

Integrals of certain Dirichlet series

Integrales de ciertas series de Dirichlet

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ABSTRACT. We compute in closed form the integrals of certain expressions involving a class of Dirichlet series. This is a generalization of a formula of Jonathan Borwein to a problem stated (and solved) by A. Ivić.

Key words and phrases. Dirichlet Series, Integrals.

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RESUMEN. Calculamos en forma cerrada integrales de ciertas expresiones que involucran una clase de series de Dirichlet. Esto es una generalización de una fórmula de Jonathan Borwein a un problema enunciado (y resuelto) por A. Ivić.

Palabras y frases clave. Integrales, Series de Dirichlet.

1. Evaluation of certain integrals

We write as usual $s = \sigma + it$ with σ, t real numbers and let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function. A. Ivić [3] proved the following remarkable integral evaluation:

$$\int_0^{\infty} \frac{\{3 - 2\sqrt{2} \cos(t \ln 2)\} |\zeta(1/2 + it)|^2}{t^2 + 1/4} dt = \pi \ln 2.$$

Inspired by this J. Borwein proved that the integrals of expressions involving a certain class of Dirichlet series could be evaluated [1]. If one sets

$$\lambda(s) := \sum_1^{\infty} \frac{\lambda_n}{n^s},$$

he proved the following result.

Theorem 1.1. (J. Borwein) *Assume that the above Dirichlet series is absolutely convergent for fixed $\sigma > 0$ and λ_n are real numbers. Then*

$$\int_0^\infty \left| \frac{\lambda(s)}{s} \right|^2 dt = \frac{\pi}{2\sigma} \sum_{n=1}^\infty \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}, \quad (1)$$

where $\Lambda_n := \sum_{k=1}^n \lambda_k$, $\Lambda_0 := 0$.

The aim of this note is to extend (a bit) the last two results.

Theorem 1.2. *Assume that the above Dirichlet series is absolutely convergent for fixed $\sigma > 0$ and λ_n are real. Then*

$$\begin{aligned} j_\lambda(\sigma) &:= \int_0^\infty \left| \frac{\lambda(s)}{s^2} \right|^2 dt = \\ &= \frac{\pi}{2\sigma^2} \sum_{n=1}^\infty \frac{1}{n^{2\sigma}} \left\{ \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{2\sigma} + \lambda_n \Lambda_n \ln n - \lambda_n \sum_{m=1}^n \lambda_m \ln m \right\}. \end{aligned} \quad (2)$$

Proof. Set

$$\lambda_N(s) := \sum_{n=1}^N \frac{\lambda_n}{n^s},$$

and recall that if a, b are real and positive numbers then ([2] Formula 3729 p. 410)

$$\int_0^\infty \frac{\cos ax}{(b^2 + x^2)^2} dx = \frac{\pi}{4b^3} (1 + ab) e^{-ab}.$$

Then

$$\begin{aligned} \int_0^\infty \frac{|\lambda_N(s)|^2}{(\sigma^2 + t^2)^2} dt &= \int_0^\infty \frac{\sum_{0 < n, m \leq N} \lambda_n \lambda_m n^{-\sigma+it} m^{-\sigma-it}}{(\sigma^2 + t^2)^2} dt \\ &= 2 \sum_{0 < m < n \leq N} \frac{\lambda_n \lambda_m}{(nm)^\sigma} \int_0^\infty \frac{\cos(t \ln n/m)}{(\sigma^2 + t^2)^2} dt + \sum_{n=1}^N \frac{\lambda_n^2}{n^{2\sigma}} \int_0^\infty \frac{1}{(\sigma^2 + t^2)^2} dt \\ &= \frac{\pi}{2\sigma^3} \sum_{0 < m < n \leq N} \frac{\lambda_n \lambda_m}{(nm)^\sigma} \frac{(1 + \sigma \ln n/m)}{(n/m)^\sigma} + \frac{\pi}{4\sigma^3} \sum_{n=1}^N \frac{\lambda_n^2}{n^{2\sigma}}. \end{aligned}$$

Now, the first sum in the last expression is equal to

$$\begin{aligned} \sum_{0 < m < n \leq N} \frac{\lambda_n \lambda_m}{(nm)^\sigma} \frac{(1 + \sigma \ln n/m)}{(n/m)^\sigma} &= \sum_{n=1}^N \frac{\lambda_n}{n^{2\sigma}} \left\{ \sum_{m=1}^n \lambda_m (1 + \sigma \ln n/m) - \lambda_n \right\} \\ &= \sum_{n=1}^N \frac{\lambda_n (\Lambda_n - \lambda_n)}{n^{2\sigma}} + \sigma \sum_{n=1}^N \frac{\lambda_n}{n^{2\sigma}} \sum_{m=1}^n \lambda_m \ln n/m \\ &= \sum_{n=1}^N \frac{\lambda_n (\Lambda_n - \lambda_n)}{n^{2\sigma}} + \sigma \sum_{n=1}^N \frac{\lambda_n \Lambda_n \ln n}{n^{2\sigma}} - \sigma \sum_{n=1}^N \frac{\lambda_n}{n^{2\sigma}} \sum_{m=1}^n \lambda_m \ln m. \end{aligned}$$

Inserting the last expression into the above formula and observing that $\lambda_n \Lambda_n - \lambda_n^2/2 = \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{2}$ yields

$$\begin{aligned} &\int_0^\infty \frac{|\lambda_N(s)|^2}{(\sigma^2 + t^2)^2} dt \\ &= \frac{\pi}{2\sigma^3} \sum_{n=1}^N \frac{\lambda_n (\Lambda_n - \lambda_n)}{n^{2\sigma}} + \frac{\pi}{2\sigma^2} \sum_{n=1}^N \frac{\lambda_n \Lambda_n \ln n}{n^{2\sigma}} - \frac{\pi}{2\sigma^2} \sum_{n=1}^N \frac{\lambda_n}{n^{2\sigma}} \sum_{m=1}^n \lambda_m \ln m + \frac{\pi}{4\sigma^3} \sum_{n=1}^N \frac{\lambda_n^2}{n^{2\sigma}} \\ &= \frac{\pi}{4\sigma^3} \sum_{n=1}^N \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}} + \frac{\pi}{2\sigma^2} \sum_{n=1}^N \frac{\lambda_n \Lambda_n \ln n}{n^{2\sigma}} - \frac{\pi}{2\sigma^2} \sum_{n=1}^N \frac{\lambda_n}{n^{2\sigma}} \sum_{m=1}^n \lambda_m \ln m. \end{aligned}$$

The last expression tends to $j_\lambda(\sigma)$ as $N \rightarrow \infty$ and yields the result. \checkmark

As an example if one takes $\lambda(s) = \zeta(s)$, that is $\lambda_n = 1$, $\Lambda_n = n$, then ($1 < \sigma$)

$$\int_0^\infty \left| \frac{\zeta(s)}{s^2} \right|^2 dt = \frac{\pi}{2\sigma^2} \sum_{n=1}^\infty \frac{1}{n^{2\sigma}} \left\{ \frac{2n-1}{2\sigma} + \ln \left(\frac{n^n}{n!} \right) \right\}.$$

2. A remark

We note the following result which is the analogue of Ivić's integral.

Lemma 2.1. *Set*

$$S := \frac{\ln \{1/2\}}{2.3} + \frac{\ln \{1.3/2.4\}}{4.5} + \frac{\ln \{1.3.5/2.4.6\}}{6.7} + \dots.$$

Then the following evaluation holds:

$$I := \int_0^\infty \left\{ 3 - 2\sqrt{2} \cos(t \ln 2) \right\} \frac{|\zeta(1/2 + it)|^2}{(1/4 + t^2)^2} dt = 2\pi (\ln 2 + S).$$

Proof. We use $(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$ ($0 < \sigma$). Now if $1/2 < \sigma$ then

$$\begin{aligned}
2I &= \int_{-\infty}^{\infty} \left| \frac{(1 - 2^{1-s})\zeta(s)}{s^2} \right|^2 dt = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \left(\frac{m}{n} \right)^{it} \frac{dt}{(\sigma^2 + t^2)^2} \\
&= \frac{\pi}{2\sigma^3} \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} \sum_{n < m} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \frac{\cos(t \ln m/n) dt}{(\sigma^2 + t^2)^2} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^m m^{-\sigma} \sum_{n=1}^{m-1} (-1)^n n^{-\sigma} (1 + \sigma \ln m/n) e^{-\sigma \ln m/n} \right\} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^m m^{-2\sigma} \sum_{n=1}^{m-1} (-1)^n (1 + \sigma \ln m/n) \right\} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma)(1 - 2^{1-2\sigma}) + 2\sigma \sum_{m=1}^{\infty} (-1)^m m^{-2\sigma} \sum_{n=1}^{m-1} (-1)^n \ln m/n \right\}
\end{aligned}$$

The desired result follows letting $\sigma \rightarrow 1/2$ observing that $\lim_{\sigma \rightarrow 1/2} \zeta(2\sigma)(1 - 2^{1-2\sigma}) = \ln 2$ and

$$S = \sum_{m=1}^{\infty} (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n.$$

The last equality follows observing that the inner sum in the last equation with $m = 1$ is zero and

$$\begin{aligned}
\sum_{m=2}^3 (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n &= \frac{\ln \{1/2\}}{2.3}, \\
\sum_{m=4}^5 (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n &= \frac{\ln \{1.3/2.4\}}{4.5},
\end{aligned}$$

and so on. The proof is completed. \checkmark

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