Completion of premetric spaces

Completación de espacios premétricos

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Abstract. We study the concept of a premetric space introduced by F. Richman in the context of constructive mathematics, and present a method for completing them.

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Resumen. Estudiamos el concepto de espacio premétrico introducido por F. Richman en el contexto de las matemáticas constructivas, y presentamos un método para completarlos.

Palabras y frases clave. Completación, espacios premétricos, familias regulares, familias de Cauchy, matemáticas constructivas.

1. Introducción

F. Richman [8] addressed the problem of completing a metric space in constructive mathematics without the axiom of countable choice. In constructive mathematics the underlying logic is intuitionistic instead of classical. The axiom of countable choice (CC) says that any countable collection of non empty sets $(X_n)_n$ has a choice function, i.e., a function $f : \mathbb{N} \to \bigcup_n X_n$ such that $f(n) \in X_n$ for all $n \in \mathbb{N}$. This weak form of choice suffices for the proof of several classical results, for instance, CC is equivalent to the Baire category theorem for certain class of metric spaces (the full Baire theorem requires the axiom of dependent choice, see [2] for a complete treatment of these issues). F. Richman [7] presented an interesting account of the arguments in favor and

also against CC in the context of constructive mathematics. In a choiceless environment, he argued that the reals numbers should not be defined as Cauchy sequences [6, 8]. Lubarsky has shown [3] that in constructive mathematics without choice it cannot be proved that the Cauchy reals are complete. To point out why CC is needed, we recall briefly, for the readers convenience, the usual completion of a metric space (see for instance [10]). Given a metric space X, we consider the quotient Z obtained by setting in the set of all Cauchy sequences in X, the natural equivalence relation where two Cauchy sequences are equivalent if its elements get as closed to 0 as we wish. Then Z is endowed with a metric which makes it a complete metric space containing a dense isometric copy of X. To prove that Z is indeed complete, we pick a sequence $([s]_i)_i$ in Z, then the axiom of countable choice give us a sequence $(x_i)_i$ in X such that $d(x_n^*, [s]_n) \leq 1/n$, where x^* is the element in the dense copy of X inside Z corresponding to a point $x \in X$. If $([s]_i)_i$ is Cauchy, then so is $(x_i)_i$ and its equivalence class is the limit of $([s]_i)_i$.

Without CC, sequences are not the proper objects to represent real numbers. Another natural approach is through Dedekind cuts which is only applicable to ordered structures (see [4, 5] for a comparison of different approaches in the context of constructive mathematics). Richman [8] replaced the notion of a metric space (which assumes the existence of $\mathbb{R}$) by a structure he called a premetric space which only needs the rational numbers and presented a method for completing a premetric space. He then used these ideas to define the real numbers in constructive mathematics without CC. We will present a method for completing premetric spaces similar to Richman's but simpler.

Now we recall the basic definitions and the proposed completion introduced by Richman. Let X be a nonempty set. A binary relation d between $X \times X$ and the nonnegative rationals is a *premetric* on X, and we write $d(x, y) \leq q$ instead of $((x, y), q) \in d$, if it satisfies the following conditions for all $x, y, z \in X$ and all nonnegative rational numbers p, q:

(1) $d(x, y) \leq 0$ if and only if $x = y$.

(2) if $d(x, y) \leq q$ then $d(y, x) \leq q$.

(3) if $d(x, z) \leq p$ and $d(z, y) \leq q$ then $d(x, y) \leq p + q$ (triangular inequality).

(4) $d(x, y) \leq p$ if and only if $d(x, y) \leq q$ for all $q > p$ (upper continuity).

A set with a premetric is called a *premetric space*, and we use the notation (X, d) to specify the set and its premetric. Also, we write $d(x, y) \nleq q$ instead of $((x, y), q) \notin d$. When a relation d satisfies 2, 3, 4 above and also $d(x, x) \leq 0$ for all $x \in X$, then d is called a *pseudo-premetric*. When (X, d) is a pseudo-premetric space, then the relation $x \sim y$ if $d(x, y) \leq 0$ is an equivalence relation. Thus we can define the corresponding quotient and get a premetric space where the induced premetric is given by $d([x], [y]) \leq q$ if $d(x, y) \leq q$. The proof that $\sim$

is an equivalence relation, and that the induced premetric is indeed a premetric depends only on the first three defining conditions of a pseudo-premetric.

In general, common notions defined for metric spaces can be copied on premetric spaces. If $D \subseteq X$ and X is premetric space, then D is said to be *dense* in X when given any $\varepsilon > 0$ and $x \in X$, there exists $y \in D$ such that $d(y, x) \leq \varepsilon$. A map $f : X \to Y$ between two premetric spaces (X, d_X) and (Y, d_Y) is an *isometric embedding* if for all $x, x' \in X$ and q a non negative rational, $d_X(x, x') \leq q$ if, and only if, $d_Y(f(x), f(x')) \leq q$. When f is in addition onto, then it is called an *isometry*. The metric notion of diameter of a set is also defined on a premetric space as a binary relation, namely, we write diam $A \leq q$ for $A \subseteq X$ and a nonnegative rational number q, if for all $x, y \in A$, $d(x, y) \leq q$.

Richman introduced the following notions. A family $\{S_q : q \in \mathbb{Q}^+\}$ of subsets of X is *regular* if $d(x, y) \leq p + q$ for all $x \in S_p$ and $y \in S_q$. Two regular families $S = \{S_q : q \in \mathbb{Q}^+\}$ and $T = \{t_q : q \in \mathbb{Q}^+\}$ are *equivalent* if $d(x, y) \leq p + q$ for all $x \in S_p$ and $y \in T_q$. Let $\widehat{X}_R$ be the quotient of all regular families under that equivalence relation. The natural identification $i_R : X \to \widehat{X}_R$ is defined by $i_R(x) = [S^x]$, where $S^x = \{S_q : q \in \mathbb{Q}^+\}$ with $S_q = \{x\}$ for all $q \in \mathbb{Q}^+$. Thus we have the notion of completeness introduced by Richman: A premetric space (X, d) is *R-complete* if the map i_R is onto. And finally, a premetric on $\widehat{X}_R$ is defined as follows: $\widehat{d}_R([S], [T]) \leq q$, if for all $\varepsilon > 0$, there are $a, b, c \in \mathbb{Q}^+$ and $s \in S_a$, $t \in T_b$ such that $a + b + c < q + \varepsilon$ and $d(s, t) \leq c$. Then Richman proved that $i_R : X \to \widehat{X}_R$ is an isometric embedding of X into $\widehat{X}_R$ with dense image.

On his review of Richman's paper [8], A. Setzer [9] remarked that there was not a formal verification that $(\widehat{X}_R, \widehat{d}_R)$ was indeed a R-complete premetric space. Motivated in part by Setzer's remark, we define a premetric space $\widehat{X}$ using, instead of regular families, some collections of subsets of X similar to Cauchy filters but simpler. We introduce a notion of completeness and show that $\widehat{X}$ is a completion of X. Our method is somewhat similar to that of completing a uniform space. Finally, we show that $(\widehat{X}_R, \widehat{d}_R)$ is isometric to ours and it is R-complete, thus filling a gap left in Richman's paper.

2. Completeness

In order to simplify the treatment of completeness developed by Richman, we use, instead of regular families, a notion similar to that of a Cauchy filter but much weaker. Similar ideas about the completion of uniform spaces can be found in §3, Chapter II of [1].

Definition 2.1. Let X be a premetric space and $F \subseteq \mathcal{P}(X)$. We say that F is a *Cauchy family* on X if it satisfies the following:

i) $S \cap T \neq \varnothing$ for all $S, T \in F$.

ii) for all $\varepsilon > 0$ there exists $S \in F$ such that $\operatorname{diam} S \leq \varepsilon$.

A trivial example of Cauchy family is $\{\{x\}\}$, where x is any point of a premetric space. A Cauchy filter is a just a Cauchy family which is also a filter. Notice that a Cauchy family does not need to be even a filter base, as it could fail to have the finite intersection property. For instance, consider $\mathbb{Q}$ as a premetric space with its natural premetric given by the relation $|x - y| \leq q$. Let $F = \{S_q : q \in \mathbb{Q}^+\} \cup \{\mathbb{Q}_0^+, \mathbb{Q}_0^-\}$, where $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$, $\mathbb{Q}_0^- = \mathbb{Q}^- \cup \{0\}$ and

$$S_q = \{t \in \mathbb{Q} : 0 < |t| \leq q\}$$

for $q \in \mathbb{Q}^+$. Then F is a Cauchy family that does not satisfy the finite intersection property as $S_p \cap \mathbb{Q}_0^+ \cap \mathbb{Q}_0^- = \varnothing$.

There is no problem in using Cauchy filters instead of regular families, but the notion of a Cauchy family is sufficient for our purposes.

Now we introduce a pseudo-premetric on the collection of Cauchy families.

Definition 2.2. Let F and F' be Cauchy families on X, we say that $d(F, F') \leq q$, if for all $\varepsilon > 0$ there exist $S \in F$ and $T \in F'$ such that:

i) $\operatorname{diam} S, \operatorname{diam} T \leq \varepsilon$.

ii) $\operatorname{diam}(S \cup T) \leq q + \varepsilon$.

The intuition behind the premetric on the collection of Cauchy families is as follows. Considering each Cauchy family F as a point, then the elements of F are "good approximations" of F. Thus smaller the diameter of an element $S \in F$, the better an approximation it is. Thus, roughly speaking, two Cauchy families F, F' are at distance at most q, when there are approximations S and T of F and F', respectively, such that S and T are at distance at most q.

Here we abuse the notation a bit, we use d for both the premetric on X and the ternary relation just defined above.

Theorem 2.3. *Let (X, d) be a premetric space. The relation d on the collection of Cauchy families is a pseudo-premetric.*

Proof. Let F and F' be two Cauchy families and q a nonnegative rational number. Applying directly conditions $i)$ and $ii)$ from Definition 2.2 we have that $d(F, F) \leq q$ and $d(F, F') \leq q$ if $d(F', F) \leq q$.

To prove the triangular inequality, suppose that $d(F, G) \leq p$ and $d(G, H) \leq q$. Then for each $\varepsilon > 0$ there are subsets $S \in F$, $U \in H$ and $T, T' \in G$ such that $\operatorname{diam} S, \operatorname{diam} U, \operatorname{diam} T, \operatorname{diam} T' \leq \frac{\varepsilon}{2}$, and such that $\operatorname{diam}(S \cup T) \leq p + \frac{\varepsilon}{2}$ and $\operatorname{diam}(T' \cup U) \leq q + \frac{\varepsilon}{2}$. By condition $i)$ of the definition of a Cauchy family, there is $t \in T \cap T'$. Thus $d(x, t) \leq p + \frac{\varepsilon}{2}$ and $d(t, y) \leq q + \frac{\varepsilon}{2}$ for all $x \in S$ and $y \in U$, so by the triangular inequality on X we get $d(x, y) \leq p + q + \varepsilon$, and

diam $(S \cup U) \leq p + q + \varepsilon$. Also, by upper continuity on X, diam S, diam $U \leq \varepsilon$, so we have shown that $d(F, H) \leq p + q$.

Finally to show upper continuity, suppose first that $d(F, F') \leq p$ and $q > p$. Then for each $\varepsilon > 0$ we can find $S \in F$ and $T \in F'$ such that diam S, diam $T \leq \varepsilon$, satisfying moreover that diam $(S \cup T) \leq p + \varepsilon$. Since $p + \varepsilon < q + \varepsilon$ the upper continuity of d on X implies that also diam $(S \cup T) \leq q + \varepsilon$, and thus by definition, $d(F, F') \leq q$. Conversely, suppose $d(F, F') \leq q$ for all $q > p$. Let $\varepsilon > 0$ and set $q = p + \frac{\varepsilon}{2}$, there are $S \in F$ and $T \in F'$ such that diam S, diam $T \leq \frac{\varepsilon}{2}$ and such that diam $(S \cup T) \leq q + \frac{\varepsilon}{2} = p + \varepsilon$. By applying again upper continuity (on X) we have that diam S, diam $T \leq \varepsilon$ and thus $d(F, F') \leq p$. ☑

Now we introduce the premetric space of Cauchy families of a given premetric space.

Definition 2.4. Let X be a premetric space. The premetric space obtained from the quotient of the set of all Cauchy families on X, induced by its respective pseudo-premetric, is denoted by $(\widehat{X}, \widehat{d})$. Also, we use the notation x^* for the equivalence class of the family $\{\{x\}\}$.

As usual, we denote the elements of $\widehat{X}$ as $[F]$, the equivalence class of the Cauchy family F.

Theorem 2.5. *Let X be a premetric space. The function $i : X \to \widehat{X}$ defined as $i(x) = x^*$ is an isometric embedding with dense image.*

Proof. To verify that i is an isometric embedding, suppose first $x, y \in X$ and $d(x, y) \leq q$. Since diam $(\{x\} \cup \{y\}) \leq q \leq q + \varepsilon$ for each $\varepsilon > 0$, then $d(\{\{x\}\}, \{\{y\}\}) \leq q$ and thus $\widehat{d}(x^*, y^*) \leq q$. On the other hand, suppose $\widehat{d}(x^*, y^*) \leq q$, then diam $(\{x\} \cup \{y\}) \leq q + \varepsilon$ for each $\varepsilon > 0$, and therefore $d(x, y) \leq q + \varepsilon$ for each $\varepsilon > 0$. So by upper continuity we conclude $d(x, y) \leq q$.

Now let us show that $i(X)$ is dense in $\widehat{X}$. Suppose $\varepsilon > 0$ and $[F] \in \widehat{X}$. By the definition of a Cauchy family, there exists $S \in F$ such that diam $S \leq \varepsilon$. Since $S \neq \varnothing$, fix an element $s \in S$. We claim that $d(F, \{\{s\}\}) \leq \varepsilon$. In fact, let $\delta > 0$, there exists $T \in F$ such that diam $T \leq \delta$. Let $t \in T \cap S$, as $S \cap T \neq \varnothing$. Then for all $u \in T$ we have that $d(u, t) \leq \delta$ and $d(t, s) \leq \varepsilon$, and therefore $d(u, s) \leq \varepsilon + \delta$. That is, diam $(T \cup \{\{s\}\}) \leq \varepsilon + \delta$ and $d(F, \{\{s\}\}) \leq \varepsilon$. Thus we have found an element $s \in X$ such that $\widehat{d}([F], s^*) \leq \varepsilon$. ☑

Following Richman, we now introduce the notion of a complete premetric space, in terms of its canonical map.

Definition 2.6. Let X be a premetric space. We say that X is *complete*, if the map $i : X \to \widehat{X}$ is onto.

3. Completion

Having at hand the formal definition of completeness, we now introduce the natural notion of a completion of a premetric space.

Definition 3.1. Let X and Y be premetric spaces. We say that Y is a *completion* of X, if Y is complete, and there exists an isometric embedding $j : X \to Y$ with dense image.

By Theorem 2.5, the existence of a completion of a premetric space X will be immediately guaranteed once we show that $(\widehat{X}, \widehat{d})$ is complete. To show that $\widehat{X}$ is complete, we prove an extension theorem. We remark that the heart of the argument is this extension theorem, the rest of the proof of the completeness of $\widehat{X}$ and the uniqueness of the completion will be achieved by algebraic manipulation of diagrams.

Theorem 3.2. *Let $(A, d_A), (B, d_B)$ and (C, d_C) be premetric spaces, $f : A \to B$ an isometric embedding, and $h : A \to C$ an isometric embedding with dense image. Then there exists a unique isometric embedding $g : C \to \widehat{B}$ such that the following diagram commutes*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} \\
C & \xrightarrow{\ g\ } & \widehat{B}
\end{array}
$$

Proof. We define first for each $c \in C$ and each positive rational q, the set

$$
S_q^c = \{f(x) \in B : x \in A,\ d_C(h(x), c) \leq q/2\}.
$$

Thereby we set $S^c = \{S_q^c : q \in \mathbb{Q}^+\}$. We will show that S^c is a Cauchy family on B for all $c \in C$. Indeed, since h has dense image, given two nonnegative rationals μ and ν, there exists $x \in A$ such that $d_C(h(x), c) \leq \frac{1}{2}\min\{\mu, \nu\}$. Therefore $f(x) \in S_\mu^c \cap S_\nu^c$ and we have shown that $S_\mu^c \cap S_\nu^c \neq \varnothing$. On the other hand, to check the second condition of a Cauchy family, it suffices to show that $\operatorname{diam} S_\varepsilon^c \leq \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$, then for all $x, y \in A$ such that $d_C(h(x), c) \leq \varepsilon/2$ and $d_C(h(y), c) \leq \varepsilon/2$, we have, by the triangular inequality, that $d_C(h(x), h(y)) \leq \varepsilon$. Since h and f are isometric embeddings we also have $d_A(x, y) \leq \varepsilon$ and $d_B(f(x), f(y)) \leq \varepsilon$. Thus $\operatorname{diam} S_\varepsilon^c \leq \varepsilon$ and S^c is a Cauchy family.

Now define $g(c) = [S^c]$ for each $c \in C$. To prove that g is an isometric embedding suppose $d_C(c, c') \leq p$. To see that $\widehat{d}_B([S^c], [S^{c'}]) \leq p$, fix $\varepsilon > 0$, since $\operatorname{diam} S_\varepsilon^c \leq \varepsilon$ and $\operatorname{diam} S_\varepsilon^{c'} \leq \varepsilon$, it suffices to show that $\operatorname{diam}(S_\varepsilon^c \cup S_\varepsilon^{c'}) \leq p + \varepsilon$. This follows immediately from the triangular inequality since for every $f(x) \in S_\varepsilon^c$ and $f(y) \in S_\varepsilon^{c'}$ we have $d_B(f(x), f(y)) \leq p + \varepsilon$. Conversely,

suppose $\widehat{d}_B([S^c], [S^{c'}]) \leq p$. To see that $d_C(c, c') \leq p$ it suffices to show that $d_C(c, c') \leq p + \varepsilon$ for all $\varepsilon > 0$. Then, let $\varepsilon > 0$ and choose μ and ν such that $\operatorname{diam}(S_\mu^c \cup S_\nu^{c'}) \leq p + \varepsilon/3$. Since the image of h is dense in A, there exist $a, a' \in A$ such that $d_C(h(a), c) \leq \frac{1}{3}\min\{\mu, \varepsilon\}$ and $d_C(h(a'), c') \leq \frac{1}{3}\min\{\nu, \varepsilon\}$. As $d_C(h(a), c) \leq \varepsilon/3$ and $d_C(h(a'), c') \leq \varepsilon/3$, then $h(a) \in S_\mu^c$ and $h(a') \in S_\nu^{c'}$ and therefore $d_C(h(a), h(a')) \leq p + \varepsilon/3$. Hence by the triangular inequality, $d_C(c, c') \leq p + \varepsilon$ for all $\varepsilon > 0$, and by upper continuity we conclude $d_C(c, c') \leq p$.

Finally observe that for all $a \in A$ we have $f(a) \in S_\varepsilon^{h(a)}$ for all $\varepsilon > 0$, thereby $\operatorname{diam}(\{f(a)\} \cup S_\varepsilon^{h(a)}) \leq \varepsilon$ for all $\varepsilon > 0$. Thus $\{\{f(a)\}\}$ is equivalent to $S^{h(a)}$, that is, $i(f(a)) = (f(a))^* = [S^{h(a)}] = g(h(a))$. Therefore $g \circ h = i \circ f$ and the diagram commutes.   ☑

Finally, we show that $\widehat{X}$ is indeed a completion and it is unique up to isometry.

Theorem 3.3. $(\widehat{X}, \widehat{d})$ *is a complete premetric space for every premetric space* (X, d) *and thus every premetric space admits a completion which is unique up to isometry.*

Proof. We will show that $\widehat{X}$ is complete. By Theorem 2.5, we have the isometric embeddings $i : X \to \widehat{X}$ and $j : \widehat{X} \to \widehat{\widehat{X}}$ each one with dense image. Thus the map $k = j \circ i : X \to \widehat{\widehat{X}}$ has also dense image. Thereby we can apply the Theorem 3.2 to find a map l such that the following diagram commutes

$$
\begin{array}{ccccc}
X & \xrightarrow{\mathrm{id}} & X & \xrightarrow{i} & \widehat{X} \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
\widehat{\widehat{X}} & \xrightarrow{l} & \widehat{X} & \xrightarrow{j} & \widehat{\widehat{X}}
\end{array}
$$

Contracting the diagram we have

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \widehat{X} \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} \\
\widehat{\widehat{X}} & \xrightarrow{jl} & \widehat{\widehat{X}}
\end{array}
$$

By the uniqueness given by Theorem 3.2 $j \circ l = \mathrm{id}$, and this proves that j is onto, that is, that $\widehat{X}$ is complete.

To prove uniqueness, suppose Y is a complete premetric space and $h : X \to Y$ is an isometric embedding with dense image. Let $j : Y \to \widehat{Y}$ and $i : X \to \widehat{X}$ be the natural embeddings. Since Y is complete, j is a bijection. Then by

Theorem 3.2 there are maps u, v such that the following diagram commutes.

$$
\begin{array}{ccccc}
X & \xrightarrow{\;\text{id}\;} & X & \xrightarrow{\;jh\;} & \widehat{Y} \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j^{-1}} \\
Y & \xrightarrow{\;u\;} & \widehat{X} & \xrightarrow{\;v\;} & Y
\end{array}
$$

Contracting the diagram we have

$$
\begin{array}{ccc}
X & \xrightarrow{\;jh\;} & \widehat{Y} \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j^{-1}} \\
Y & \xrightarrow{\;vu\;} & Y
\end{array}
$$

By uniqueness we have $v \circ u = \text{id}$. Thus the map u is an isometry between Y and $\widehat{X}$.   ☑

As we said in the introduction, Richman proposed in [8] a way of completing a premetric space but he did not formally verified that it was indeed a complete premetric space. Now we are going to show that $(\widehat{X}_R, \widehat{d}_R)$ is R-complete. The definition was given in the introduction.

Let $\widehat{X}_R$ be the quotient of all regular families under the equivalence relation defined in the introduction and $\widehat{d}_R$ be the premetric on $\widehat{X}_R$. The natural identification $i_R : X \to \widehat{X}_R$ is defined by $i_R(x) = [S^x]$, where $S^x = \{S_q : q \in \mathbb{Q}^+\}$ with $S_q = \{x\}$ for all $q \in \mathbb{Q}^+$. Richman proved that i is an isometric embedding with dense image (see [8, Theorem 2.2]). Let $\widehat{\widehat{X}}_R$ denote the quotient of all regular families of $(\widehat{X}_R, \widehat{d}_R)$ and $\widehat{i}_R : \widehat{X}_R \to \widehat{\widehat{X}}_R$ the natural isometric embedding. We will show that $\widehat{i}_R$ is onto.

We need the following lemma proved by Richman.

Lemma 3.4. [8, Lemma 2.1]. *Let X be a premetric space and $S = \{S_q : q \in \mathbb{Q}^+\}$ a regular family on X. Then $d_R(i_R(x), [S]) \leq q$ for any $x \in S_q$ and $q \in \mathbb{Q}^+$.*

Theorem 3.5. *Let X be a premetric space, then*

   (i) *There exists an isometry $\varphi : (\widehat{X}_R, \widehat{d}_R) \to (\widehat{X}, \widehat{d})$.*

   (ii) *The map $\widehat{i}_R : \widehat{X}_R \to \widehat{\widehat{X}}_R$ is onto.*

Proof. (i) As the natural map $i : X \to \widehat{X}_R$ is an isometric embedding with dense image, then by Theorem 3.2 there exists an isometric embedding φ such

that the next diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathrm{id}\ } & X \\
\downarrow{\scriptstyle i_R} & & \downarrow{\scriptstyle j} \\
\widehat{X}_R & \xrightarrow{\ \varphi\ } & \widehat{X}
\end{array}
$$

We prove now that φ is onto. Let $s \in \widehat{X}$, we define for each positive rational q the set

$$
S_q = \{x \in X : \widehat{d}(j(x), s) \le q\}. \tag{1}
$$

Since j is an isometric embedding, it is easy to verify (using the triangular inequality) that $S = \{S_q : q \in \mathbb{Q}^+\}$ is a regular family. By Lemma 3.4 $\widehat{d}_R([S], i_R(x)) \le q$, and thus $\widehat{d}(\varphi([S]), \varphi(i_R(x))) \le q$, as φ is an isometric embedding. By the commutativity of the diagram, we have that $\widehat{d}(\varphi([S]), j(x)) \le q$. Since $x \in S_q$ we have also $\widehat{d}(j(x), s) \le q$, so by triangular inequality on $\widehat{X}$ we conclude $\widehat{d}(\varphi([S]), s) \le 2q$ for any $q \in \mathbb{Q}^+$, that is, $\varphi([S]) = s$ and φ is onto.

(ii) Let $\widehat{j} : \widehat{X} \to \widehat{\widehat{X}}$ be the natural map, $\varphi : \widehat{X}_R \to \widehat{X}$ the map defined in part (i) and $\widehat{i}_R : \widehat{X}_R \to \widehat{\widehat{X}}_R$ the natural map. By Theorem 3.2, there is an isometric embedding $\psi : \widehat{\widehat{X}}_R \to \widehat{\widehat{X}}$ such that the following diagram commutes

$$
\begin{array}{ccc}
\widehat{X}_R & \xrightarrow{\ \varphi\ } & \widehat{X} \\
\downarrow{\scriptstyle \widehat{i}_R} & & \downarrow{\scriptstyle \widehat{j}} \\
\widehat{\widehat{X}}_R & \xrightarrow{\ \psi\ } & \widehat{\widehat{X}}
\end{array}
$$

Since $\widehat{j}$, φ are onto maps, then as the diagram commutes $\widehat{i}_R$ is also onto. ☑

Richman [8] implicitly introduced a notion of maximal regular family by saying that a regular family $\{T_q\}_q$ is maximal if for any other equivalent regular family $\{T'_q\}_q$ we have $T'_q \subseteq T_q$ for each $q \in \mathbb{Q}^+$. For instance, in $\mathbb{Q}$ with the usual premetric, the regular family $\{T_q\}_q$ with $T_q = \{q\}$ is equivalent to the maximal regular family $\{S_q\}_q$ with $S_q = \{t \in \mathbb{Q} : |t| \le q\}$. He observed that every equivalence class contains a maximal element which he called its canonical representative. The regular family given by (1) is maximal. Moreover, it is also a Cauchy family. It follows from the proof of the previous theorem that if T is a regular family and F is a Cauchy family such that $[F] = \varphi[T]_R$, then $[T]_R \cap [F] \ne \varnothing$ (here we use $[T]_R$ to denote the equivalence class in the sense of Richman and $[F]$ in the sense used for $\widehat{X}$).

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