [13]. These conditions are the motivation for the conditions we are imposing on the polynomial in the present paper. We discuss several concrete examples and its applications to knot theory. In many of our examples the p_μ are complex numbers, and therefore, we obtain subgroups of $\mathrm{SL}(2,\mathbb{C})$; and $\mathrm{PSL}(2,\mathbb{C})$ is isomorphic to the group of orientation preserving isometries of the hyperbolic space H^3. Our applications to knot theory use the fact that many knots K have groups with representations in $\mathrm{PSL}(2,\mathbb{C})$ and therefore $S^3 - K$ admits the structure of a hyperbolic 3-manifold, [19]. The use of the indeterminate ξ, however, allows us to arrange matrix elements according to the degree of polynomials. We introduce the subgroup Π_1 of index 4 which is generated by the parabolic matrices $A(z)$ and $C(z) = BA(z)B^{-1}$. For $m = 1$ and $p_1 = \xi$ this generalizes the group studied in [14]. As an example, we consider two-bridge and three-bridge knots. Using an idea of Riley [18] we show that at least some of these knots lead to subgroups of Π_1 generated by four or less parabolic matrices. An example is the "figure-eight knot" [11, p.60], the matrix group that represent its fundamental group is generated by $A(1), C(\omega)$, $\omega = e^{2\pi i/3}$ which is a subgroup of $\Pi_1[1, \omega]$.

2. The group Π

2.1. Let $m \in \mathbb{N}$ and $t_1, \ldots, t_m \in \mathbb{C} \setminus \{0\}$ be given and let

$$\begin{aligned} &M := \{r = k_1 t_1 + \ldots + k_m t_m : k_1, \ldots, k_m \in \mathbb{Z}\}, \\ &\text{assuming that } \; r = 0 \implies k_1 = \ldots = k_m = 0\,. \end{aligned} \tag{1}$$

The set M does not depend on the order or the signs of the t_μ. Let

$$A(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \; (z \in \mathbb{C}), \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $A(z)^n = A(nz)$. We shall study the group defined by the presentation

$$\Pi = \Pi[t_1\xi, \ldots, t_m\xi] := \big\langle A(t_1\xi), \ldots, A(t_m\xi), B\,;\, B^4 = I \big\rangle \tag{2}$$

where ξ will always be the indeterminate of polynomials in $\mathbb{C}[\xi]$. The group Π does not depend on the order and signs of the parameters $t_1, .., t_m$.

The notation $\Pi(\zeta)$ means that the indeterminate ξ is replaced by the complex number ζ. Hence we have the presentation

$$\Pi(\zeta) := \big\langle A(t_1\zeta), \ldots, A(t_m\zeta), B\,;\, B^4 = I \big\rangle \in \mathrm{SL}(2, \mathbb{C}).$$

2.2. The following theorem was proved in [16, Th.2.1] except for the statement about uniqueness which is true here [16, Th.2.5] because Π has the special form (2).

Theorem 2.1. *Every $W \in \Pi$ can be written uniquely as*

$$W = B^\kappa U_n B^\lambda, \quad \kappa = 0, 1, 2, 3,\; \lambda = 0, 1,\; n \in \mathbb{N}_0\,, \tag{3}$$

$$U_n = A(r_n\xi)B \cdots A(r_1\xi)B\,, \;\; U_0 = I \tag{4}$$

with $r_\nu \in M$ $(\nu = 1, \ldots, n)$, see (1).

Let $r_n \in M$ $(n \in \mathbb{N})$ be given. We recursively define polynomials α_n and β_n by

$$\begin{aligned} \alpha_0 = 1,\, \alpha_1 = r_1\xi, \quad & \alpha_{n+1} = r_{n+1}\xi\alpha_n - \alpha_{n-1}\,, \\ \beta_0 = 0,\, \beta_1 = -1, \quad & \beta_{n+1} = r_{n+1}\xi\beta_n - \beta_{n-1}\,. \end{aligned} \tag{5}$$

Then we have [16, Prop.2.2]

$$U_n(\xi) = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix}, \ \ \alpha_n = r_n \cdots r_1\xi^n + \dots \ \ (n \in \mathbb{N})\,. \tag{6}$$

We will often write $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Multiplying by B it is easy to check that

$$W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ WB = \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right),\ BWB = \left(\begin{smallmatrix} -d & c \\ b & -a \end{smallmatrix}\right),\ BW = \left(\begin{smallmatrix} -c & -d \\ a & b \end{smallmatrix}\right). \tag{7}$$

3. Critical points

A point $\zeta_0 \in \mathbb{C}$ is called a *critical point* if there exists a non-constant word $V \in \Pi$ such that $V(\zeta_0) = I$ and we then say that the critical point ζ_0 is *associated to* V. Then we obtain the presentation

$$\Pi(\zeta_0) = \langle A(t_1\zeta_0), \dots, A(t_m\zeta_0), B\,;\, B^4 = V(\zeta_0) = I\rangle.$$

Since Π has only a countable number of words it follows that $B^4 = I$ is the only relation of $\Pi(\zeta)$ except for a countable number of $\zeta \in \mathbb{C}$.

***Remark* 3.1.** Writing $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we have $1 = \alpha\delta - \beta\gamma$. We differentiate and use that $\beta = \gamma = 0$ because $V(\zeta_0) = I$. Hence we have $0 = \alpha\delta^{'} + \alpha^{'}\delta - \beta^{'}\gamma - \gamma^{'}\beta = \alpha^{'} + \delta^{'}$. It follows that $V^{'}(\zeta_0) = 0$.

In this section we always write

$$W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Pi,\ tr\ W = a + b.$$

We shall present two methods to obtain critical points.

(i) The first method use the *Riley operator* $W \mapsto W^{\sim}$. By definition the Riley operator replaces all $A(r\xi)$ by $A(-r\xi)$ and all $B^{\pm}$ by $B^{\mp}$ without changing the order of these matrices in W. We have

$$W^{\sim} = QWQ^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad \text{where } Q := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{8}$$

This follows by the repeated application of the identities $QA(r\xi)Q^{-1} = A(-r\xi)$ and $QBQ^{-1} = -B$. See [13, p.117].

Theorem 3.2. *If the polynomial $a(\zeta) - d(\zeta)$ is not constant then there exists $\zeta_0 \in \mathbb{C}$ such that*

$$W(\zeta_0)W^{\sim}(\zeta_0) = W(\zeta_0)QW(\zeta_0)Q^{-1} = I \tag{9}$$

so that ζ_0 is a critical point associated to $V(\zeta_0) := W(\zeta_0)QW(\zeta_0)Q^{-1}$.

Proof. It follows from (8) that

$$V = W \cdot W^{\sim} = \begin{pmatrix} 1 + a(a-d) & -b(a-d) \\ c(a-d) & 1 - d(a-d) \end{pmatrix}.$$

Since $a - d$ is not constant by assumption, there exist one or more ζ_0 with $a(\zeta_0) - d(\zeta_0) = 0$ and therefore with $V(\zeta_0) = I$. ☑

(ii) We will use the monic *Chebyshev polynomials* [11, p.112] [2] $t_n(x) = x^n + \dots$ and $u_n(x) = x^n + \dots$ defined by $t_n(x) = 2T_n(x/2)$ and $u_n(x) = U_n(x/2)$ where $T_n(x)$ and $U_n(x)$ are the classical Chebyshev polynomials.

For instance we have

$$\begin{aligned} t_2(x) = x^2 - 2,\ t_3(x) = x^3 - 3x,\ t_4(x) = x^4 - 4x^2 + 2, \\ u_2(x) = x^2 - 1,\ u_3(x) = x^3 - 2x,\ u_4(x) = x^4 - 3x^2 + 1. \end{aligned} \tag{10}$$

The monic Chebyshev polynomials appear under various names.

Since $t_n(2\cos\theta) = 2\cos(n\theta)$ and $u_n(2\cos\theta) = (1/n)\, t_n'(2\cos\theta)$ we have

$$t_n(2\cos\frac{2\nu\pi}{n}) = +2,\ u_{n-1}(2\cos\frac{2\nu\pi}{n}) = 0 \text{ for } \nu = 1, \dots, \lfloor (n-1)/2 \rfloor. \tag{11}$$

For $W \in \mathrm{SL}(2, \mathbb{C})$ and $\tau := \mathrm{tr}\ W$ we have

$$W^n = \frac{1}{2} t_n(\tau) I + u_{n-1}(\tau)(W - \frac{1}{2}\tau I). \tag{12}$$

Theorem 3.3. *If $n \geq 3$ and $\mathrm{tr}\ W = a + d$ is not constant then there exist distinct $\zeta_\nu \in \mathbb{C}$ such that*

$$W^n(\zeta_\nu) = I \quad for\ \ \nu = 1, \dots, \lfloor (n-1)/2 \rfloor. \tag{13}$$

Hence the ζ_ν are critical points associated to $V(\zeta_\nu) := W^n(\zeta_\nu)$.

Proof. There are $\zeta_\nu \in \mathbb{C}$ such that

$$\tau(\zeta_\nu) = 2\cos(2\nu\pi/n) \text{ for } \nu = 1, \dots, \lfloor \frac{n-1}{2} \rfloor.$$

Hence it follows from (11) that $t_n(\tau(\zeta_\nu)) = 2,\ u_{n-1}(\tau(\zeta_\nu)) = 0$ so that $W^n(\zeta_\nu) = I$. ☑

The first possible case is $n = 3$. By (11) and (12) we have

$$W^3(\tau) = \frac{1}{2}\left(\tau^3 - 3\tau\right) I + \left(\tau^2 - 1\right)\left(W(\tau) - \frac{\tau}{2} I\right).$$

We want that $W^3(\tau) = I$. This holds if and only if $\tau = -1$.

As an *example* we consider the Picard group $\Pi[1, i]$ with $r_1 = 1 + i$, $r_2 = r \in \mathbb{Z}$, $r_3 = 1 - i$. For $W = U_3$ we have $a - d = \xi(2r\xi^2 + r - 2)$. The solutions of $a(\zeta) - d(\zeta) = 0$ are $\zeta = 0, \pm\sqrt{1/2}$ if $r = 1$ but only $\zeta = 0$ if $r = 2$. Note that the solution $\zeta = 0$ is of little interest because $W(0) = -B$.

Now we briefly discuss the assumption that the polynomials $a - d$ or $a + d$ are not constant. The following result was proved in [13, Pro.3.1], see also [16, Sect.2.4] and [7].

Proposition 3.4. *The polynomials $a \pm d$ are constant if and only if*

$$r_{n-\nu+1} = \pm r_\nu \ (\nu = 1, \ldots, \lfloor n/2 \rfloor). \tag{14}$$

If $a \pm d$ is constant then $a \pm d \in \{0, 2, -2\}$.

4. The singular set

4.1. As before let $\Pi = \Pi[t_1\xi, \ldots, t_m\xi]$. For $W \in \Pi$ we define

$$S_0(W) := \{\zeta \in \mathbb{C} : \ \mathrm{tr}\ W(\zeta) \in [-2, +2]\} \quad \text{if tr } W \text{ is not constant} \tag{15}$$

and $S_0(W) := \varnothing$ if tr W is constant, see Proposition 3.4 for the $W \in \Pi$ with constant trace. If ζ_0 is a critical point associated with W then $\zeta_0 \in S_0(W)$.

***Remark* 4.1.** Let $\tau(\zeta) := \mathrm{tr}\ W(\zeta)$ and $n = \deg(\tau)$. Then $[-2, +2]$ has n pre-images under the map $\zeta \mapsto \tau(\zeta)$ except for the finitely many ζ with $\tau'(\zeta) = 0$. Each pre-image is an arc of $S_0(W)$ and the endpoints ζ of these arcs satisfy $\tau(\zeta) = \pm 2$. If ζ is a critical point then $\tau(\zeta) = 2$ and $\tau'(\zeta) = 0$ by Remark 3.1.

The *singular set* of Π is defined as

$$S = S(\Pi) = \bigcup_{W \in \Pi} S_0(W) \tag{16}$$

and it is the union of countably many analytic arcs.

The following result is closely related to a theorem proved by Robert Riley in a larger context, see Theorem 1.1. Using the monic Chebyshev polynomials introduced in Section 3.1 we construct explicit critical points.

Theorem 4.2. *The critical points are dense in the closure $\bar{S}$ of S.*

Proof. Let $W \in \Pi$ and $k = \deg(\mathrm{tr}\ W)$. The structure of $S_0(W)$ was described in Remark 4.1. As in part (ii) of the proof of Theorem 3.2 we construct points

$$\zeta_{n\nu j} \in S_0(W) \quad \text{with } n \geq 3,\ j = 1, \dots, k\,,\ \nu = 1, \dots, \lfloor (n-1)/2 \rfloor \tag{17}$$

for which $\tau(\zeta_{n\nu j}) = 2\cos(2\nu\pi/n)$.

We obtain from (12) that $W^n(\zeta_{n\nu j}) = I$ so that $\zeta_{n\nu j}$ is a critical point. By (11) their union is dense in $S_0(W)$. Now it follows from (16) that the critical points are dense in S and therefore dense in the closure $\bar{S}$. ☑

4.2. Let $\Pi = \Pi[t_1\xi, \dots, t_m\xi]$ and $r \in M$. Starting with $W_0 = W$ we define

$$W_n = [\,W_{n-1}, A(r\zeta)] =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \Pi \quad (n \in \mathbb{N}) \tag{18}$$

where $[\cdot,\cdot]$ denotes the commutator. As in [13, Th.5.1] we obtain

$$W_{n+1} = \begin{pmatrix} 1 - r\zeta a_n c_n & r\zeta(a_n^2 - 1) + (r\zeta)^2 a_n c_n \\ -r\zeta c_n^2 & 1 + r\zeta a_n c_n + (r\zeta c_n)^2 \end{pmatrix}. \tag{19}$$

By induction it follows that

$$r\zeta c_n(\zeta) = -(r\zeta c(\zeta))^{2^n}, \ \mathrm{tr}\ W_n = 2 + (r\zeta c(\zeta))^{2^{n+1}}. \tag{20}$$

Theorem 4.3. *Let $\rho := \inf\{|r| : r \in M, r \neq 0\}$ and $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Pi$. If the polynomial c is not constant then*

$$\{\zeta \in \mathbb{C} : |\zeta c(\zeta)| \leq \frac{1}{\rho}\} \subset \bar{S}. \tag{21}$$

The same is true with $c(\zeta)$ replaced by $a(\zeta), b(\zeta)$ or $d(\zeta)$.

Proof. Let $r \in M$. For $n \in \mathbb{N}$ and $0 < k \leq 2^{n+1}$ let

$$L_{n,k} = \{\zeta \in \mathbb{C} : |r\zeta c(\zeta)| \leq 1,\ \arg(r\zeta c(\zeta)) = \frac{2k-1}{2^{n+1}}\pi\}. \tag{22}$$

For $\zeta_k \in L_{n,k}$ we therefore have

$$0 \geq (r\zeta_k c(\zeta_k))^{2^{n+1}} = -|(r\zeta_k c(\zeta_k))^{2^{n+1}}| \geq -1\,.$$

Hence it follows from (20) that

$$\mathrm{tr}\ W_n(\zeta_k) = 2 + (r\zeta_k c(\zeta_k))^{2^{n+1}} \in [1, 2]$$

and therefore from (15) and (16) that $\zeta_k \in S_0(W_n) \subset S$.

Now we let $\zeta \in \mathbb{C}$ and $|r\zeta c(\zeta)| \leq 1$. Then there exists $\zeta_k \in L_{n,k}$ with $|\zeta_k| = |\zeta|$ and $|\arg \zeta_k - \arg \zeta| \leq 2^{-n}$. Since $\zeta_k \in S$ it follows that $\zeta \in \bar{S}$. Hence we have proved that $\{\zeta \in \mathbb{C} : |\zeta c(\zeta)| \leq 1/r\} \subset \bar{S}$ which implies (22) by the definition of ρ.

To conclude the proof we apply the above result to the matrices BW, BWB and BW using (7) and we obtain (21) with a, b, d instead of c ☑

4.3. Now we state some geometric properties of the components $S_0(W)$ of the singular set.

Theorem 4.4. *Let $S_0(W) \neq \varnothing$, see* (15). *(i) The logarithmic capacity is*

$$4 \text{ cap } S_0(W) = |r_n \cdots r_1|^{1/n}. \tag{23}$$

(ii) If $S_0(W) \subset \{z \in \mathbb{C} : |z| \leq q\}$ then the length satisfies

$$\text{len } S_0(W) \leq (1 + q^2)\pi n. \tag{24}$$

Proof. (i) Using (6) the first assertion follows from a theorem of Fekete [5] and the fact that cap $[-2, +2] = 1$, see [13, p.122].

(ii) The second assertion is a consequence of an important result of Eremenko and Hayman [4, Th.2]:

Let f be a polynomial of degree n and let $F_p = \{z \in \mathbb{C} : f(z) \in [-p, p]\}$ for $0 < p \leq \infty$. Then

$$\int_{F_\infty} \frac{1}{1 + |z|^2} |dz| \leq \pi n. \tag{25}$$

We apply this result to the polynomial $f(z) := \text{tr } W(z)$ and $F_2 = S_0(W)$. By (25) we obtain

$$\text{len } S_0(w) = \int_{F_2} |dz| \leq \int_{F_2} \frac{1 + q^2}{1 + |z|^2} |dz| \leq \int_{F_\infty} \frac{1 + q^2}{1 + |z|^2} |dz| \leq (1 + q^2)\pi n.$$

☑

5. Discrete groups

5.1. The following result about discrete groups complements Theorem 4.3.

Proposition 5.1. *Let $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Pi$ and let $r \in M$, $\zeta_0 \in \mathbb{C}$, $\zeta_0 \neq 0$. If the polynomial $c(\zeta)$ is not constant and if*

$$|r\zeta c(\zeta)| < 1 \quad \textit{with} \quad r \in M\,. \tag{26}$$

then the group $\Pi(\zeta)$ is not discrete in a neighbourhood of ζ_0 except that $\Pi(\zeta_0)$ may be discrete if $c(\zeta_0) = 0$. The same holds for a, b, d instead of c.

Proof. We use the notation and results at the beginning of Section 4.2. Because of (26) we may assume by continuity that $0 < |r\zeta c(r\zeta)| < 1$ holds in a punctured neighbourhood of ζ_0. Now we show by induction that

$$|1 - a_n| \leq (|a| + n - 1)|r\zeta c|^{2^{n-1}} \quad (n \in \mathbb{N}). \tag{27}$$

This is true for $n = 1$ because $1 - a_1 = r\zeta ac$ by (19). Now let (27) be true for n. By (27) and since $|r\zeta c| \leq 1$, we have $|a_n| \leq |1-a_n|+1 \leq (|a|+n-1)+1 = |a|+n$. Also, by (20) we have $|r\zeta c_n| = |r\zeta c|^{2^n}$. It follows from (30) that

$$|a_{n+1}| = |a_n||r\zeta c_n| \leq (|a| + n)|r\zeta c|^{2^n}.$$

This proves (27) for $n + 1$.

Furthermore we have $|r\zeta c_n| \leq |r\zeta c|^{2^n} \to 0$ as $n \to \infty$. Using (19) we therefore obtain

$$a_{n+1} \to 1,\ b_{n+1} \to 0,\ c_{n+1} \to 0,\ d_{n+1} \to 1$$

with $c_{n+1} \neq 0$ because $r\zeta c \neq 0$, see (20). It follows that $I \neq W_{n+1}(\zeta) \to I$ as $n \to \infty$. Since $W_{n+1}(\zeta) \in \Pi(\zeta)$ we conclude that $\Pi(\zeta)$ is not discrete. ☑

5.2. Let $\Pi = \Pi[t_1\xi, \ldots, t_m\xi]$. We will use the following sufficient condition of Beardon [1, p.14] where $\|\ldots\|$ is the matrix norm.

Lemma 5.2. *The group* $\Pi(\zeta)$ *is discrete if*

$$\inf\{\|W - I\| : W \in \Pi(\zeta),\, W \neq I\} > 0\,.$$

Theorem 5.3. *The groups*

$$\Pi[\sqrt{p}\,, i\sqrt{q}\,] \quad (p, q \in \mathbb{N}) \tag{28}$$

are discrete.

For $p = 1, q = 2$ we obtain the well-known fact that the group $\Pi[1, i\sqrt{2}] = \mathrm{SL}(2, O_2)$ is discrete. For $p = q = 1$ we obtain again that the Picard group $\Pi[1, i] = \mathrm{SL}(2, O_1)$ is discrete, the often used additional generator $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ is redundant, see e.g. [3][16, Cor.4.3]. Note that we have put $\xi = 1$. For general $\xi \in \mathbb{C}$ these groups are not always discrete by Proposition 5.4.

Proof. (a) The first of the following two sets was defined in (1). Now we define

$$\begin{aligned} M &:= \{k\sqrt{p} + il\sqrt{q} : k, l \in \mathbb{Z}\}, \\ M' &:= \{k + il\sqrt{pq}) : k, l \in \mathbb{Z}\}. \end{aligned} \tag{29}$$

Now let $\rho_1, \rho_2 \in M$ and $\rho_1', \rho_2' \in M'$. Then it follows from (29) that

$$\begin{aligned} \rho_1\rho_2 &= (k_1k_2p - l_1l_2q) + i(k_1l_2 + l_1k_2)\sqrt{pq} \in M' \\ \rho_1'\rho_2' &= (k_1'k_2' - l_1'l_2'pq) + i(k_1'l_2' + l_1'k_2')\sqrt{pq} \in M' \\ \rho_1\rho_2' &= (k_1k_2' - l_1l_2'q)\sqrt{p} + i(l_1k_2' + k_1l_2'p)\sqrt{q} \in M \end{aligned} \tag{30}$$

and similarly for $\rho_1'\rho_2$. Here we have used that $p, q \in \mathbb{N}$.

(b) Let $W \in \Pi[\sqrt{p}, i\sqrt{q}]$. By (3) and (4) we have

$$W = B^\kappa U_n B^\lambda,\ \ U_n = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix} = A(r_n)B \cdots A(r_1)B \tag{31}$$

with $r_\nu \in M$. We claim that

$$\alpha_{2\nu-1}, \beta_{2\nu} \in M\,,\ \ \alpha_{2\nu}, \beta_{2\nu-1} \in M' \quad \text{for } \nu \in \mathbb{N}\,. \tag{32}$$

By (31),(5) and (30) we have $a_1 = r_1 \in M$, $\alpha_2 = r_1r_2 - 1 \in M'$, furthermore $\beta_1 = -1 \in M'$, $\beta_2 = r_1 \in M$. This proves (32) for $\nu = 1$.

Suppose that (32) is true for some ν. Then we have

$$\alpha_{2\nu+1} = r_{2\nu+1}\alpha_{2\nu} - \alpha_{2\nu-1} \in M'$$

by the induction hypothesis (32) and by (30) with $\rho_1 = r_{2\nu+1} \in M$, $\rho_2 = \alpha_{2\nu} \in M'$ and we obtain

$$\alpha_{2\nu+2} = r_{2\nu+2}\alpha_{2\nu+1} - \alpha_{2\nu} \in M$$

by (32) and by (30) with $\rho_1 = r_{2\nu+2} \in M'$, $\rho_2 = \alpha_{2\nu-1} \in M$ and by (30). This proves (32) for $\nu + 1$.

(c) Now let $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Pi[\sqrt{p}, i\sqrt{q}]$. The factors B in (31) only rotate the matrix W. Hence it follows from (6) and (32) that the a and d form a mesh ≥ 1 in the $(k,l)-plane$, see (29)

We want to apply Lemma 5.2 and therefore consider

$$|W - I|^2 = |a-1|^2 + |b|^2 + |c|^2 + |d-1|^2.\ W \neq I.$$

If $|b|^2 + |c|^2 > 0$ then it follows from (29) that $|W - I|^2 \geq 1$ because $p, q \geq 1$. Now let $b = c = 0$. Since $\det W = 1$ we have $ad = 1$. Since $W \neq I$ we cannot have $a = d = 1$. Hence $|1 - a| > \delta$ with some constant δ with $0 < \delta < 1$. It follows that $|W - I| > \delta$. Hence Π is a discrete group by Lemma 5.2. ☑

5.3. Finally we give some examples of groups that are not discrete. The first two examples are related to Theorem 5.3 and their proofs rely on Proposition 5.1.

Proposition 5.4. *Let $q \in \mathbb{N}$. If $0 < |\zeta - 1| \leq \frac{1}{4}$ then the group $\Pi[\zeta, i\sqrt{q}\zeta]$ is not discrete. If $0 < |\zeta - 1| \leq \frac{1}{20}$ then $\Pi[\sqrt{2}\zeta, i\sqrt{q}\zeta]$ is not discrete.*

Note that, by Theorem 5.3, the above groups are discrete for $\zeta = 1$, the center of the above disks.

Proof. (a) With $r_1 = r_2 = 1$ it follows from (5) that $\alpha_2(\zeta) = \zeta^2 - 1$. Hence we have

$$0 < |\zeta\alpha_2| = |\zeta(1+\zeta)(1-\zeta)| < 1 \text{ for } 0 < |\zeta - 1| \leq \tfrac{1}{4}.$$

Therefore $\zeta\alpha_2(\zeta)$ satisfies (26) so that $\Pi[1, i\sqrt{q}](\zeta)$ is not discrete.

(b) With $r_1 = r_2 = r_3 = \sqrt{2}$ it follows from (5) that $\alpha_3(\zeta) = 2\sqrt{2}(\zeta^3 - \zeta)$. Hence

$$0 < |\sqrt{2}\zeta\alpha_3(\zeta)| = 4|\zeta|^2|\zeta+1||\zeta-1| < 1 \text{ for } 0 < |\zeta - 1| \leq 0.05.$$

Therefore $\sqrt{2}\zeta\alpha_2(\zeta)$ satisfies (26) so that $\Pi[\sqrt{2}, \sqrt{q}](\zeta)$ is not discrete. ☑

Proposition 5.5. *If $t_1, t_2 \in \mathbb{R}$ and $\Pi = \Pi[t_1\xi, t_2\xi]$ then the group $\Pi(\zeta)$ is not discrete for any $\zeta \in \mathbb{C}, \zeta \neq 0$.*

Proof. We need a classical result about diophantine approximation [10, Th.5]: If $\alpha \in \mathbb{R}$ is irrational then there exist sequences $p_n, q_n \in \mathbb{Z}$ such that

$$|q_n\alpha - p_n| < 1/q_n, \;\; q_n \to \infty \;\; (n \to \infty). \tag{33}$$

First we show that $\alpha := t_1/t_2$ is irrational. Otherwise we would have $\alpha = k/l$ with $k, l \in \mathbb{Z}$ and therefore $lt_1 - kt_2 = 0$ which contradicts (1).

Now let p_n, q_n be as in (33) and $r_n := q_nt_1 - p_nt_2$. Then we have $0 \neq |r_n| = |t_1||q_n\alpha - p_n| \leq |t_1/p_n| \to 0$ and thus

$$I \neq W_n(\zeta) := A(r_n\zeta) = \begin{pmatrix} 1 & r_n\zeta \\ 0 & 1 \end{pmatrix} \to I \;\; (n \to \infty)$$

for $\zeta \neq \infty$. Hence $\Pi(\zeta)$ is not discrete. ☑

6. Some subgroups

Let $\Pi = \Pi[t_1\xi, \ldots, t_m\xi]$ and

$$C(z) := \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} = BA(z)B^{-1} \;\; (z \in \mathbb{C}). \tag{34}$$

Let the word $W \in \Pi$ contain precisely the matrices $A(r_\nu\xi)$ where $\nu = 1, \ldots n$, $0 \leq n < \infty$ and $r_\nu = k_{1,\nu}t_1 + \ldots + k_{m,\nu}t_m$, compare (1). Then we define $\sigma(W) := \sum_{\nu=1}^{n}(k_{1,\nu} + \ldots + k_{m,\nu})$. Furthermore we define $\tau(W)$ as the number of B in W where B^{-1} counts as -1. Then we have $\sigma(W_1W_2) = \sigma(W_1) + \sigma(W_2)$

and $\sigma(W^{-1}) = -\sigma(W)$, correspondingly for τ. In the special case $\Pi[\xi]$ a much more detailed free group structure could be proved, see [15].

All the following congruences $\equiv$ will be modulo 4. We shall consider the subgroups

$$\Pi_1 := \{W \in \Pi : \tau(W) \equiv 0\}, \quad \Pi_2 := \{W \in \Pi : \tau(W) \equiv \sigma(W)\}. \tag{35}$$

It follows that $\Pi_1 \cap \Pi_2 = \{W \in \Pi : \sigma(W) \equiv \tau(W) \equiv 0\}$. In [16, Th.3.2] it was proved that

$$\Pi_1 = \big\langle A(t_1\xi), \ldots, A(t_m\xi), C(t_1\xi), \ldots, C(t_m\xi)\big\rangle. \tag{36}$$

Proposition 6.1. *The groups* Π_1 *and* Π_2 *are normal subgroups of* Π *with indices* $|\Pi : \Pi_1| = |\Pi : \Pi_2| = 4$ *and* $|\Pi : \Pi_1 \cap \Pi_2| = 16$.

Proof. It follows from (34) that Π_1 and Π_2 are normal subgroups of Π. For $k = 0, 1, 2, 3$ the cosets

$$B^k\Pi_1 = \{W \in \Pi : \tau(W) \equiv k\}, \;\; B^k\Pi_2 = \{W \in \Pi : \tau(W) \equiv \sigma(W) + k\}$$

are distinct and their union is Π. Hence Π_1 and Π_2 have index 4 in Π.

Furthermore we have

$$B^l(\Pi_1 \cap \Pi_2) = \{W \in \Pi_2 : \tau(W) = l\} \;\; (l = 0, 1, 2, 3).$$

It follows that $|\Pi_2 : \Pi_1 \cap \Pi_2| = 4$ and therefore that $|\Pi : \Pi_1 \cap \Pi_2| = |\Pi : \Pi_2|\,|\Pi_2 : \Pi_1 \cap \Pi_2| = 16$. ☑

Let $W \in \Pi$. The Riley operator $W \mapsto W^{\sim}$ was introduced in Section 3. Since $W^{\sim}$ exchanges the exponents of B and $A(r\xi)$ we obtain that $\tau(W) = 0$, $\sigma(W) = 0$. It follows that

$$WW^{\sim} \in \Pi_1 \cap \Pi_2. \tag{37}$$

Proposition 6.2. *The commutator subgroup* Π' *of* $\Pi[t_1\xi, \ldots, t_m\xi]$ *is a normal subgroup of* Π_1 *with infinite index.*

Proof. Since $\tau(W) = 0$ holds for all $W \in \Pi'$ we have $\Pi' \subset \Pi_1$. Let $r \in M$, $r \neq 0$ and $W_n = (A(r\xi)B)^{4^n}$ for $n \in \mathbb{N}$. Then we have

$$W_{n+\nu}W_n^{-1} = (A(r\xi)B)^{(4^\nu-1)4^n} \notin \Pi' \;\; \text{for } n, \nu \in \mathbb{N}.$$

Hence the $W_n\Pi'$ form an infinite system of disjoint cosets in Π_1. ☑

The situation can be different for $\Pi'(\zeta)$ with $\zeta \in \mathbb{C}$. Let $\Pi = \Pi[\xi, i\xi]$. Then $\Pi(1)$ is the Picard group $\mathrm{SL}(2, \mathbb{Z}[i])$. In [6, Th.2] it was proved that $\Pi'(1)$ is the only normal subgroup of $\Pi(1)$ with index 4. It follows that $\Pi'(1) = \Pi_1(1)$.

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Institut für Mathematik
Technische Universität Berlin
D-19623
Berlin, Germany
e-mail: pommeren@math.tu-berlin.de

Escuela de Matemáticas
Universidad Nacional de Colombia, Sede Medellín
Facultad de Ciencias
Calle 59a N.63-20
Medellín, Colombia
e-mail: mmtoro@unal.edu.co