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Extinction and survival in competitive Lotka-Volterra systems with constant coefficients and infinite delays

Extinción y sobrevivencia en sistemas competitivos de Lotka-Volterra con coeficientes constantes y retardos infinitos

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Abstract. The qualitative properties of a nonautonomous competitive Lotka-Volterra system with infinite delays are studied.

By using a result of matrix theory and the fluctuation lemma, we establish a series of easily verifiable algebraic conditions on the coefficients and the kernel, which are sufficient to ensure the survival and the extinction of a determined number of species. The surviving part is stabilized around a globally stable critical point of a subsystem of the system under study. These conditions also guarantee the asymptotic behavior of the system.

Key words and phrases. Lotka-Volterra system, extinction, competition, stability, delay, persistence.

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Resumen. Se estudian las propiedades cualitativas de un sistema competitivo no aútonomo de Lotka-Volterra con retardo infinito.

Mediante el uso de un resultado de la teoría de matrices y del lema de fluctuaciones, se establecen una serie de condiciones algebraicas, fácilmente verificables, sobre los coeficientes y los núcleos, que son suficientes para garantizar la extinción y la sobrevivencia de un determinado número de especies. La parte sobreviviente se estabiliza alrededor de un punto de equilibrio globalmente estable de un subsistema del sistema en estudio. Estas condiciones también garantizan el comportamiento asintótico del sistema.

Palabras y frases clave. Sistemas de Lotka-Volterra, extinción, sobrevivencia, estabilidad, retardo, persistencia.

1. Introduction

This article deals with nonautonomous competitive systems of integro-differential equations with infinite delays,

$$x_i'(t) = x_i(t)\left[b_i - a_{ii}x_i(t) - \sum_{j=1 j\neq i}^{n} a_{ij}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds\right], \quad t\geq 0,\ i=1,\dots,n, \tag{1}$$

with the initial conditions

$$x_i(t) = \varphi_i(t), \quad t\leq 0, \quad i=1,\dots,n, \tag{2}$$

where

$$\varphi_i \in BC^+ = \{\varphi \in [(-\infty,0],[0,\infty)] : \varphi(0)>0 \text{ and } \varphi \text{ is bounded }\},\ i=1,\dots,n.$$

$$x_i'(t) = x_i(t)\left[b_i - a_{ii}x_i(t) - \sum_{j=1 j\neq i}^{r} a_{ij}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds\right], \quad t\geq 0,\ i=1,\dots,r, \tag{3}$$

r being an integer such that $1<r<n$, the initial condition

$$x_i(t) = \varphi_i(t), \quad t\leq 0, \quad i=1,\dots,r, \quad \text{where } \varphi_i\in BC^+, \tag{4}$$

Here the initial functions $\varphi_i(s)$, $i=1,\dots,n$, are the histories of the sizes of the populations in the past. The coefficients $b_i,\ a_{ij},$ for all $1\leq i,j\leq n$, are positive real numbers and the delay kernels $K_{ij}\colon [0,+\infty)\to[0,+\infty)$, $1\leq i,j\leq n$, are continuous functions such that $\int_0^{+\infty} K_{ij}(t)dt = 1$. Integrals represent the hereditary or accumulated inhibitory effect in the past that the population size of one species has had upon the other species.

The following hypotheses are also considered

H_1 : For every $k=2,\dots,n$, there exists an integer $i_k<k$ such that

$$\frac{b_k}{a_{kj}} < \frac{b_{i_k}}{a_{i_k j}}, \quad j=1,\dots,k.$$

$H_2:$ For every $i = 1, \dots r$, the inequality $b_i > \sum_{j=1, j\neq i}^{r} a_{ij}\left(\frac{b_j}{a_{jj}}\right)$ is hold.

$H_3: \ \mu_{ij} = \int_0^\infty sK_{ij}(s)ds < \infty, \ 1 \leq i, j \leq n.$

$H_4:$ For every integer $k = r+1, \dots, n$, there exists an integer $i_k < k$ such that

$$\frac{b_k}{a_{kj}} < \frac{b_{i_k}}{a_{i_k j}}, \quad j = 1, \dots, k.$$

Observation 1.1. It is easy to see that there is a natural correspondence between the solutions of the r-dimensional system (3) and those solutions of (1) for which $x_j(t) = 0$ for all $j = r+1, \dots, n$. That is, $col(x_1(t), \dots, x_r(t))$ is a solution of (3) if and only if $col(x_1(t), \dots, x_r(t), 0, \dots, 0)$ is solution of (1).

Observation 1.2. Ahmad and Lazer in [21] proved that if $b_i > 0$, $a_{ij} > 0$ and if hypothesis H_2 is satisfied; that is,

$$b_i > \sum_{j=1, j\neq i}^{r} a_{ij}\left(\frac{b_j}{a_{jj}}\right), \ 1 \leq i \leq r,$$

then there exists a unique solution $col(u_1^*, \dots, u_r^*)$ of the system of linear equations

$$\sum_{j=1}^{r} a_{ij}x_j = b_i, \ \ i = 1, \dots, r, \tag{5}$$

with $u_i^* > 0$, $1 \leq i \leq r$, and the matrix $r \times r$ (a_{ij}) is invertible or non-singular.

Observation 1.3. It can be easily proved that the following points are equilibrium points for the system (1) if and only if

$$x_i^* = 0, \quad \text{or} \quad \sum_{j=1}^{n} a_{ij}x_j^* = b_i, \ i = 1, \dots, n.$$

The positive critical points are the positive solutions of the linear system of n equations with n unknowns

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, \dots, n.$$

It can be easily showed that the following points are equilibrium points for the system (1): $col\left(\frac{b_1}{a_{11}}, 0, \dots, 0\right)$, $col\left(0, \frac{b_2}{a_{22}}, 0, \dots, 0\right), \dots,$ $col\left(0, \dots, 0, \frac{b_n}{a_{nn}}\right)$, $x^* = col(u_1^*, \dots, u_r^*, 0, \dots, 0) = col(u^*, 0)$ where $0 = col(0, \dots, 0)$, $u^* = col(u_1^*, \dots, u_r^*)$, $2 \leq r < n$, and u^* is a solution of the linear system (5).

Observation 1.4. The hypothesis H_1 implies that the only critical points with non-negative components of the system (1) are $col(0,\dots,0)$, $col\left(\frac{b_1}{a_{11}},0,\dots,0\right)$, $col\left(0,\frac{b_2}{a_{22}},0,\dots,0\right),\dots,col\left(0,\dots,0,\frac{b_n}{a_{nn}}\right)$. For a proof see [27].

This work is included within the area of mathematical biology, particularly in population dynamics, for we study non-autonomous competitive Lotka-Volterra systems of n-species with infinite delays. These models with delay are also called hereditary models with memory. They are introduced in biology, for example, in biokinetics, which study the theories of biological equilibrium mathematically based on the struggle for existence between species competing on the ground for the same resources. The evolution of these systems is influenced significantly not only by the present but also by its most remote past. There are delayed effects in these systems due to the memory functions, the delay kernel. The use of these models with delay dates back to the time of the Italian mathematician Vito Volterra (1860-1940) and the French mathematician Marcel Brelot (1903-1987), who studied the classical prey-predator model with continuous infinite delay in their publications [25] and [4] respectively. Over the last 50 years and especially the last 20 years, these systems have been and continue to be investigated intensely, as can be seen in the references cited in this work.

The system (1) describes the competition between n species, where $x_i(t)$ denotes the population density of the i-th species at time t. It is well known, by the fundamental theory of systems of functional differential equations [15], that (1) has only one solution $x(t) = col(x_1(t),\dots,x_n(t))$ which satisfies the initial conditions (2). It is easy to verify that the solutions of the system (1) corresponding to the initial conditions (2) are defined on $[0,+\infty)$ and remain positive for all $t \geq 0$.

There is a substantial amount of research dealing with the global asymptotic stability of the Lotka-Volterra systems with infinite delay developed in [11], [14], [16], [24] and in the references cited there. Other studies on competition models have been done by several authors, see [1], [2], [3], [5], [8], [12], [13], [18], [22] [26], [28], [29]. In [18], [22], [28], [29], the authors provide conditions on the coefficients that imply permanence or extinction in the non-autonomous Lotka-Volterra systems, but without delay.

It should be mentioned that, during the last decade, there were many works on the competitive non-autonomous Lotka-Volterra systems with infinite delay and feedback controls by various researchers (see, e.g., [6], [7], [19], [20] and references cited there).

Montes de Oca and Pérez, by constructing Lyapunov-type functionals, established in [9] and [10] a series of algebraic conditions on the coefficients and kernel which guarantee the survival and extinction of a number of species. In this work the same results are proved without using the construction of Lyapunov functionals. The following results are obtained:

(1) If H_1 and H_3 hold, then any solution $x(t) = col\,(x_1(t), x_2(t), \ldots, x_n(t))$ of the system (1), with initial condition (2), has the property that $x_i(t) \to 0$ exponentially for all i, $2 \leq i \leq n$ as $t \to +\infty$ and $x_1(t) \to \dfrac{b_1}{a_{11}}$, where $\dfrac{b_1}{a_{11}}$ is an equilibrium point of the logistics equation $x'(t) = x(t)[b_1 - a_{11}x(t)]$. Here the point $(\dfrac{b_1}{a_{11}}, 0, \ldots, 0)$ is a global attractor of the system (1).

(2) If H_2, H_3 and H_4 hold, then any solution $x(t) = col\,(x_1(t), x_2(t), \ldots, x_n(t))$ of the system (1), with initial condition (2) has the property that $x_i(t) \to 0$ exponentially for all $i = r+1, \ldots, n$ as $t \to \infty$ and for all i, $1 \leq i \leq r$, $x_i(t) \to u_i^*$ as $t \to \infty$, where $u^* = col\,(u_1^*, \ldots, u_r^*)$ is a positive equilibrium point of the system (3). The point $col\,(u_1^*, \ldots, u_r^*, 0, \ldots, 0)$ is a global attractor of the system (1).

2. Preliminaries

The following two Lemmas are established in [11].

Lemma 2.1. Let $x : \mathbb{R} \to \mathbb{R}$ be a bounded non-negative continuous function, and let $K : [0, +\infty) \to [0, +\infty)$ be a continuous kernel such that $\int_0^\infty K(s)ds = 1$. Then

$$\begin{aligned}\underline{x} = \liminf_{t\to+\infty} x(t) &\leq \liminf_{t\to+\infty} \int_{-\infty}^t K(t-s)x(s)ds \\ &\leq \limsup_{t\to+\infty} \int_{-\infty}^t K(t-s)x(s)ds \leq \limsup_{t\to+\infty} x(t) = \overline{x}.\end{aligned}$$

Lemma 2.2. Let $x : \mathbb{R} \to \mathbb{R}$ be a bounded non-negative continuous function, and let $K : [0, +\infty) \to [0, +\infty)$, be a continuous kernel such that $\int_0^\infty K(s)ds = 1$. If $\lim\limits_{t\to+\infty} x(t) = x_0$, then $\lim\limits_{t\to+\infty} \int_{-\infty}^t K(t-s)x(s)ds = x_0$.

The following lemma is fully proved in the papers [23], [11] and [17].

Lemma 2.3. (**Fluctuation Lemma**) Let $x(t)$ be a derivable and bounded function on (α, ∞). Then there exist sequences $\tau_n \to \infty$, $\sigma_n \to \infty$ such that

i) $x'(\sigma_n) \to 0 \quad$ and $\quad x(\sigma_n) \to \overline{x} = \limsup\limits_{t\to+\infty} x(t), \quad$ as $n \to \infty$.

ii) $x'(\tau_n) \to 0 \quad$ and $\quad x(\tau_n) \to \underline{x} = \liminf\limits_{t\to+\infty} x(t), \quad$ as $n \to \infty$.

Lemma 2.4. Let $b = col(b_1, b_2, \ldots, b_r)$ and let $A = (a_{ij})$ be the matrix with $a_{ij} \geq 0$, and $a_{ii}\,, b_i > 0, \quad 1 \leq i, j \leq r$. Suppose that hypothesis H_2 is satisfied. Then the matrix $2D - A$, where $D = diag\,(a_{11}, \ldots, a_{rr})$ is the diagonal matrix of A, has inverse and is given by $\sum_{i=1}^{\infty} \left(AD^{-1} - I\right)^i$; moreover, it is non-negative.

Proof. See ([24], Lemma 4.1). ☑

Lemma 2.5. Suppose that hypotheses H_2 and H_3 hold. If $x(t) = col\,(x_1(t), x_2(t), \ldots, x_r(t))$ is a solution of the system (3) with the initial conditions (4), then for all i, $1 \leq i \leq r$,

$$\limsup_{t\to+\infty} x_i(t) \leq \frac{b_i}{a_{ii}}, \text{ and } \liminf_{t\to+\infty} x_i(t) \geq \frac{b_i - \sum\limits_{j=1, j\neq i}^{r} a_{ij}\left(\dfrac{b_j}{a_{jj}}\right)}{a_{ii}e^{L_i}},$$

where $L_i = \sum_{j=1}^{r} a_{ij} M_j \mu_{ij}$, and $M_i > \max\left\{\varphi_i^u, \dfrac{2b_i}{a_{ii}}\right\}$, $1 \leq i \leq r$. This means that the system (3) is permanently coexistent.

Proof. See Lemmas 2.6 and 2.7 in [10]. ☑

Proposition 2.6. Suppose that hypotheses H_1 and H_3 hold. If $x(t) = col(x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of the system (1) with the initial conditions (2), then $x_i(t) \to 0$ exponentially for all i, $2 \leq i \leq n$, as $t \to +\infty$. Moreover, there exists a positive number $\alpha = \alpha(x)$ such that $x_1(t) \geq \alpha$ for all $t \geq 0$.

Proof. The proof of the proposition is analogous to the proof of the proposition 2.1 in [9]. The proof method used in [9] is based on using mathematical induction and proving first that $x_n \to 0$ exponentially as $t \to +\infty$. ☑

Proposition 2.7. Suppose that hypotheses H_2, H_3 and H_4 hold. If $x(t) = col\,(x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of the system (1) with the initial conditions (2), then $x_i(t) \to 0$ exponentially for all $i = r+1, \ldots, n$ as $t \to \infty$. Moreover, there exists a positive number $\overline{\alpha} = \overline{\alpha}(x)$ such that $x_i(t) > \overline{\alpha}$ for all $i = 1, \ldots, r$ and $t \geq 0$.

Proof. The proof is analogous to the proof of the Proposition 2.3 and 2.4 in [10]. ☑

3. Main theorems

Theorem 3.1. Suppose that hypotheses H_1 and H_3 hold. If $x(t) = col\,(x_1(t), x_2(t), \ldots, x_n(t))$ is a solution of the system (1) with the initial conditions (2), then $x_i(t) \to 0$ exponentially for all i, $2 \le i \le n$ as $t \to +\infty$ and $x_1(t) \to \dfrac{b_1}{a_{11}}$, where $\dfrac{b_1}{a_{11}}$ is an equilibrium point of the logistics equation $x'(t) = x(t)[b_1 - a_{11}x(t)]$; in other words, the point $col\left(\dfrac{b_1}{a_{11}}, 0, \ldots, 0\right)$ is a global attractor of the system (1).

***Proof*.** By Proposition 2.6, there exists a positive number $\alpha = \alpha(x)$ such that $x_1(t) \ge \alpha$ for all $t \ge 0$ and $x_i(t) \to 0$ exponentially for i, $2 \le i \le n$, as $t \to +\infty$. From this fact and by virtue of the lemma 2.2, it follows that

$$\lim_{t\to+\infty} \int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds = 0, \qquad j = 2, \ldots, n. \tag{6}$$

On the other hand, by Lemma 2.3 (Fluctuation Lemma), there exists a sequence $\tau_n^1 \to +\infty$ as $n \to +\infty$ such that $x_1'(\tau_n^1) \to 0$ and $x_1(\tau_n^1) \to \underline{x}_1$ as $n \to +\infty$, where $\underline{x}_1 = \liminf\limits_{t\to+\infty} x_1(t)$. Combining the above with the equation (6) we obtain

$$\lim_{n\to+\infty} \int_{-\infty}^{\tau_n^1} K_{ij}(\tau_n^1 - s)x_j(s)ds = 0 \text{ for } i = 2, \ldots, n.$$

Replacing t by τ_n^1 in the first equation of the system (1), it follows that

$$x_1'(\tau_n^1) = x_1(\tau_n^1)\left[b_1 - a_{11}x_1(\tau_n^1) - \sum_{j=2}^{n} a_{1j} \int_{-\infty}^{\tau_n^1} K_{1j}(\tau_n^1 - s)x_j(s)ds\right]. \tag{7}$$

Letting $n \to \infty$ we get $0 = \underline{x}_1[b_1 - a_{11}\underline{x}_1]$. Since $\underline{x}_1 > 0$, we have that

$$\underline{x}_1 = \liminf_{t\to+\infty} x_1(t) = \frac{b_1}{a_{11}}. \tag{8}$$

On the other hand, $x_1'(t) < x_1(t)\,[b_1 - a_{11}x_1(t)]$, from the Comparison Theorem it follows that $0 \le x_1(t) < v_1(t)$ for all $t \ge 0$, where $v_1(t)$ is the solution of the logistic equation $z_1'(t) = z_1(t)\,[b_1 - a_{11}z_1(t)]$, with $v_1(0) = x_1(0) = \varphi_1(0)$. Thus we conclude that

$$\limsup_{t\to+\infty} x_1(t) \le \limsup_{t\to+\infty} v_1(t) = \lim_{t\to+\infty} v_1(t) = \frac{b_1}{a_{11}}. \tag{9}$$

From (8) and (9) we obtain

$$\liminf_{t\to+\infty} x_1(t) = \frac{b_1}{a_{11}} \le \limsup_{t\to+\infty} x_1(t) \le \frac{b_1}{a_{11}}.$$

Hence $\lim\limits_{t\to+\infty} x_1(t) = \dfrac{b_1}{a_{11}}$. This completes the proof of the theorem. ☑

Theorem 3.2. Suppose that hypotheses H_2, H_3 and H_4 hold. If $x(t) = col\,(x_1(t), x_2(t), \dots, x_r(t))$ is a solution of the system (3) with the initial conditions (4), then $x_i(t) \to u_i^*$ for all i, $1 \leq i \leq r$ as $t \to \infty$, where $u^* = col\,(u_1^*, \dots, u_r^*)$ is a point of equilibrium of the system (3).

Proof. By Proposition 2.7, there exists a positive number $\overline{\alpha} = \overline{\alpha}(x)$ such that $x_i(t) \geq \overline{\alpha} > 0$ for all i, $1 \leq i \leq r$, $t \geq 0$. On the other hand, by Lemma 2.5, $x_i(t)$ is bounded above for all i, $1 \leq i \leq r$, and $t \geq 0$. Therefore, the components $x_i(t)$ for $1 \leq i \leq r$, are bounded above and below by positive constants. By Lemma 2.3 (Fluctuation Lemma) there are sequences $\tau_n^i \to \infty$ and $\sigma_n^i \to \infty$ as $n \to \infty$, such that for all i, $1 \leq i \leq r$,

$$x_i'(\sigma_n^i) \to 0 \quad \text{and} \quad x_i(\sigma_n^i) \to \overline{x}_i = \limsup_{t\to+\infty} x_i(t), \quad \text{as } n \to \infty,$$

$$x_i'(\tau_n^i) \to 0 \quad \text{and} \quad x_i(\tau_n^i) \to \underline{x}_i = \liminf_{t\to+\infty} x_i(t), \quad \text{as } n \to \infty.$$

On the other hand, since $\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds$ for i, j, $1 \leq i, j \leq r$ and $t \geq 0$ is bounded, without loss of generality we assume that for all i, j, $1 \leq i, j \leq r$, $\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)$ and $\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds$ both converge.

Replacing τ_n^i and σ_n^i in the system (3), we obtain for all i, $1 \leq i \leq r$,

$$x_i'(\tau_n^i) = x_i(\tau_n^i)\left[b_i - a_{ii}x_i(\tau_n^i) - \sum_{j=1, j\neq i}^{r} a_{ij}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds\right],$$

$$x_i'(\sigma_n^i) = x_i(\sigma_n^i)\left[b_i - a_{ii}x_i(\sigma_n^i) - \sum_{j=1, j\neq i}^{r} a_{ij}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds\right].$$

As $n \to +\infty$ we get

$$0 = \underline{x}_i\left[b_i - a_{ii}\underline{x}_i - \sum_{j=1, j\neq i}^{r} a_{ij}\lim_{n\to+\infty}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds\right], \tag{10}$$

$$0 = \overline{x}_i\left[b_i - a_{ii}\overline{x}_i - \sum_{j=1, j\neq i}^{r} a_{ij}\lim_{n\to+\infty}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds\right]. \tag{11}$$

By Lemma 2.1 and the definition of upper and lower limit respectively, we have

$$\lim_{n\to+\infty}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds \leq \limsup_{t\to+\infty}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds \leq \overline{x}_j,$$

$$\underline{x}_j \leq \liminf_{t\to+\infty}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds \leq \lim_{n\to+\infty}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds.$$

Substituting these inequalities in (10) and (11) we obtain

$$0 \geq \underline{x}_i \left[b_i - a_{ii}\underline{x}_i - \sum_{j=1, j\neq i}^{r} a_{ij}\overline{x}_j \right] \quad \text{for } i = 1, \ldots, r.$$

$$0 \leq \overline{x}_i \left[b_i - a_{ii}\overline{x}_i - \sum_{j=1, j\neq i}^{r} a_{ij}\underline{x}_j \right] \quad \text{for } i = 1, \ldots, r.$$

Since $\overline{x}_i \geq \underline{x}_i > \alpha > 0$, we get

$$b_i \leq \left[a_{ii}\underline{x}_i + \sum_{j=1, j\neq i}^{r} a_{ij}\overline{x}_j \right] \quad \text{para } i = 1, \ldots, r, \tag{12}$$

$$b_i \geq \left[a_{ii}\overline{x}_i + \sum_{j=1, j\neq i}^{r} a_{ij}\underline{x}_j \right] \quad \text{para } i = 1, \ldots, r. \tag{13}$$

Writing these systems in terms of matrices and vectors as

$$b \leq \left[(A - D)\, \overline{x} + D\underline{x} \right],$$

$$b \geq \left[(A - D)\, \underline{x} + D\overline{x} \right],$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix}$, $D = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & a_{rr} \end{pmatrix}$,
$b = col\,(b_1, \ldots, b_r)$, $\overline{x} = col\,(\overline{x}_1, \ldots, \overline{x}_r)$ and $\underline{x} = col\,(\underline{x}_1, \ldots, \underline{x}_r)$.

Subtracting the first inequality from the second yields

$$0 \leq (2D - A)\,(\underline{x} - \overline{x})\,.$$

By Lemma 2.4, $2D - A$ is invertible and its inverse is nonnegative, so $(\underline{x} - \overline{x}) \geq 0$, this implies that $\underline{x} \geq \overline{x}$ and consequently $\underline{x} = \overline{x}$. From (12) and (13) we have $b_i = \sum_{j=1}^{r} a_{ij}\overline{x}_i$, $1 \leq i \leq r$. Because of $u^* = col\,(u_1^*, \ldots, u_r^*)$ is the unique solution of the system of linear equations (5), we have $\underline{x}_i = \overline{x}_i = x_i^* = \lim_{t\to+\infty} x_i(t)$ for all $i = 1, \ldots, r$. Hence $x^* = col(x_1^*, \ldots, x_r^*)$ is a positive equilibrium point that attracts all solutions of the system (3) with initial condition (4). This completes the proof of the theorem. ☑

Theorem 3.3. Suppose that hypotheses H_2, H_3 and H_4 hold. If $x(t) = col\,(x_1(t), x_2(t), \ldots, x_n(t))$ is a solution (1), with initial condition of the system (2), then for all $i = r+1, \ldots, n$, $x_i(t) \to 0$ exponentially as $t \to \infty$ and for all i, $1 \leq i \leq r$, $x_i(t) \to u_i^*$ as $t \to \infty$, where $u^* = col\,(u_1^*, \ldots, u_r^*)$ is the only equilibrium point of the system

$$x_i'(t) = x_i(t)\left[b_i - a_{ii}x_i(t) - \sum_{j=1, j\neq i}^{r} a_{ij}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds\right], \qquad t \geq 0,\ i = 1, \ldots, r,$$

which is guaranteed by the hypothesis H_2, (see observation 1.2). The point $col\,(u_1^*, \ldots, u_r^*, 0, \ldots, 0)$ is a global attractor of the system (1).

Proof. By Proposition 2.7, there exists a positive number $\overline{\alpha} = \overline{\alpha}(x)$ such that $x_i(t) \geq \overline{\alpha} > 0$ for all i, $1 \leq i \leq r$, $t \geq 0$ and $x_i(t) \to 0$ exponentially for $i = r+1, \ldots, n$, as $t \to +\infty$. From this fact and the Lemma 2.2, it follows that

$$\lim_{t\to+\infty}\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds = 0, \quad j = r+1, \ldots, n. \tag{14}$$

On the other hand, by lemma 2.5, $x_i(t)$ is bounded above for all i, $1 \leq i \leq r$, and $t \geq 0$. Therefore, the components $x_i(t)$ for $1 \leq i \leq r$, are bounded above and below by positive constants.

By Lemma 2.3 (Fluctuation Lemma) there exist sequences $\tau_n^i \to \infty$ and $\sigma_n^i \to \infty$ such that for all i, $1 \leq i \leq r$,

$$x_i'(\sigma_n^i) \to 0 \quad \text{and} \quad x_i(\sigma_n^i) \to \overline{x}_i = \limsup_{t\to+\infty} x_i(t), \quad \text{as } n \to \infty, \quad \text{and}$$

$$x_i'(\tau_n^i) \to 0 \quad \text{and} \quad x_i(\tau_n^i) \to \underline{x}_i = \liminf_{t\to+\infty} x_i(t), \quad \text{as } n \to \infty.$$

By (14),

$$\lim_{n\to+\infty}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds = 0, \quad j = r+1, \ldots, n, \tag{15}$$

and

$$\lim_{n\to+\infty}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds = 0, \quad j = r+1, \ldots, n. \tag{16}$$

On the other hand, $\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds$ for i, j, $1 \leq i, j \leq r$ and $t \geq 0$ are bounded, without loss of generality we may assume that for all i, j, $1 \leq i, j \leq r$, $\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)$ and $\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds$ both converge.

Substituting the sequences τ_n^i and σ_n^i into the system (1), we obtain for each i, $1 \leq i \leq r$,

$$x_i'(\tau_n^i) = x_i(\tau_n^i)\left[b_i - a_{ii}x_i(\tau_n^i) - \sum_{j=1,j\neq i}^{n} a_{ij}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds\right],$$

$$x_i'(\sigma_n^i) = x_i(\sigma_n^i)\left[b_i - a_{ii}x_i(\sigma_n^i) - \sum_{j=1,j\neq i}^{n} a_{ij}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds\right].$$

When $n \to +\infty$,

$$0 = \underline{x}_i\left[b_i - a_{ii}\underline{x}_i - \sum_{j=1,j\neq i}^{r} a_{ij}\lim_{n\to+\infty}\int_{-\infty}^{\tau_n^i} K_{ij}(\tau_n^i - s)x_j(s)ds\right],$$

$$0 = \overline{x}_i\left[b_i - a_{ii}\overline{x}_i - \sum_{j=1,j\neq i}^{r} a_{ij}\lim_{n\to+\infty}\int_{-\infty}^{\sigma_n^i} K_{ij}(\sigma_n^i - s)x_j(s)ds\right].$$

As in the proof of the theorem 3.2, the desired conclusion is reached. This completes the proof of the theorem. ☑

4. Examples

In this section we shall give two examples to illustrate the conclusions of Theorems 3.1 and 3.2. In the first one the conclusions of Theorem 3.1 will be illustrated and in the second one, those of Theorem 3.2.

Example 4.1. Consider the system

$$x_i'(t) = x_i(t)\left(b_i - a_{ii}x_i(t) - \sum_{j=1,j\neq i}^{3} a_{ij}\int_0^{+\infty} k_{ij}(s)x_j(t-s)ds\right),\ i = 1,2,3 \tag{17}$$

where

$$\begin{aligned} b_1 &= 9/2, & b_2 &= 5/2, & b_3 &= 7/2,\\ a_{11} &= 2, & a_{12} &= 2, & a_{13} &= 3,\\ a_{21} &= 9/2, & a_{22} &= 9/2, & a_{23} &= 3/7,\\ a_{31} &= 64, & a_{32} &= 64, & a_{33} &= 32, \end{aligned}$$

$$\begin{aligned} K_{12} &= \beta_{12}e^{-\beta_{12}t}, & K_{13}(t) &= \beta_{13}e^{-\beta_{13}t}, & K_{21}(t) &= \beta_{21}e^{-\beta_{21}t},\\ K_{23} &= \beta_{23}e^{-\beta_{23}t}, & K_{31}(t) &= \beta_{31}e^{-\beta_{31}t}, & K_{32}(t) &= \beta_{32}e^{-\beta_{32}t}, \end{aligned}$$

and β_{ij}, $1 \leq i, j \leq 3$, $i \neq j$ are positive numbers.Thus for $k = 2$ there exists $i_2 = 1$ such that

$$b_2 a_{11} - b_1 a_{21} = \frac{5}{2}(2) - \frac{9}{2}\left(\frac{9}{2}\right) = -\frac{61}{4} < 0.$$

And for $k = 3$ there exists $i_3 = 2$ such that

$$b_3 a_{21} - b_2 a_{31} = \frac{7}{2}\left(\frac{9}{2}\right) - \left(\frac{5}{2}\right) 64 = -\frac{577}{4} < 0,$$

$$b_3 a_{22} - b_2 a_{32} = \frac{7}{2}\left(\frac{9}{2}\right) - \left(\frac{5}{2}\right) 64 = -\frac{577}{4} < 0,$$

$$b_3 a_{23} - b_2 a_{33} = \frac{7}{2}\left(\frac{3}{7}\right) - \left(\frac{5}{2}\right) 32 = -\frac{157}{2} < 0.$$

Hence hypothesis H_1 is fulfilled. It is clear that $\mu_{ij} = \dfrac{1}{\beta_{ij}}$ for $1 \leq i, j \leq 3$ and $i \neq j$. This shows that assumption H_3 holds. Therefore, all conditions in Theorem 3.1 are satisfied. It is to be noted that, by the remark 1.3, we know that the equilibrium points for the system (17) are given by $col\left(\frac{b_1}{a_{11}}, 0, 0\right) = col\left(\frac{9}{4}, 0, 0\right)$, $col\left(0, \frac{b_2}{a_{22}}, 0\right) = col\left(0, \frac{5}{9}, 0\right)$, $col\left(0, 0, \frac{b_3}{a_{33}}\right) = col\left(0, 0, \frac{7}{64}\right)$ and $col\,(0,0,0)$; it is also known, by the remark 1.3, that the positive critical points are the positive solutions of the linear system of 3 equations with 3 unknowns, solving the system

$$\sum_{j=1}^{3} a_{ij} x_j = b_i, \;\; i = 1, 2, 3.$$

It can be easily verified that the only critical points with non-negative components are those generated by the remark 1.4. By Theorem 3.1 we conclude that for any solution $x(t) = col(x_1(t), x_2(t), x_3(t))$ of the system (17), with initial conditions (2), it has the property that $x_2(t)$ and $x_3(t)$ are extinct and $\lim_{t \to +\infty} x_1(t) = \frac{9}{4}$, that is, the point $col\left(\frac{9}{4}, 0, 0\right)$ is the only global attractor of the system (17). □

Example 4.2. Consider the system

$$x_i'(t) = x_i(t)\left(b_i - a_{ii} x_i(t) - \sum_{j=1, j\neq i}^{4} a_{ij} \int_0^{+\infty} K_{ij}(s) x_j(t-s) ds\right), \; i = 1, 2, 3, 4, \tag{18}$$

where

$$
\begin{array}{lll}
b_1 = 16, & b_2 = 16, & b_3 = 2, \\
b_4 = 2, & a_{11} = 5, & a_{12} = \frac{1}{2}, \\
a_{13} = 1, & a_{14} = 1, & a_{21} = \frac{1}{2}, \\
a_{22} = 5, & a_{23} = 1, & a_{24} = 1, \\
a_{31} = 5, & a_{32} = 1, & a_{33} = 1, \\
a_{34} = 1 & a_{41} = 4, & a_{42} = 1, \\
a_{43} = 1, & a_{44} = 1, &
\end{array}
$$

$$
\begin{array}{lll}
K_{12} = 200e^{-200t}, & K_{13}(t) = \beta_{13}e^{-\beta_{13}t}, & K_{14}(t) = \beta_{14}e^{-\beta_{14}t}, \\
K_{21}(t) = 200e^{-200t}, & K_{23}(t) = \beta_{23}e^{-\beta_{23}t}, & K_{24}(t) = \beta_{24}e^{-\beta_{24}t}, \\
K_{31}(t) = \beta_{31}e^{-\beta_{31}t}, & K_{32}(t) = \beta_{32}e^{-\beta_{32}t}, & K_{34}(t) = \beta_{34}e^{-\beta_{34}t}, \\
K_{41}(t) = \beta_{41}e^{-\beta_{41}t}, & K_{42}(t) = \beta_{42}e^{-\beta_{42}t}, & K_{43}(t) = \beta_{43}e^{-\beta_{43}t}.
\end{array}
$$

and β_{ij}, $1 \leq i, j \leq 4$, $i \neq j$ are positive numbers. Thus for $k = 3$, there exists $i_3 = 1$ such that

$$b_3a_{11} - b_1a_{31} = (2)(5) - (16)(5) = -70 < 0,$$

$$b_3a_{12} - b_1a_{32} = 2\left(\frac{1}{2}\right) - (16)(1) = -15 < 0,$$

$$b_3a_{13} - b_1a_{33} = 2(1) - (16)(1) = -14 < 0,$$

And for $k = 4$ there exists $i_4 = 1$ such that

$$b_4a_{11} - b_1a_{41} = 2(5) - (16)(4) = -54 < 0,$$

$$b_4a_{12} - b_1a_{42} = 2\left(\frac{1}{2}\right) - (16)(1) = -15 < 0,$$

$$b_4a_{13} - b_1a_{43} = b_4a_{14} - b_1a_{44} = 2(1) - (16)(1) = -14 < 0.$$

Hence hypothesis H_1 is satisfied. On the other hand, for $i = 1, 2$,

$$b_1 = 16 > a_{12}\left(\frac{b_2}{a_{22}}\right) = \left(\frac{1}{2}\right)\left(\frac{16}{5}\right) = \frac{8}{5},$$

$$b_2 = 16 > a_{21}\left(\frac{b_1}{a_{11}}\right) = \left(\frac{1}{2}\right)\left(\frac{16}{5}\right) = \frac{8}{5}.$$

So the hypothesis H_2 is fulfilled. Clearly $\mu_{ij} = \frac{1}{\beta_{ij}} < \infty$ for $1 \leq i, j \leq 4$ and $i \neq j$. This shows that assumption H_3 holds. Therefore all conditions of Theorem 3.3 are satisfied. By observation 1.3, it is known that the equilibrium points for the system (18) are given by $col\left(\frac{b_1}{a_{11}}, 0, 0, 0\right) = col\left(\frac{16}{5}, 0, 0, 0\right)$, $col\left(0, \frac{b_2}{a_{22}}, 00\right) = col\left(0, \frac{16}{5}, 0, 0\right)$, $col\left(0, 0, \frac{b_3}{a_{33}}, 0\right) = col\,(0, 0, 2, 0)$, $col\left(0, 0, 0, \frac{b_4}{a_{44}}\right) = col\,(0, 0, 0, 2)$ and $col\,(0, 0, 0)$. In addition the positive critical points are the positive solutions of the linear system of 4 equations with 4 unknowns, solving the system

$$\sum_{j=1}^{4} a_{ij} x_j = b_i, \; i = 1, 2, 3, 4.$$

It can be easily verified that the only critical point with non-negative components is $col\left(\frac{32}{11}, \frac{32}{11}, 0, 0\right)$. Therefore, we conclude by Theorem 3.3 that for any solution $x(t) = col\,(x_1(t), x_2(t), x_3(t), x_4(t))$ of the system (18), with initial conditions (2), it has the property that species $x_3(t)$ and $x_4(t)$ are extinct and $x_1(t) \to x_1^*$, $x_2(t) \to x_2^*$ as $t \to +\infty$, where $x^* = col\,(x_1^*, x_2^*) = col\left(\frac{32}{11}, \frac{32}{11}\right)$ is the solution

$$\begin{cases} x_1'(t) = x_1(t)\left[16 - 5x_1(t) - \frac{1}{2}\int_0^{+\infty} 200 e^{-200s} x_2(t-s)ds\right], \\ x_2'(t) = x_2(t)\left[16 - 5x_2(t) - \frac{1}{2}\int_0^{+\infty} 200 e^{-200s} x_1(t-s)ds\right]. \end{cases}$$

that is, the point $col\left(\frac{32}{11}, \frac{32}{11}, 0, 0\right)$ is the only global attractor of the system (18).

References

[1] S. Ahmad, *On the nonautonomous Volterra–Lotka competition equations*, Proc. Amer. Math. Soc. **117** (1993), 199–204.

[2] ______, *Extinction of species in nonautonomous Lotka–Volterra systems*, Proc. Amer. Math.Soc **127** (1999), 2905–2910.

[3] A. Battauz and F. Zanolin, *Coexistence states for periodic competitive Kolmogorov systems*, J. Math. Anal. Appl **219** (1998), 179–199.

[4] M. Brelot, *Sur le probleme biologique hereditaire de deux especes devorante et devoree*, Ann. Mat. Pura Appl.Ser, 1931.

[5] R. S. Cantrell and C. Cosner, *On the steady-state problem for the Volterra-Lotka competition model with difusion*, Houston J. Math **13** (1987), 337–352.

[6] F. Chen, Z. Li, and Y. Huang, *Note on the permanence of a competitive system with infinite delay and feedback controls*, Nonlinear Analysis: Real World Applications **8** (2007), 680–687.

[7] F. Chen, C. Shi, and Zhong Li, *Extinction in a nonautonomous Lotka-Volterra competitive system with infinite delay and feedback controls*, Nonlinear Analysis: Real World Applications **13** (2012), 2214–2226.

[8] C. Cosner and A. C. Lazer, *Stable Coexistence States in the Volterra-Lotka Competition Model with Difusion*, SIAM J. Appl. Math **44** (1984), 1112–1132.

[9] F. Montes de Oca and L. Pérez, *Extinction in nonautonomous competitive Lotka-Volterra systems with infinite delay*, Nonlinear Analysis: Series A:Theory and Methods **75** (2012), 758–768.

[10] ______, *Balancing Survival and Extinction in nonautonomous competitive Lotka-Volterra systems with infinite delay*, Discrete and Continuous Dynamical Systems: Series B **20** (2015), 2663–2690.

[11] F. Montes de Oca and M. Vivas, *Extinction in a two dimensional lotka–volterra systems with infinite delay*, Nonlinear Analysis: Real world Applications **7** (2006), 1042–1047.

[12] L. Dung and H. L. Smith, *A Parabolic System Modeling Microbial Competition in an Unmixed Bio-reactor*, Journal of Diferential Equations **130** (1996), 59–91.

[13] ______, *Steady states of models of microbial growth and competition with chemotaxis*, J Math. Anal. Appl **229** (1999), 295–318.

[14] K. Gopalsamy, *Stability and Oscillations in Delay Diferential Equations of population dynamics*, Mathematics and Its Applications, Kluwer Academic Publishers. Boston, 1992.

[15] J. Hale and S. Verduyn Lunel, *Introduction to functional diferential equations*, Applied Mathematical Sciences, 1993.

[16] X. He, *Almost periodic solutions of a competition system with dominated infinite delays*, Tohoku Math. J **50** (1998), 71–89.

[17] W. Hirsch and J. Hanisch, *Differential equation models of some parasitic infection-methods for the study of asymptotic behavior*, Comm. Pure Appl. Math **38** (1995), 733–753.

[18] Z. Hou, *Permanence, global attraction and stability*, De Gruyter Ser.in Math. and Life Sc, 2013.

[19] H. Hu, Z. Teng, and S. Gao, *Extinction in nonautonomous Lotka-Volterra competitive system with pure-delays and feedback controls*, Nonlinear Analysis: Real World Applications **10** (2009), 2508–2520.

[20] H. Hu, Z. Teng, and H. Jiang, *On the permanence in non-autonomous lotka-volterra competitive system with pure-delays and feedback controls*, Nonlinear Analysis: Real World Applications **10** (2009), 1803–1815.

[21] A. C. Lazer and S. Ahmad, *Average growth extinction in a competitive Lotka-Volterra system*, Nonlinear Analysis **62** (2005), 545–557.

[22] Z. Teng, *On the nonautonomous Lotka-Volerra N-species competing systems*, Appl. Math.Comp **114** (2000), 175–185.

[23] A. Tineo, *Asymptotic behavior of positive solutions of the nonautonomous Lotka-Volterra competetions equations*, Differential and Integral Equations **2** (1993), 449–457.

[24] ______, *Necessary and sufficient conditions for extinction of one species*, Advanced Nonlinear Studies **5** (2005), 57–71.

[25] V. Volterra, *Lecon sur la theorie mathematique de la lutte por la vie*, Gauthier Villars, Paris, 1931.

[26] F. Zanolin, *Permanence and positive periodic solutions for kolmogorov competing species systems*, Results Math **21** (1992), 224–250.

[27] M. L. Zeeman, *Extinction in competiive Lotka-Volterra Systems*, Proc. Amer.Math.Soc **123** (1995), 87–96.

[28] J. Zhao, L. Fu, and J. Ruan, *Extinction in a nonautonomous competitive Lotka-Volterra system*, Appl. Math. Letters **22** (2009), 766–770.

[29] J. Zhao and J. Tiang, *Average conditions for permanence and extinction in nonautonomous Lotka- Voltera systems*, JMAA **299** (2004), 663–675.

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