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Minimal prime ideals of skew PBW extensions over 2-primal compatible rings

Ideales primos minimales de extensiones PBW torcidas sobre anillos compatibles 2-primal

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Abstract. In this paper, we characterize the units of skew PBW extensions over compatible rings. With this aim, we recall the transfer of the property of being 2-primal for these extensions. As a consequence of our treatment, the results established here generalize those corresponding for commutative rings and Ore extensions of injective type. In this way, we present new results for several noncommutative rings of polynomial type not considered before in the literature.

Key words and phrases. Minimal prime ideal, 2-primal ring, unit, skew PBW extension.

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Resumen. En este artículo, caracterizamos las unidades de las extensiones PBW torcidas sobre anillos compatibles. Con este propósito, recordamos la transferencia de la propiedad 2-primal para estas extensiones. Como una consecuencia de nuestro tratamiento, los resultados establecidos aquí generalizan aquellos correspondientes para anillos conmutativos y extensiones de Ore de tipo inyectivo. De esta manera, presentamos nuevos resultados para anillos no conmutativos de tipo polinomial no considerados antes en la literatura.

Palabras y frases clave. Ideal primo minimal, anillo 2-primal, unidad, extensión PBW torcida.

1. Introduction

Throughout the paper, for a ring B, the *lower nil radical* or the *prime radical* (the intersection of all prime ideals in B), the *upper radical* (the sum of all nil ideals), the set of *nilpotent elements* of B, and the *Jacobson radical* of B are denoted by $\mathrm{Nil}_*(B), \mathrm{Nil}^*(B), \mathrm{Nil}(B)$ and $J(B)$, respectively. Now, as it is well-known in the literature, a ring B is called *2-primal*, if $\mathrm{Nil}_*(B) = \mathrm{Nil}(B)$, i.e., if the prime radical is completely semiprime (an ideal I of B is completely semiprime, if $a^2 \in I$ implies $a \in I$). The importance of 2-primal rings is that they can be considered as a generalization of commutative rings and reduced rings (a ring B is *reduced*, if B has no nonzero nilpotent elements). Commutative and reduced rings are strictly contained in 2-primal rings (see Marks [30] for a beautiful and detailed exposition about the relations between these rings). Several results about 2-primal rings can be found in the literature. For instance, Shin [49], Proposition 1.11, shows that a ring B is 2-primal if and only if every minimal prime ideal P of B is completely prime (i.e., B/P is a domain). He also proved that the minimal prime spectrum of a 2-primal ring is a Hausdorff space with a basis of closed-and-open sets ([49], Proposition 4.7).

With respect to Ore extensions defined by Ore [34], Ferrero and Kishimoto [8], Example 2.1, proved that if B is 2-primal, the differential polynomial ring $B[x;\delta]$ need not to be 2-primal. For a 2-primal ring B, Marks [28] investigated conditions on ideals of B that ensure that a skew polynomial ring $B[x;\sigma]$ or a differential polynomial ring $B[x;\delta]$ be 2-primal. On the other hand, Marks [29] considered the 2-primal property of the Ore extension $B[x;\sigma,\delta]$ where B is a local ring and σ is an automorphism of B. Marks shows that for a local ring with a nilpotent maximal ideal, the Ore extension $B[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of B. If $B[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; if $B[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition. In particular, Marks [29], Example 2.2, shows that Ore extensions of automorphism type $B[x;\sigma]$ (i.e., σ is an automorphism of B) need not be 2-primal. About minimal prime ideals of 2-primal rings, Kim and Kwak [20] is one of the most important works.

Now, with the aim of studying the radicals mentioned above, the notion of *rigidness* introduced by Krempa [21] has been very useful. Let us say a few words about this notion. Following [21], an endomorphism σ of a ring B is called to be *rigid*, if $a\sigma(a) = 0$ implies $a = 0$, for $a \in B$. B is said to be σ-*rigid*, if there exists a rigid endomorphism σ of B. One can see that any rigid endomorphism of a ring is a monomorphism, and σ-rigid rings are reduced rings (c.f. Hong et al., [16], p. 218). On the other hand, Hong et al., [17] considered the notion of σ-*rigid ideal*. They investigated relations between σ-rigid ideals of B and the related ideals of Ore extensions, and also studied connections between $\mathrm{Nil}_*(B)$ (resp. $\mathrm{Nil}^*(B)$) and $\mathrm{Nil}_*(B[x;\sigma,\delta])$ (resp. $\mathrm{Nil}^*(B[x;\sigma,\delta])$). For example, they proved that if $\mathrm{Nil}_*(B)$ (resp. $\mathrm{Nil}^*(B)$) is a σ-rigid δ-ideal of B, then $\mathrm{Nil}_*(B[x;\sigma,\delta]) \subseteq \mathrm{Nil}_*(B)[x;\sigma,\delta]$ (resp. $\mathrm{Nil}^*(B[x;\sigma,\delta]) \subseteq \mathrm{Nil}^*(B)[x;\sigma,\delta]$).

The above notion of rigidness was generalized by Annin in his PhD Thesis [1] (see also [2]; compare with Hashemi and Moussavi [14]) in the following way. For a ring B with an endomorphism σ and a σ-derivation δ, B is said to be *σ-compatible*, if for every $a, b \in B$, we have $ab = 0$ if and only if $a\sigma(b) = 0$ (necessarily, the endomorphism σ is injective); B is said to be *δ-compatible*, if for each $a, b \in B$, the equality $ab = 0$ implies $a\delta(b) = 0$. If B is both σ-compatible and δ-compatible, B is called *(σ, δ)-compatible*. The σ-rigid rings are (σ, δ)-compatible rings but the converse is false (see [15], Lemma 3.3, Examples 2.1, 2.2 and 2.3). Nevertheless, Hashemi et al., [14], Lemma 2.2, proved that a ring B is (σ, δ)-compatible and reduced if and only if B is σ-rigid. Hence σ-compatible rings generalize σ-rigid rings in the case B is not assumed to be reduced. The compatibility has been also defined for ideals by Hashemi [10], as a generalization of σ-rigid ideals. Precisely, there, he investigated the relations between $\mathrm{Nil}_*(B)$ (resp. $\mathrm{Nil}^*(B)$) and $\mathrm{Nil}_*(B[x; \sigma, \delta])$ (resp. $\mathrm{Nil}^*(B)$) assuming that $\mathrm{Nil}_*(B)$ (resp. $\mathrm{Nil}^*(B)$) is a (σ, δ)-compatible ideal of B and gave a generalization of Hong et al.'s results [17].

More recently, Nasr-Isfahani [31] investigated the radicals of Ore extensions assuming that the ring of coefficients is (σ, δ)-compatible. He proved that if B is (σ, δ)-compatible, then $B[x; \sigma, \delta]$ is 2-primal if and only if B is 2-primal if and only if $\mathrm{Nil}_*(B; \sigma, \delta) = \mathrm{Nil}(B)$ if and only if $\mathrm{Nil}_*(B[x; \sigma, \delta]) = \mathrm{Nil}(B)$ if and only if $\mathrm{Nil}(B)[x; \sigma, \delta] = \mathrm{Nil}_*(B[x; \sigma, \delta])$. He also proved that for a (σ, δ)-compatible ring B, $B[x; \sigma, \delta]$ is 2-primal if and only if for each $f \in B[x; \sigma, \delta]$, $f\sigma(f) \in \mathrm{Nil}_*(B[x; \sigma, \delta])$ if and only if $\mathrm{Nil}_*(B[x; \sigma, \delta])$ (see Definition of $\sigma(f)$ in Section 3). In his paper, he provided a generalization of Shin's Theorem above mentioned, and proved that for a (σ, δ)-compatible ring B, $B[x; \sigma, \delta]$ is 2-primal if and only if every minimal (σ, δ)-prime ideal of B is completely prime. Now, more recently, about Nasr-Isfahani's article [31], Wang and Chen [54] considered the relationship between prime ideals in B and the ones in $B[x; \sigma, \delta]$. They proved that for a (σ, δ)-compatible ring B, B is 2-primal if and only if for every minimal prime ideal P in $B[x; \sigma, \delta]$ there exists a minimal prime ideal P' in B such that $P = P'[x; \sigma, \delta]$, and that $f(x) \in B[x; \sigma, \delta]$ is a unit if and only if its constant term is a unit and other coefficients are nilpotent. They also proved that the Jacobson radical of $B[x; \sigma, \delta]$, $J(B[x; \sigma, \delta])$, is equal to $\mathrm{Nil}_*(B[x; \sigma, \delta])$, and that the stable range of $B[x; \sigma, \delta]$ is different from one.

Considering all above results and with the aim of extending of all them to the more general setting of skew PBW extensions introduced by Gallego and Lezama [9] (in Section 2, we say a few words about these objects), Hashemi et al., [12] studied under certain conditions the connections of the prime radical and the upper nil radical of a ring R with the prime radical and the upper nil radical of a skew PBW extension A over R. They also considered the transfer of several properties such as being prime, semiprime and the characterization of minimal prime ideals. Precisely, the purpose of this paper is to contribute to the study started in [12].

We describe the structure of the article. In Section 2, we establish some useful results about skew PBW extensions and (Σ, Δ)-compatible rings for the rest of the paper. We present some examples of these noncommutative rings where the results obtained in Section 3 can be applied. Section 3 contains the results about skew PBW extensions over 2-primal (Σ, Δ)-compatible rings which were proved by Hashemi et al., [12]. We include the proofs in order to make the article self-contained. These results extend all presented by Nasr-Isfahani [31] for Ore extensions. In Section 4, the original and new results of this paper about minimal prime ideals in 2-primal skew PBW extensions are presented. These results extend all established by Wang and Chen [54] for Ore extensions but now in the context of skew PBW extensions. Finally, the last section presents a possible line of research about the topics considered here.

We have to say that the techniques used in this paper are fairly standard and follow the same path as other texts on the subject, and hence the results are relatively new for skew PBW extensions, being similar to others existing in the literature. As a matter of fact, this paper continues the study of ideals of these extensions realized by several authors in [11], [12], [23], [33], [35], [44] and [47]. Therefore, our paper can be considered as a modest contribution to [12] and also to the study of the 2-primal property and minimal prime ideals for noncommutative rings of polynomial type which can not be expressed as Ore extensions of injective type.

Throughout the paper, the word ring means an associative ring (not necessarily commutative) with unit. The letter $\Bbbk$ denotes a field. $\mathbb{C}$ denotes the field of complex numbers. The symbol $\subset$ denotes that a set is strictly contained in other set.

2. Skew PBW extensions and (Σ, Δ)-compatible rings

Skew PBW extensions are a direct generalization of PBW extensions introduced by Bell and Goodearl [6]. They also are strictly more general than Ore extensions of injective type (see [47], Example 1, [40] or [41] for a list of noncommutative rings which are skew PBW extensions but not Ore extensions of injective type), and other families of noncommutative rings studied in the literature such as remarkable algebras appearing in representation theory, Hopf algebras, quantum groups, noncommutative algebraic geometry and other algebras of interest in the context of mathematical physics (e.g., [26], [40] and [52] for more details). Several ring theoretical and homological properties of these extensions have been studied by some people in the context of noncommutative algebra and noncommutative algebraic geometry (e.g., [3], [13], [24], [26], [27], [32], [38], [45] and [53]).

Next, we mention briefly some examples of skew PBW extensions (see [42] for a detailed reference of every example): (1) Universal enveloping algebras of finite dimensional Lie algebras. (2) Almost normalizing extensions. (3) Solvable polynomial rings. (4) Diffusion algebras. (5) 3-dimensional skew polynomial algebras studied by Rosenberg [48] (see also [41]). The advantage of skew PBW extensions is that they do not require the coefficients to commute with the variables and, moreover, the coefficients need not come from a field (see Definition 2.1). In fact, the skew PBW extensions share examples of algebras with generalized Weyl algebras defined by Bavula [5] (also known as hyperbolic algebras by Rosenberg [48]), with G-algebras and some PBW algebras defined by Bueso et al., [7], (both G-algebras and PBW algebras take coefficients in fields and assume that coefficientes commute with variables), Auslander-Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Artin-Schelter regular algebras, some Koszul and augmented Koszul algebras, quantum polynomials, some quantum universal enveloping algebras, some graded skew Clifford algebras and others (e.g., [18], [51] and [52]). As we can see, skew PBW extensions include a considerable number of noncommutative rings of polynomial type, so a theory of units of these extensions will establish results for algebras not considered before and, of course, it will cover also several treatments in the literature.

In this section we recall some results about skew PBW extensions and (Σ, Δ)-compatible rings which are important for the rest of the paper.

2.1. Skew PBW extensions

Definition 2.1 ([9], Definition 1)**.** Let R and A be rings. We say that A is a *skew PBW extension* (also known as *σ-PBW extension*) *over* R, which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same multiplicative identity element.

(ii) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis $\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, and $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements $1 \leq i, j \leq n$, there exists an element $d_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$ (i.e., there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)}$ of R with $x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{l=1}^{n} r_l^{(i,j)} x_l$).

Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $d_{i,j}$ are unique, ([9], Remark 2).

Proposition 2.2 ([9], Proposition 3)**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. From now on, we write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$, and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

***Remark* 2.3** ([9], Section 3)**.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension.

(i) Consider the families Σ and Δ in Proposition 2.2. Throughout the paper, for any element $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $\sigma^\alpha := \sigma_1^{\alpha_1} \circ \cdots \circ \sigma_n^{\alpha_n}$, $\delta^\alpha = \delta_1^{\alpha_1} \circ \cdots \circ \delta_n^{\alpha_n}$, where $\circ$ denotes composition, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) Given the importance of monomial orders in the proofs of some results presented in Sections 3 and 4, next we recall some key facts about these for skew PBW extensions.

Let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. If f is a nonzero element of A, then f can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R$, and $X_m \succ \cdots \succ X_1$ (eventually, we use expressions as $f = a_0 + a_1Y_1 + \cdots + a_mY_m$, with $a_i \in R$, and $Y_m \succ \cdots \succ Y_1$). With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^m$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Following [9], Definition 11, if $\succeq$ is a total order on $\mathrm{Mon}(A)$, we say that $\succeq$ is a *monomial order* on $\mathrm{Mon}(A)$, if the following conditions hold:

- For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$, $x^\beta \succeq x^\alpha$ implies $\mathrm{lm}(x^\gamma x^\beta x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\alpha x^\lambda)$ (the total order is compatible with multiplication).
- $x^\alpha \succeq 1$, for every $x^\alpha \in \mathrm{Mon}(A)$.
- $\succeq$ is degree compatible, i.e., $|\beta| \succeq |\alpha|$ implies $x^\beta \succeq x^\alpha$.

Monomial orders are also called *admissible orders*. The third condition of the monomial order is used in the proof of the fact that every monomial order on $\mathrm{Mon}(A)$ is a well order, that is, there are not infinite decreasing chains in $\mathrm{Mon}(A)$ (see [9], Proposition 12). Nevertheless, this hypothesis is not really needed to get a well ordering if a more elaborated argument, based upon Dickson's Lemma, is developed (see [7]). The importance of considering monomial orders on $\mathrm{Mon}(A)$ can be appreciated in [9], [19] and [25] where the Gröbner theory for left ideals, left modules and projective modules over skew PBW extensions was studied.

(iii) If R is a division ring, Definition 2.1 is a particular case of the notion of left PBW ring as defined in [7], where the tails in the relations appearing in (iv) do not need to be linear, using weighted admissible orderings. For general multifiltered extensions or left PBW rings, even the tails in the relations appearing in (iii) do not need to be in R.

Proposition 2.4 ([9], Theorem 7). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(1) for each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.

(2) For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $d_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = d_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $d_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

***Remark* 2.5** ([36], Proposition 2.9 and Remark 2.10 (iv)). If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension, then the following assertions hold:

(a) If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and r is an element of R, then

$$\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\
&+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))))) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R \ \text{ for } \ 1 \leq j \leq n.
\end{aligned}$$

(b) If $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, and a_i, b_j are elements of R, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i. This assertion follows from the expression:

$$\begin{aligned}
a_i X_i b_j Y_j &= a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b_j)} x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b_j} x^{\beta_j}.
\end{aligned}$$

Since we are concerned with the property of being 2-primal over skew PBW extensions, we need to establish a criterion which allows us to extend the family Σ of injective endomorphisms, and the family of Σ-derivations Δ of the ring R

(Proposition 2.2) to a skew PBW extension A over R. With this aim, for the next result consider the injective endomorphisms $\sigma_i \in \Sigma$, and the σ_i-derivations $\delta_i \in \Delta$ $(1 \leq i \leq n)$ formulated in Proposition 2.2.

Proposition 2.6 ([42], Theorem 5.1)**.** *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. Suppose that $\sigma_i\delta_j = \delta_j\sigma_i$, $\delta_i\delta_j = \delta_j\delta_i$, and $\delta_k(d_{i,j}) = \delta_k(r_l^{(i,j)}) = 0$, for $1 \leq i, j, k, l \leq n$, where $d_{i,j}$ and $r_l^{(i,j)}$ are the elements established in Definition 2.1. If $\overline{\sigma_k} : A \to A$ and $\overline{\delta_k} : A \to A$ are the functions given by $\overline{\sigma_k}(f) := \sigma_k(a_0)+\sigma_k(a_1)X_1+\cdots+\sigma_k(a_m)X_m$ and $\overline{\delta_k}(f) := \delta_k(a_0)+\delta_k(a_1)X_1+\cdots + \delta_k(a_m)X_m$, for every $f = a_0 + a_1X_1 + \cdots + a_mX_m \in A$ and each k, respectively, then $\overline{\sigma_k}$ is an injective endomorphism of A and $\overline{\delta_k}$ is a $\overline{\sigma_k}$-derivation of A.*

2.2. (Σ, Δ)-compatible rings

Following Krempa [21], an endomorphism σ of a ring B is called to be *rigid*, if $a\sigma(a) = 0$ implies $a = 0$, for $a \in B$. A ring B is said to be σ-rigid, if there exists a rigid endomorphism σ of B. It is clear that any rigid endomorphism of a ring is a monomorphism, and σ-rigid rings are reduced (see Hong et al., [16], p. 218). Properties of σ-rigid rings have been studied by several authors (e.g., [16] and [21]; see [36] for a detailed list of references). On the other hand, following Annin [1] and [2] (c.f. Hashemi and Moussavi [14]), for a ring B with an endomorphism σ and a σ-derivation δ, B is said to be σ-*compatible*, if for every $a, b \in B$, we have $ab = 0$ if and only if $a\sigma(b) = 0$; B is called δ-*compatible*, if for each $a, b \in B$, then $ab = 0$ implies $\delta(b) = 0$. If B is both σ-compatible and δ-compatible, then B is called (σ, δ)-*compatible*. In this case, the endomorphism σ is injective. Hashemi and Moussavi ([14], Lemma 2.2) proved that a ring B is (σ, δ)-compatible and reduced if and only if B is σ-rigid. Hence (σ, δ)-compatible rings generalize σ-rigid rings for the case B is not assumed to be reduced.

The natural task was to extend this notion of compatibility to the more general contexts of skew PBW extensions; this is the content of Definition 2.8. Before, we recall the notion of Σ-rigid ring which extends the σ-rigid rings above. Consider the notation in Remark 2.3 (i).

Definition 2.7. ([36], Definition 3.2) Let B be a ring and Σ a finite family of endomorphisms of B. Σ is called a *rigid endomorphisms family*, if for elements $r \in B$ and $\alpha \in \mathbb{N}^n$, the equality $r\sigma^{\alpha}(r) = 0$ implies $r = 0$. A ring B is said to be Σ-*rigid*, if there exists a rigid endomorphisms finite family Σ of B.

Note that if Σ is a rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then we have the equalities $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, that is, B is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see [16], Example 9). Ring theoretical properties of Σ-rigid rings have been studied in [32], [38], [39], [44], [45] and [47].

With the aim of establishing relations between skew PBW extensions and (Σ, Δ)-compatible rings, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations of a ring R (see Proposition 2.2). Definition 2.8 was introduced independently by Hashemi et al., [11] and Reyes and Suárez [43].

Definition 2.8 ([11], Definition 3.1; [43], Definition 3.2)**.** Consider a ring R with a finite family of endomorphisms Σ and a finite family of Σ-derivations Δ. Following the notation established in Remark 2.3 (i), we have: R is said to be Σ-*compatible*, if for each $a, b \in R$, $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, for every $\alpha \in \mathbb{N}^n$; R is said to be Δ-*compatible*, if for each $a, b \in R$, $ab = 0$ implies $a\delta^{\beta}(b) = 0$, for every $\beta \in \mathbb{N}^n$. If R is both Σ-compatible and Δ-compatible, R is called (Σ, Δ)-*compatible*. From now on, we consider finite families of endomorphisms and derivations, so we say *family* to mean *finite family*.

***Remark* 2.9.**

- From [43], Proposition 3.4, we know that every Σ-rigid ring is a (Σ, Δ)-compatible ring. The converse is false as we can appreciate in [43], Example 3.6. In this way, Σ-rigid rings are contained strictly in (Σ, Δ)-compatible rings. Nevertheless, these two notions coincide when the ring is assumed to be reduced. More precisely, if A is a skew PBW extension of a ring R, then the following statements are equivalent: (1) R is reduced and (Σ, Δ)-compatible. (2) R is Σ-rigid. (3) A is reduced ([11], Lemma 3.5; [43], Theorem 3.9).
- (Σ, Δ)-compatible rings extend the compatible rings defined by Annin's Ph.D. Thesis [1]. As a matter of fact, (Σ, Δ)-compatible rings have been very useful in the characterization of different radicals (Wedderburn radical, lower nil radical, Levitzky radical, upper nil radical, the set of all nilpotent elements, the sum of all nil left ideals) and other ring and module theoretical properties of skew PBW extensions such as the Kothe's conjecture (e.g., [18], [37], [43] and [44]).

The next proposition generalizes [14], Lemma 2.1.

Proposition 2.10 ([11], Lemma 3.3; [43], Proposition 3.8)**.** *Let R be a (Σ, Δ)-compatible ring. For every $a, b \in R$, we have:*

(i) If $ab = 0$, then $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$, for each $\theta \in \mathbb{N}^n$.

(ii) If $\sigma^{\beta}(a)b = 0$ for some $\beta \in \mathbb{N}^n$, then $ab = 0$.

(iii) If $ab = 0$, then $\sigma^{\theta}(a)\delta^{\beta}(b) = \delta^{\beta}(a)\sigma^{\theta}(b) = 0$, for every $\theta, \beta \in \mathbb{N}^n$.

2.2.1. Examples

In this short section we present remarkable examples of skew PBW extensions over (Σ, Δ)-compatible rings. A detailed list can be found in [18], [40] and [43] and [53].

(1) If A is a skew PBW extension of a reduced ring R where the coefficients commute with the variables, that is, $x_i r = r x_j$, for every $r \in R$ and each $i = 1, \ldots, n$, or equivalently, $\sigma_i = \mathrm{id}_R$ and $\delta_i = 0$, for every i, then it is clear that R is (Σ, Δ)-compatible. Some examples of these PBW extensions are the following: PBW extensions, solvable polynomial rings, some G-algebras, some PBW algebras, some Calabi-Yau and skew Calabi-Yau algebras, some Koszul and quadratic algebras. Examples of skew PBW extensions which satisfy these conditions on the σ's and δ's are the following (see [9], [18] or [26] for a detailed definition of σ's and δ's in every example): the algebra of linear partial differential operators, the algebra of linear partial shift operators, the algebra of linear partial difference operators, the algebra of linear partial q-dilation operators, and the algebra of linear partial q-differential operators; the class of diffusion algebras; some examples of quantum algebras such as Weyl algebras, the additive analogue of the Weyl algebra, the multiplicative analogue of the Weyl algebra; some quantum Weyl algebras, the quantum algebra $\mathcal{U}'(\mathfrak{so}(3, \Bbbk))$; Dispin algebra $\mathcal{U}(osp(1, 2))$; Woronowicz algebra $\mathcal{W}_v(\mathfrak{sl}(2, \Bbbk))$; the complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$; q-Heisenberg algebra $\mathbf{H}_n(q)$; the Hayashi algebra $W_q(J)$, and the family of 3-dimensional skew polynomial algebras studied by Rosenberg [48], Definition C4.3 (see also [41]).

(2) We also encounter examples of skew PBW extensions over (Σ, Δ)-compatible rings which do not satisfy the conditions above on the σ's and δ's. For instance (i) the quantum plane $\mathcal{O}_q(\Bbbk^2)$; the algebra of q-differential operators $D_{q,h}[x, y]$; the mixed algebra D_h; the operator differential rings; and the algebra of differential operators $D_{\mathbf{q}}(S_{\mathbf{q}})$ on a quantum space $S_{\mathbf{q}}$ (see [26] for the definition of every one of these algebras). Hashemi et al., [11], Example 3.2, and Behakanira et al., [53], Section 4, present other interesting examples of skew PBW extensions over (Σ, Δ)-compatible rings. Ore extensions studied by Artamonov et al., [4] can also be considered as illustrative examples.

3. On the property of being 2-primal

In this section we study the transfer of the property of being 2-primal from a ring R to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$. All results presented in this section have been proved recently by Hashemi et al., [12]. Nevertheless, we include (with minor changes) their proofs in order to make the article self-contained. It is important to say that these results generalized those corresponding in [31] for Ore extensions.

With the aim of establishing other results on radicals of skew PBW extensions, we consider the following definition.

Definition 3.1 ([44], Definition 6; [35], Section 4)**.** Let A be a skew PBW extension over a ring R. Consider the sets of endomorphisms Σ and Δ in Proposition 2.2. (i) An ideal I of R is called Σ-*invariant*, if $\sigma_i(I) \subseteq I$, for every $1 \leq i \leq n$. Δ-*invariant* ideals are defined similarly. If I is both Σ and Δ-invariant, we say that I is (Σ, Δ)-*invariant*. (ii) A proper Σ-invariant ideal of R is Σ-*prime*, if whenever a product of two Σ-invariant ideals is contained in I, one of the ideals is contained in I. Δ-*prime* and (Σ, Δ)-*prime ideals* are defined similarly. We write $P_{(\Sigma,\Delta)} := \text{Spec}(R; \Sigma, \Delta)$ for the set of all (Σ, Δ)-prime ideals of R and $\text{Nil}_*(R; \Sigma, \Delta) = \bigcap_{P\in P_{(\Sigma,\Delta)}} P$ for the (Σ, Δ)-prime radical of R. R is a Σ-*prime* (resp. Σ-*semiprime*) ring, if the ideal 0 is Σ-prime (resp. if $\text{Nil}_*(R; \Sigma) = 0$). In a similar way, we define Δ-*prime*, Δ-*semiprime*, (Σ, Δ)-*prime* and (Σ, Δ)-*semiprime* rings.

For a subset $S \subseteq R$, if $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, SA denotes the set given by $SA =: \{f \in A \mid f = a_0 + a_1X_1 + \cdots + a_mX_m,\ a_i \in S,\ \text{for all } i\}$.

We start with the next proposition which extends [31], Proposition 2.2. We need to assume that the elements d's appearing in Definition 2.1 (iv) are central in the ring R.

Proposition 3.2 ([12], Proposition 4.1)**.** *Let A be a skew PBW extension over a ring R. If R is a Σ-compatible ring and* $\text{Nil}(R)$ *is a Δ-ideal of R, then* $\text{Nil}(A) \subseteq \text{Nil}(R)A$.

Proof. Let $f = \sum_{i=0}^{m} a_iX_i$ be an element of $\text{Nil}(A)$ (with $X_1 \prec X_2 \prec \cdots \prec X_m$), and let $X_m := x^{\alpha_m} = x_1^{\alpha_{m1}} \cdots x_n^{\alpha_{mn}}$. As an illustration, note that

$$\begin{aligned}
f^2 &= (a_mX_m + \cdots + a_1x_1 + a_0)(a_mX_m + \cdots + a_1x_1 + a_0)\\
&= a_mX_ma_mX_m + \text{ other terms less than } \exp(x^{2\alpha_m})\\
&= a_m[\sigma^{\alpha_m}(a_m)X_m + p_{\alpha_m,a_m}]X_m + \text{ other terms less than } \exp(x^{2\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)X_mX_m + a_mp_{\alpha_m,a_m}X_m + \text{ other terms less than } \exp(x^{2\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)[d_{\alpha_m,\alpha_m}x^{2\alpha_m} + p_{\alpha_m,\alpha_m}] + a_mp_{\alpha_m,a_m}X_m\\
&+ \text{other terms less than } \exp(x^{2\alpha_m})\\
&= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m} + \text{ other terms less than } \exp(x^{2\alpha_m}),
\end{aligned}$$

and hence,

$$\begin{aligned}f^3 &= (a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m} + \text{ other terms less than } \exp(x^{2\alpha_m}))(a_mX_m\\ &+ \cdots + a_1x_1 + a_0)\\ &= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}x^{2\alpha_m}a_mX_m + \text{ other terms less than } \exp(x^{3\alpha_m})\\ &= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}[\sigma^{2\alpha_m}(a_m)x^{2\alpha_m} + p_{2\alpha_m,a_m}]X_m\\ &+ \text{other terms less than } \exp(x^{3\alpha_m})\\ &= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)x^{2\alpha_m}X_m\\ &+ \text{other terms less than } \exp(x^{3\alpha_m})\\ &= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)[d_{2\alpha_m,\alpha_m}x^{3\alpha_m} + p_{2\alpha_m,\alpha_m}]\\ &= a_m\sigma^{\alpha_m}(a_m)d_{\alpha_m,\alpha_m}\sigma^{2\alpha_m}(a_m)d_{2\alpha_m,\alpha_m}x^{3\alpha_m}\\ &+ \text{other terms less than } \exp(x^{3\alpha_m}).\end{aligned}$$

Continuing in this way, one can show that for f^k,

$$f^k = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)d_{l\alpha_m,\alpha_m}x^{k\alpha_m} + \text{ other terms less than } \exp(x^{k\alpha_m}),$$

whence $0 = \mathrm{lc}(f^k) = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)d_{l\alpha_m,\alpha_m}$, and since the elements d's are central and left invertible in R (Proposition 2.4), we have $0 = \mathrm{lc}(f^k) = a_m\prod_{l=1}^{k-1}\sigma^{l\alpha_m}(a_m)$, and so $a_m \in \mathrm{Nil}(R)$ (Proposition 2.10). With this in mind, let $f = g + a_mX_m$, with $g = a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1} \in A$ such that $\exp(g) \prec \exp(X_m)$. It is clear that $0 = f^k = g^k + h$, for some $h \in A$, and that the coefficients of h can be written as sums of polynomials in the coefficients $a_0, a_1, \ldots, a_m$ and their images by the elements in Σ and Δ (Remark 2.5 (b)). Now, since $a_m \in \mathrm{Nil}(R)$ and this ideal is a (Σ, Δ)-ideal of R, then every coefficient of each polynomial which consists of the elements $a_0, a_1, \ldots, a_m$ is also an element of $\mathrm{Nil}(R)$, whence $h \in \mathrm{Nil}(R)A$ because $\mathrm{Nil}(R)$ is an ideal of R. In this way, $g^k \in \mathrm{Nil}(R)A$. Using the same argument as above, one can show that $a_{m-1} \in \mathrm{Nil}(R)$, and if we repeat the process then we obtain that $a_i \in \mathrm{Nil}(R)$, for every i, which completes the proof. ☑

As a consequence of Proposition 3.2, we have that if $\mathrm{Nil}(R)$ is an ideal of R, then $\mathrm{Nil}(R[x]) \subseteq \mathrm{Nil}(R)[x]$. It is important to remark that Smoktunowicz [50] presented an example of a ring B such that $\mathrm{Nil}(B)$ is an ideal of B but $\mathrm{Nil}(B)[x] \nsubseteq \mathrm{Nil}(B[x])$.

The following proposition generalizes [31], Proposition 2.4.

Proposition 3.3 ([12], Proposition 4.6). *Let A be a skew PBW extension over a Σ-compatible ring R.*

(1) If $\mathrm{Nil}(R) = \mathrm{Nil}_*(R; \Sigma, \Delta)$*, then* A *is 2-primal.*

(2) If $\mathrm{Nil}(R)A = \mathrm{Nil}_*(A)$*, then* A *is 2-primal.*

Proof. (1) Having in mind that $\mathrm{Nil}_*(R; \Sigma, \Delta)$ is a (Σ, Δ)-ideal, Proposition 3.2 guarantees that $\mathrm{Nil}(A) \subseteq \mathrm{Nil}(R)A = \mathrm{Nil}_*(R; \Sigma, \Delta)A$. Now, one can check that $\mathrm{Nil}_*(R; \Sigma, \Delta)A \subseteq \mathrm{Nil}_*(A)$, and so A is 2-primal.

(2) From the assumptions we obtain that $\mathrm{Nil}(R)$ is an ideal of R, and it is clear that $\mathrm{Nil}(R)$ is a Σ-ideal of R. Note that if a is an element of $\mathrm{Nil}(R)$, since R is Σ-compatible, then $xa \in \mathrm{Nil}(R)A$ implies that $\delta_i(a) \in \mathrm{Nil}(R)$, for every $i = 1, \dots, n$. Finally, Proposition 3.2 guarantees that $\mathrm{Nil}(A) \subseteq \mathrm{Nil}(R)A = \mathrm{Nil}_*(A)$, whence A is 2-primal.

☑

With the aim of establishing Theorem 3.4 which extends [31], Theorem 2.5, consider the following notion introduced, precisely, in [31], p. 3. For a ring B, let $\mathrm{End}(B; +)$ be the ring of additive endomorphisms of B and consider Φ a subset of $\mathrm{End}(B; +)$. A sequence $(a_0, a_1, \dots, a_t, \dots)$ of elements of B is called a Φ-m-*sequence*, if for any $i \in \mathbb{N}$, there exist elements $\varphi_i, \varphi_i' \in \Phi$ and $r_i \in B$, such that $a_{i+1} = \varphi_i(a_i) r_i \varphi_i'(a_i)$. An element $a \in B$ is said to be *strongly* Φ*-nilpotent*, if every Φ-m-sequence starting with a eventually vanishes. If $\Phi = \{\mathrm{id}_B\}$, then we obtain the notions defined by Lam et al., [22].

The following theorem generalizes [31], Theorem 2.5.

Theorem 3.4 ([12], Theorem 14)**.** *If* A *is a skew PBW extension over a* (Σ, Δ)*-compatible ring* R*, then* A *is 2-primal if and only if* $\mathrm{Nil}(R) = \mathrm{Nil}_*(R; \Sigma, \Delta)$ *if and only if* $\mathrm{Nil}(R)A = \mathrm{Nil}_*(A)$*.*

Proof. Note that if $\mathrm{Nil}(R) = \mathrm{Nil}_*(R; \Sigma, \Delta)$, then Proposition 3.3 guarantees that A is 2-primal. Conversely, suppose that A is 2-primal. We know that $\mathrm{Nil}_*(R; \Sigma, \Delta)A \subseteq \mathrm{Nil}_*(A) = \mathrm{Nil}(A)$, and hence $\mathrm{Nil}_*(R; \Sigma, \Delta) \subseteq \mathrm{Nil}(R)$. Note that if A is 2-primal, it is clear that R is 2-primal. Consider $a \in \mathrm{Nil}(R) = \mathrm{Nil}_*(R)$. It follows that a is strongly nilpotent, which means by definition, that every m-sequence starting with a eventually vanishes. Now, from Proposition 2.10, one can see that every $\{\Sigma, \Delta\}$-m-sequence starting with a eventually vanishes, i.e., a is strongly $\{\Sigma, \Delta\}$-nilpotent, and so, $a \in \mathrm{Nil}_*(R; \Sigma, \Delta)$. This fact shows that $\mathrm{Nil}(R) \subseteq \mathrm{Nil}_*(R; \Sigma, \Delta)$.

Now, if $\mathrm{Nil}(R)A = \mathrm{Nil}_*(A)$, then Proposition 3.3 implies that A is 2-primal. Conversely, suppose that A is 2-primal. We know that $\mathrm{Nil}(R) = \mathrm{Nil}_*(R; \Sigma, \Delta)$ and hence we obtain the relations $\mathrm{Nil}(R)A = \mathrm{Nil}_*(R; \Sigma, \Delta)A \subseteq \mathrm{Nil}_*(A)$. Finally, Proposition 3.2 shows that $\mathrm{Nil}_*(A) = \mathrm{Nil}(A) \subseteq \mathrm{Nil}(R)A$, which completes the proof. ☑

Corollary 3.5 ([12], Corollary 4.8)**.** *If A is a skew PBW extension over a (Σ,Δ)-compatible ring R, then A is 2-primal if and only if R is 2-primal and* $\mathrm{Nil}(R)A=\mathrm{Nil}(A)$.

Proof. If A is 2-primal, then R is 2-primal, and Theorem 3.4 guarantees that $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$. Conversely, suppose that R is 2-primal and $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$. Using a similar argument to the used in the proof of Theorem 3.4, we obtain that $\mathrm{Nil}(R) \subseteq \mathrm{Nil}_*(R;\Sigma,\Delta)$. In this way, $\mathrm{Nil}(A) = \mathrm{Nil}(R)A \subseteq \mathrm{Nil}_*(R;\Sigma,\Delta)A \subseteq \mathrm{Nil}_*(A)$. This last fact completes the proof. ☑

The following proposition is an analogue to the established in [31], Lemma 2.7; its proof uses similar arguments to the used in that lemma.

Proposition 3.6 ([12], Lemma 4.9)**.** *If R is a reduced (Σ,Δ)-compatible ring and P is a minimal (Σ,Δ)-prime ideal of R, then P is completely prime.*

Theorem 3.7 presents a generalization of [49], Proposition 1.11, and [31], Theorem 2.8.

Theorem 3.7 ([12], Theorem 4.10)**.** *If A is a skew PBW extension over a (Σ,Δ)-compatible ring R, then A is 2-primal if and only if every minimal (Σ,Δ)-prime ideal of R is completely prime.*

Proof. Suppose that A is 2-primal. From Theorem 3.4 we know that the quotient $R/\mathrm{Nil}_*(R;\Sigma,\Delta)$ is a reduced ring. Consider P a minimal (Σ,Δ)-prime ideal of R. Then $P+\mathrm{Nil}_*(R;\Sigma,\Delta)$ is a minimal $(\overline{\Sigma},\overline{\Delta})$-prime ideal of $R/\mathrm{Nil}_*(R;\Sigma,\Delta)$. Since $R/P \cong \overline{R}/\overline{P}$, Proposition 3.6 implies that P is completely prime. Suppose that every minimal (Σ,Δ)-prime ideal P of R is completely prime. Consider $\{P_i\}_{i\in I}$ the family of all minimal (Σ,Δ)-prime ideals of R. It is clear that $\mathrm{Nil}_*(R;\Sigma,\Delta)=\bigcap_{i\in I}P_i$, and so $R/\mathrm{Nil}_*(R;\Sigma,\Delta)$ embeds in $\prod_{i\in I}R/P_i$. Then $R/\mathrm{Nil}_*(R;\Sigma,\Delta)$ is reduced and hence $\mathrm{Nil}(R)\subseteq \mathrm{Nil}_*(R;\Sigma,\Delta)$. Since $\mathrm{Nil}_*(R;\Sigma,\Delta)A \subseteq \mathrm{Nil}_*(A) \subseteq \mathrm{Nil}(A)$, we obtain that $\mathrm{Nil}_*(R;\Sigma,\Delta) \subseteq \mathrm{Nil}(R)$. Therefore $\mathrm{Nil}(R)=\mathrm{Nil}_*(R;\Sigma,\Delta)$ and the assertion follows from Theorem 3.4. ☑

The following theorem is an analogue to the established in [31]. This result does not appear in [12].

Theorem 3.8. *If A is a skew PBW extension over a (Σ,Δ)-compatible ring R, where the assumptions established in Proposition 2.6 hold, then A is 2-primal if and only if for every element $f\in A$, $f\sigma_i(f)\in \mathrm{Nil}_*(A)$ implies that $f\in \mathrm{Nil}_*(A)$, for every $\sigma_i\in\Sigma$.*

Proof. Suppose that A is 2-primal and let $f=a_0+a_1X_1+\cdots+a_mX_m$ be an element of A such that $f\sigma(f)\in\mathrm{Nil}_*(A)$. Theorem 3.4 guarantees that $f\sigma(f)\in$

$\mathrm{Nil}(R)A$, and hence $a_m\sigma^{\alpha_m}(a_m) \in \mathrm{Nil}(R)$. Using that R is Σ-compatible, we have that $a_m \in \mathrm{Nil}(R)$. Now, since

$$\begin{aligned} f\sigma(f) &= (a_0 + a_1X_1 + \cdots + a_{m-1}X^{m-1})(\sigma_i(a_0) + \sigma_i(a_1)X_1 + \cdots + \sigma_i(a_{m-1})X_{m-1}) \\ &+ (a_0 + a_1X_1 + \cdots + a_{m-1}X^{m-1})\sigma_i(a_m)X_m \\ &+ a_mX_m(\sigma_i(a_1)X_1 + \cdots + \sigma_i(a_{m-1})X_{m-1}) + a_mX_m\sigma_i(a_m)X_m, \end{aligned}$$

whence $(a_0+a_1X_1+\cdots+a_{m-1}X^{m-1})(\sigma_i(a_0)+\sigma_i(a_1)X_1+\cdots+\sigma_i(a_{m-1})X_{m-1})$ is an element of $\mathrm{Nil}(R)A$. Repeating this argument one can show that $a_{m-1} \in \mathrm{Nil}(R)$, and similarly, $a_i \in \mathrm{Nil}(R)$, for all i, and hence $f \in \mathrm{Nil}(R)A$. The result follows from Theorem 3.4.

Conversely, suppose that $f\sigma_i(f) \in \mathrm{Nil}_*(A)$. Then $f \in \mathrm{Nil}_*(A)$, for every $f \in A$. Consider $f \in A$ with $f^2 = 0$. It follows that $f\sigma_i(f)\sigma_i(f\sigma_i(f)) = 0 \in \mathrm{Nil}_*(A)$, whence $f\sigma_i(f) \in \mathrm{Nil}_*(A)$. This fact means that $f \in \mathrm{Nil}_*(A)$. Therefore $\mathrm{Nil}(A) \subseteq \mathrm{Nil}_*(A)$ which completes the proof. ☑

The following theorem generalizes [31], Theorem 2.10.

Theorem 3.9 ([12], Theorem 4.11)**.** *If A is a skew PBW extension over a (Σ,Δ)-compatible ring R, then A is 2-primal if and only R is 2-primal.*

Proof. Suppose that R is 2-primal and let $a \in \mathrm{Nil}(R) = \mathrm{Nil}_*(R)$. It follows that a is strongly nilpotent, and since R is (Σ,Δ)-compatible, then a is strongly $\{\Sigma,\Delta\}$-nilpotent, and it follows that $a \in \mathrm{Nil}_*(R;\Sigma,\Delta)$. It is clear that $\mathrm{Nil}_*(R;\Sigma,\Delta) \subseteq \mathrm{Nil}(R)$, so the result follows from Theorem 3.4. ☑

From the results obtained above, we have the following corollary which resumes our treatment about 2-primal property for skew PBW extensions. This corollary is an analogue of [31], Corollary 2.11.

Corollary 3.10 ([12], Corollary 4.12)**.** *If A is a skew PBW extension over a (Σ,Δ)-compatible ring R, then the following assertions are equivalent:*

(1) *R is 2-primal.*

(2) *A is 2-primal.*

(3) $\mathrm{Nil}(R) = \mathrm{Nil}_*(R;\Sigma,\Delta)$.

(4) $\mathrm{Nil}(R)A = \mathrm{Nil}_*(A)$.

(5) *R is 2-primal and* $\mathrm{Nil}(R)A = \mathrm{Nil}(A)$.

(6) *Every minimal (Σ,Δ)-prime ideal of R is completely prime.*

4. Minimal prime ideals and units

This section contains the original results of the paper. Our purpose is to establish analogue results to those established in [54] for Ore extensions but now in the context of skew PBW extensions.

4.1. Minimal prime ideals in skew PBW extensions

We start this section with the following result which follows directly from [54], Lemma 2.2.

Proposition 4.1. *If P is a Φ-prime ideal of a ring R, then P contains a minimal Φ-prime ideal of R.*

For the next proposition, we recall the following fact mentioned in [54], p. 379: If $f : R \to S$ is an epimorphism of rings, then there exists a one-to-one correspondence between the set of all prime ideals in R that contain $\ker(f)$ and the set of all prime ideals in S, given by the correspondence $P \mapsto f(P)$. In particular, P is a minimal prime ideal containing $\ker(f)$ in R if and only if $f(P)$ is a minimal prime ideal in S.

The next proposition is a similar result to the formulated in [54], Lemma 2.3.

Proposition 4.2. *If R is a 2-primal (Σ, Δ)-compatible ring and P is a minimal prime ideal of R, then $\sigma_i(P)$, $\sigma_i^{-1}(P)$, $\delta_i(P) \subseteq P$, for every $i = 1, \dots, n$. In this way, every σ_i induces an endomorphism $\overline{\sigma_i}$, and each δ_i induces a $\overline{\sigma_i}$-derivation of $\overline{R} = R/P$, defined by $\overline{\sigma_i}(r + P) := \overline{\sigma_i(r)}$, $\overline{\delta_i}(r + p) := \overline{\delta_i(r)}$, for all $r \in R$. Hence, the ring $\overline{R}$ is a $\overline{\Sigma}$-rigid ring, and so, it is a $(\overline{\Sigma}, \overline{\Delta})$-compatible ring.*

Proof. It is clear that the assumptions above show that $\mathrm{Nil}(R) = \mathrm{Nil}_*(R)$, and so $R/\mathrm{Nil}_*(R)$ is reduced. Consider the family of applications $\widehat{\sigma_i}, \widehat{\delta_i} : R/\mathrm{Nil}_*(R) \to R/\mathrm{Nil}_*(R)$ defined by $\widehat{\sigma_i}(r + \mathrm{Nil}_*(R)) = \sigma_i(R) + \mathrm{Nil}_*(R)$, $\widehat{\delta_i}(r + \mathrm{Nil}_*(R)) = \delta_i(r) + \mathrm{Nil}_*(R)$, for all $i = 1, \dots, n$. Using the (Σ, Δ)-compatibility of R, Remark 2.5 (a) and Proposition 2.10 imply that $\mathrm{Nil}_*(R)$ is a (Σ, Δ)-ideal, which guarantees that the applications $\widehat{\sigma_i}$, $\widehat{\delta_i}$ are well-defined, for all i. Of course, $\widehat{\sigma_1}, \dots, \widehat{\sigma_n}$ is a family of endomorphisms of $R/\mathrm{Nil}_*(R)$ and $\widehat{\Delta} = \{\widehat{\delta_1}, \dots, \widehat{\delta_n}\}$ is a family of $\widehat{\Sigma}$-derivations of $R/\mathrm{Nil}_*(R)$. Let us see that $R/\mathrm{Nil}_*(R)$ is a $\widehat{\Sigma}$-rigid ring. Suppose that $(r + \mathrm{Nil}_*(R))(\widehat{\sigma^\theta}(r + \mathrm{Nil}_*(R))) = 0 + \mathrm{Nil}_*(R)$, for every $\theta \in \mathbb{N}^n$. Then $(r + \mathrm{Nil}_*(R))(\sigma^\theta(r) + \mathrm{Nil}_*(R)) = 0 + \mathrm{Nil}_*(R) = r\sigma^\theta(r) + \mathrm{Nil}_*(R)$, whence $r\sigma^\theta(r) \in \mathrm{Nil}_*(R)$, and so $r^2 \in \mathrm{Nil}_*(R)$ (Proposition 2.10). Since R is 2-primal, $r \in \mathrm{Nil}_*(R)$, i.e., $r + \mathrm{Nil}_*(R) = 0 + \mathrm{Nil}_*(R)$. Hence, $R/\mathrm{Nil}_*(R)$ is $\widehat{\Sigma}$-rigid. Now, by assumption, P is a minimal prime ideal of R, so $P + \mathrm{Nil}_*(R)$ is a minimal prime ideal in $R + \mathrm{Nil}_*(R)$ which implies that $\widehat{\sigma_i}(P + \mathrm{Nil}_*(R))$, $\widehat{\sigma_i}^{-1}(P + \mathrm{Nil}_*(R)) \subseteq P + \mathrm{Nil}_*(R)$ ([21], Lemma 3.2).

In this way, $\sigma_i(P)$ and $\sigma_i^{-1}(P)$ are contained in P, so [21], Theorem 3.3 shows that $\widehat{\delta_i}(P + \mathrm{Nil}_*(R)) \subseteq P + \mathrm{Nil}_*(R)$, for every i, and so $\delta_i(P) \subseteq P$, for all i. By Remark 2.9, $R/\mathrm{Nil}_*(R)$ is a $(\widehat{\Sigma}, \widehat{\Delta})$-compatible ring.

Finally, consider the ring R/P. Using that $\sigma_i(P), \delta_i(P) \subseteq P$, for all i, the applications $\overline{\sigma_i}$ and $\overline{\delta_i}$ as defined in the formulation of the proposition, are well-defined. Since P is a completely prime ideal in R and $\sigma_i^{-1}(P) \subseteq P$, it follows that R/P is a $\overline{\Sigma} = \{\overline{\sigma_1}, \ldots, \overline{\sigma_n}\}$-rigid ring, and hence, $(\overline{\Sigma}, \overline{\Delta})$-compatible. ☑

From Remark 2.5 (a) and Proposition 4.2 we obtain that if R is a 2-primal (Σ, Δ)-compatible ring and P is a minimal prime ideal of R, then PA is an ideal of A.

The following proposition generalizes [54], Lemma 2.4.

Proposition 4.3. *If R is a reduced (Σ, Δ)-compatible ring, then an ideal P of R is a minimal prime ideal if and only if P is a minimal (Σ, Δ)-prime ideal.*

Proof. Suppose that P is a minimal prime ideal of R. Proposition 4.2 implies that P is a (Σ, Δ)-ideal, and having in mind that R is 2-primal (because R is reduced), then P is also completely prime, which means that for all elements $a, b \in R$, if $a, b \notin P$ then $ab \notin P$. This fact shows that the element $\sigma_i(a)1\sigma_i(b) = \sigma_i(ab)$ does not belong to P (Proposition 4.2). Next, consider Φ the multiplicative semigroup with unit generated in $\mathrm{End}(R, +)$ by Σ and Δ. It follows that $R \backslash P$ is a Φ-m-system. In this way, P is a Φ-prime ideal, that is, P is a (Σ, Δ)-prime ideal. Note that if P is not minimal, then there exists a minimal (Σ, Δ)-prime ideal P' of R with $P' \subset P$. Proposition 3.6 shows that P' is completely prime, but this fact contradicts the minimality of P.

Conversely, suppose that P is a minimal (Σ, Δ)-prime ideal of R. Proposition 3.6 implies that P is completely prime. Note that if P is not minimal, then there exists a minimal prime ideal P' of R with $P' \subset P$, whence P' is (Σ, Δ)-prime, a contradiction, of course. This completes the proof. ☑

The following proposition extends [54], Lemma 2.5.

Proposition 4.4. *If A is a skew PBW extension over a reduced and (Σ, Δ)-compatible ring R, then P is a minimal prime ideal in A if and only if there exists a minimal prime ideal P' in R such that $P = P'A$.*

Proof. From Remark 2.9 we know that R is Σ-rigid, so A is reduced. Suppose that P is a minimal prime ideal of A. By [49], Proposition 1.11, we can assert that P is completely prime. Let $Q := R \cap P$. It is clear that Q is an ideal of R. Since we have that $ab \in P$ implies $a \in P$ or $b \in P$, for any elements $a, b \in Q$, then Q is completely prime. Thus, there exists a minimal prime ideal P' of R such that $P' \subseteq Q$ and R/P' is a domain. Using these facts, the idea is to prove that $P = P'A$. First of all, note that Remark 2.5 and Proposition 4.2

imply that $P'A$ is an ideal of A. Using Proposition 4.2 we can observe that R/P' is a $(\overline{\Sigma}, \overline{\Delta})$-compatible ring, where $\overline{\Sigma}$ and $\overline{\Delta}$ are as in the formulation of Proposition 4.2. Let $h : A = \sigma(R)\langle x_1, \dots, x_n\rangle \to \sigma(R/P)\langle x_1, \dots, x_n\rangle$ the application defined in the natural way as $h(\sum_{i=0}^{m} a_i X_i) = \sum_{i=0}^{m}(a_i + P')X_i$, for any element $\sum_{i=0}^{m} a_i X_i \in A$. One can check that h is surjective and that it is an additive and multiplicative map. Now, since $\ker(h) = P'A$, then there exists a ring isomorphism between $A/P'A$ and $\sigma(R/P')\langle x_1, \dots, x_n\rangle$. Having in mind that R/P' is a domain, $\sigma(R/P')\langle x_1, \dots, x_n\rangle$ also is, so $P'A$ is a completely prime ideal of A. Now, using that $P' \subseteq P$, it follows that $P'A \subseteq P$, and hence the minimality of P guarantees that $P = P'A$.

Conversely, suppose that P is a minimal prime ideal of R. Using the argument above we obtain that PA is a completely prime ideal of A. To see that PA is minimal, note that if this is not the case, then there exists a minimal prime ideal I in A with $I \subset PA$. By the arguments above, we can see that there exists a minimal prime ideal P' in R with $I = P'A$, and so $P'A \subset PA$, whence $P' \subset P$, which is a contradiction. Therefore, PA is a completely prime minimal ideal of A, which completes the proof. ☑

The following theorem is the important result of this section. This is a similar result to the established in [54], Theorem 2.6.

Theorem 4.5. *If A is a skew PBW extension over a 2-primal (Σ, Δ)-compatible ring R, then P is a minimal prime ideal in A if and only if there exists a minimal prime ideal P' in R such that $P = P'A$.*

Proof. From Theorem 3.9 we know that A is 2-primal. Suppose that P is a minimal prime ideal in A. It is clear that the ideal $P + \text{Nil}_*(A)$ is minimal prime in the quotient ring $A/\text{Nil}_*(A)$. Using that A is 2-primal, we can observe that $\text{Nil}_*(A) = \text{Nil}_*(R)A$ (Corollary 3.10). If we consider the applications $\widehat{\sigma_i}, \widehat{\delta_i} : R/\text{Nil}_*(R) \to \text{Nil}_*(R)$ formulated in Proposition 4.2, we obtain that $R/\text{Nil}_*(R)$ is $(\widehat{\Sigma}, \widehat{\Delta})$-compatible. Now, for the ring homomorphism $h : A = \sigma(R)\langle x_1, \dots, x_n\rangle \to \sigma(R/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$ defined by $h(\sum_{i=0}^{m} a_i X_i) = \sum_{i=0}^{m}(a_i + \text{Nil}_*(R))X_i$, for every element $\sum_{i=0}^{m} a_i X_i$ of A, we have that $\ker(h) = \text{Nil}_*(R)A = \text{Nil}_*(A)$, and thus we obtain the ring isomorphism $k : A/\text{Nil}_*(A) \xrightarrow{\cong} \sigma(R/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$. This isomorphism shows that $k(P + \text{Nil}_*(A))$ is a minimal prime ideal in $\sigma(R/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$, whence there exists a minimal prime ideal P' in R with $k(P + \text{Nil}_*(A)) = \sigma(P'/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$ (Proposition 4.4). Equivalently, $P + \text{Nil}_*(A) = P'A/\text{Nil}_*A$, by the definition of k, and so $P = P'A$.

Conversely, suppose that P is a minimal prime ideal of R. It is clear that $P + \text{Nil}_*(R)$ is a minimal prime ideal of the reduced ring $R/\text{Nil}_*(R)$, whence $(P + \text{Nil}_*(R))A$ is a minimal prime ideal in $\sigma(R/\text{Nil}_*(R))$ (Proposition 4.4). Using the ring isomorphism k above, we obtain that $PA/\text{Nil}_*(A)$ is a minimal prime ideal in $A/\mathrm{Nil}_*(A)$. Therefore PA is a minimal prime ideal in A, which completes the proof. ☑

Corollary 4.6. *If R is a 2-primal (Σ, Δ)-compatible ring and $\{P_i\}_{i\in I}$ are all minimal prime ideals of R, then $N_*(A) = \bigcap_{i\in I} P_iA$.*

Proof. The assertion follows from Proposition 4.1 and Theorem 4.5. ☑

Corollary 4.7. *Let A be a skew PBW extension over a (Σ, Δ)-compatible ring R. A is 2-primal if and only if for every minimal prime ideal P in A there exists a minimal prime ideal P' in R such that $P = P'A$.*

Proof. One direction is a consequence of Theorem 4.5. For the converse, the assumption guarantees that for any minimal prime ideal P in A, $P = P'A$ for some minimal prime ideal P' in R. In this way, $A/P \cong \sigma(R/P)\langle x_1, \dots, x_n\rangle$ is a domain, which implies that P is a completely prime ideal of A. Therefore, Shin [49], Proposition 1.11 shows that A is a 2-primal ring. ☑

4.2. Units in 2-primal skew PBW extensions

Finally, in this section we study the units in skew PBW extensions over 2-primal (Σ, Δ)-compatible rings. We start with the following proposition which extends [54], Lemma 3.1.

Proposition 4.8. *Let A be a skew PBW extension over a reduced and (Σ, Δ)-compatible ring R. If $f = \sum_{i=0}^{m} a_iX_i$ and $g = \sum_{j=0}^{t} b_jY_j$ are elements of A such that $fg = c \in R$, then $a_0b_0 = c$ and $a_ib_j = 0$, for all elements i, j with $i + j > 0$.*

Proof. Let us prove that $a_ib_j = 0$, for $i + j > 0$. If this is not the case, then there exist elements $f = \sum_{i=0}^{m} a_iX_i$, $g = \sum_{j=0}^{t} b_jY_j$ with $m + t$ minimal such that $fg = c \in R$, but the other sentences are not satisfied. Of course, the case $m + t = 0$ gives a contradiction, and hence we consider $m + t > 0$. With this in mind, consider $m+n > 0$. Note that the leading coefficient of fg is $a_m\sigma^{\theta_m}(b_j) = 0$, and by the (Σ, Δ)-compatibility of R, we obtain that $a_mb_t = 0$, that is, $b_ta_m = 0$ (because R is reduced). This fact shows that $b_tfg = b_tc$, but the minimality of the number we have that for any i, j with $i + j > 0$, the equality $b_ta_ib_j = 0$ holds, whence $(b_ta_i)^2 = 0$, and so $a_ib_t = 0$, for $i > 0$ (R is reduced). By the (Σ, Δ)-compatibility of R we obtain that $a_iX_ib_t = 0$, that is, $f(g - b_tX_t) = c$. The minimality of $m + t$ gives us the assertion. Continuing in this way, one can see that $c_0 = a_0b_0 = fg = c$, and $a_ib_j = 0$, for all elements i, j with $i + j > 0$. ☑

***Remark* 4.9.** The following two results are direct consequence of Proposition 4.8. Both correspond to the established in [54], Corollaries 3.2 and 3.3, respectively. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension. Then:

(1) Let R be a (Σ, Δ)-compatible ring. R is a domain if and only if A is a domain.

(2) Let R be a reduced (Σ, Δ)-compatible ring. If $f = \sum_{i=0}^{m} a_i X_i$ is a unit of A, then a_0 is a unit and $a_i = 0$, for all $i \geq 1$.

The next theorem is the important result of this section and is an analogue to the proved in [54], Theorem 3.4.

Theorem 4.10. *If A is a skew PBW extension over a 2-primal (Σ, Δ)-compatible ring R, then an element $f = \sum_{i=0}^{m} a_i X_i \in A$ is a unit if and only if a_0 is a unit of R and other coefficients are nilpotent.*

Proof. By assumption, R is 2-primal and (Σ, Δ)-compatible, so A is 2-primal and $\text{Nil}_*(A) = \text{Nil}_*(R)A$ (Corollary 3.10). From the proof of Proposition 4.2 we observe that $R/\text{Nil}_*(R)$ is reduced $(\widehat{\Sigma}, \widehat{\Delta})$-compatible. Considering the notation used in that proposition, we can assert that there exists a ring homomorphism $A \to \sigma(R/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$ given by $f = \sum_{i=0}^{m} a_i X_i \mapsto \widehat{f} = \sum_{i=0}^{m}(a_i + \text{Nil}_*(R))X_i$. Now, note that if $f \in A$ is a unit, then $\widehat{f}$ also is in $\sigma(R/\text{Nil}_*(R))\langle x_1, \dots, x_n\rangle$. From Remark 4.9 (2) we obtain that $a_0 + \text{Nil}_*(R)$ is a unit and $a_l + \text{Nil}_*(R) = 0 + \text{Nil}_*(R)$, for every $l = 1, \dots, m$, whence a_0 is a unit of R and a_i is nilpotent for al l.

The converse is clear having in mind that f is a sum of a unit and an element of $\text{Nil}_*(A)$. ☑

The second important result of this section is a theorem which generalizes [54], Theorem 3.5.

Theorem 4.11. *If A is a skew PBW extension over a 2-primal (Σ, Δ)-compatible ring R, then $J(A) = \text{Nil}_*(A)$.*

Proof. Consider an element $f = a_0 + a_1 X_1 + \cdots + a_m X_m \in J(A)$. If $1 + fx$ is a unit, then Theorem 4.10 implies that $a_i \in \text{Nil}_*(R)$, for all $i \geq 0$. In this way, Corollary 3.10 shows that $f \in \text{Nil}_*(A)$, and so $J(A) \subseteq \text{Nil}_*(A)$. The other inclusion is clear. ☑

***Remark* 4.12.**
- It is not hard to see that if $A = \sigma(R)\langle x_1, \dots, x_n, \rangle$ is a bijective skew PBW extension over a domain R, then A is also a domain and $J(A) = \{0\}$ (and hence $\text{Nil}_*(A) = \{0\}$), and that if A is a bijective skew PBW extension of a prime ring R, then A is also a prime ring and $\text{Nil}_*(A) = \{0\}$. Now, since domains are reduced rings which are contained strictly in 2-primal rings (see [30] for more details), and (Σ, Δ)-compatible rings are more general than Σ-rigid rings (which are also reduced), we can think of Theorem 4.11 as a generalization of these two facts for skew PBW extensions over rings which are more general than domains.

- In [54], Theorem 3.6 and Corollary 3.7, it was proved that if B is a reduced (σ,δ)-compatible ring, then its stable range $\mathrm{sr}(B)$ is different from 1, and if B is a 2-primal (σ,δ)-compatible ring, then $\mathrm{sr}(B)\neq 1$. For the class of skew PBW extensions, in [25], Proposition 72, it was shown that if R is a left Noetherian ring with finite left Krull dimension $\mathrm{lKdim}(R)$ and $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ is a bijective skew PBW extension (following the notation presented in Definition 2.1, A is called *bijective*, if σ_i is bijective for each $1\leq i\leq n$, and $d_{i,j}$ is invertible, for any $1\leq i,j\leq n$), then $1\leq \mathrm{sr}(A)\leq \mathrm{lKdim}(R)+n+1$. In this way, if we adapt the proofs presented in [54], Theorem 3.6 and Corollary 3.7 for the context of skew PBW extensions, we can show that the stable range of a skew PBW extension $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ over a (Σ,Δ)-compatible ring R satisfying other conditions is different from one, and hence combining with the result presented in [25], Proposition 72, we have that the stable range of these objects is greater than one and less or equal than $\mathrm{lKdim}(R)+n+1$. As one can see, this result agrees with [25], Example 73.

5. Future work

Recently, in [46] the second author has considered skew PBW extensions over weak compatible rings. These structures are more general than compatible rings in the sense studied in this paper, so a natural task is to investigate minimal prime ideals and units of skew PBW extensions over this class of rings. A similar task can be formulated for the study of modules over these extensions with the aim of extending the results obtained in [37].

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