Revista Colombiana de Matemáticas
Volumen 54(2020)2, páginas 93-108

L^∞-BMO bounds for pseudo-multipliers associated with the harmonic oscillator

Continuidad L^∞-BMO para pseudomultiplicadores asociados con el oscilador armónico

Duván Cardona

Ghent University, Ghent, Belgium

Abstract. In this note we investigate some conditions of Hörmander-Mihlin type in order to assure the L^∞-BMO boundedness for pseudo-multipliers of the harmonic oscillator. The H^1-L^1 continuity for Hermite multipliers also is investigated.

Key words and phrases. Harmonic oscillator, Pseudo-multiplier, Hermite expansion, Littlewood-Paley theory, BMO.

2020 Mathematics Subject Classification. 81Q10.

Resumen. En esta nota se investigan condiciones de tipo Hörmander-Mihlin para garantizar la continuidad L^∞-BMO de pseudomultiplicadores asociados con el oscilador armónico. También se estudia la continuidad de tipo H^1-L^1 para multiplicadores de Hermite.

Palabras y frases clave. Oscilador armónico, pseudomultiplicador, expansión de Hermite, teoría de Littlewood-Paley, BMO.

1. Introduction

The aim of this paper is to investigate the boundedness from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$ for pseudo-multipliers associated with the harmonic oscillator (see *e.g.* S. Thangavelu [20, 21]). As it was observed by M. Ruzhansky in [16], from the point of view of the theory of pseudo-differential operators, pseudo-multipliers would be the special case of the symbolic calculus developed in M. Ruzhansky and N. Tokmagambetov [22, 17] (see also Remark 2.2). Let us consider the (Hermite operator) quantum harmonic oscillator $H := -\Delta_x + |x|^2$, (where Δ_x is the standard Laplacian) which extends to an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$. It is a well known fact, that the Hermite functions[1] ϕ_ν, $\nu \in \mathbb{N}_0^n$, are the L^2-eigenfunctions of H, with corresponding eigenvalues satisfying: $H\phi_\nu = (2|\nu| + n)\phi_\nu$. The system $\{\phi_\nu\}_{\nu\in\mathbb{N}_0^n}$, which is a subset of the Schwartz class $\mathscr{S}(\mathbb{R}^n)$, provides an orthonormal basis of $L^2(\mathbb{R}^n)$. So, the spectral theorem for unbounded operators implies that

$$Hf(x) = \sum_{\nu\in\mathbb{N}_0^n} (2|\nu| + n)\widehat{f}(\phi_\nu),\ \ f \in \mathrm{Dom}(H), \tag{1}$$

where $\widehat{f}(\phi_\nu)$ is the Fourier-Hermite transform of f at ϕ_ν, which is given by

$$\widehat{f}(\phi_\nu) = \int_{\mathbb{R}^n} f(x)\phi_\nu(x)dx. \tag{2}$$

If $G \subset \mathbb{R}^n$ is the complement of a subset of zero Lebesgue measure in $\mathbb{R}^n$, the pseudo-multiplier associated with a function $m : G \times \mathbb{N}_0^n \to \mathbb{C}$ is defined by

$$Af(x) = \sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\widehat{f}(\phi_\nu)\phi_\nu(x),\ \ x \in G,\ f \in \mathrm{Dom}(A). \tag{3}$$

In this sense we say that A is the pseudo-multiplier associated to the function m, and that m is the symbol of A. In this paper the main goal is to give conditions on m in order that A can be extended to a bounded operator from L^∞ to BMO. The problem of the boundedness of pseudo-multipliers is an interesting topic in harmonic analysis (see *e.g.* J. Epperson [9], S. Bagchi and S. Thangavelu [1], D. Cardona and M. Ruzhansky [16] and references therein). The problem was initially considered for multipliers of the harmonic oscillator

$$Af(x) = \sum_{\nu\in\mathbb{N}_0^n} m(\nu)\widehat{f}(\phi_\nu)\phi_\nu(x),\ \ f \in \mathrm{Dom}(A).^2 \tag{4}$$

Indeed, an early result due to S. Thangavelu (see [19, 20]) states that if m satisfies the following discrete Marcienkiewicz condition

$$|\Delta_\nu^\alpha m(\nu)| \leq C_\alpha(1 + |\nu|)^{-|\alpha|},\ \ \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq [\frac{n}{2}] + 1, \tag{5}$$

where Δ_ν is the usual difference operator, then the corresponding multiplier $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ extends to a bounded operator for all $1 < p < \infty$.

[1] Each Hermite function ϕ_ν has the form $\phi_\nu := \Pi_{j=1}^n \phi_{\nu_j}$ $\phi_{\nu_j}(x_j) = (2^{\nu_j}\nu_j!\sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j)e^{-\frac{1}{2}x_j^2}$, where $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}_0^n$, and $H_{\nu_j}(x_j) := (-1)^{\nu_j} e^{x_j^2} \frac{d^k}{dx_j^k}(e^{-x_j^2})$ denotes the Hermite polynomial of order ν_j.

[2] $\mathrm{Dom}(A) = \{f \in L^2(\mathbb{R}^n) : \sum_{\nu\in\mathbb{N}_0^n} |m(\nu)\widehat{f}(\phi_\nu)|^2 < \infty\}$ is a dense subset of $L^2(\mathbb{R}^n)$. Indeed, note that $\{\phi_\nu\}_\nu \subset \mathrm{Dom}(A)$, and consequently $L^2(\mathbb{R}^n) = \overline{\mathrm{span}(\{\phi_\nu\}_\nu)} \subset \overline{\mathrm{Dom}(A)}$.

In view of Theorem 1.1 of S. Blunck [2] (see also P. Chen, E. M. Ouhabaz, A. Sikora, and L. Yan, [7, p. 273]), if we restrict our attention to spectral multipliers $A = m(H)$, the boundedness on $L^p(\mathbb{R}^n)$, can be assured if m satisfies the Hörmander condition of order s,

$$\|m\|_{l.u.H^s} := \sup_{r>0} \|m(r\cdot)\eta(|\cdot|)\|_{H^s(\mathbb{R}^n)} = \sup_{r>0} r^{s-\frac{n}{2}} \|m(\cdot)\eta(r^{-1}|\cdot|)\|_{H^s(\mathbb{R}^n)} < \infty, \tag{6}$$

where $\eta \in \mathscr{D}(0,\infty)$ and $s > \frac{n+1}{2}$, for all $p \in [p_0, \frac{p_0}{p_0-1}]$, for some $p_0 \in (1,2)$. If $|\nu| = \nu_1 + \cdots + \nu_n$, for spectral pseudo-multipliers

$$Ef(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, 2|\nu|+n)\widehat{f}(\phi_\nu)\phi_\nu(x), \;\; f \in \mathrm{Dom}(E), \tag{7}$$

under one of the following conditions

- J. Epperson [9]: $n = 1$, E bounded on $L^2(\mathbb{R})$ and

$$|\Delta_\nu^\gamma m(x, 2\nu+1)| \leq C_\gamma (2\nu+1)^{-\gamma}, \;\; 0 \leq \gamma \leq 5, \tag{8}$$

- S. Bagchi and S. Thangavelu [1]: $n \geq 2$, E bounded on $L^2(\mathbb{R}^n)$ and

$$|\Delta_\nu^\gamma m(x, 2|\nu|+1)| \leq C_\gamma (2|\nu|+1)^{-\gamma}, \;\; 0 \leq |\gamma| \leq n+1, \tag{9}$$

the operator E extends to an operator of weak type $(1,1)$. This means that $E : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ admits a bounded extension (we denote by $L^{1,\infty}(\mathbb{R}^n)$ the weak L^1-space[3]). In view of the Marcinkiewicz interpolation Theorem it follows that E extends to a bounded linear operator on $L^p(\mathbb{R}^n)$, for all $1 < p \leq 2$.

We can note that in the previous results the L^2-boundedness of pseudo-multipliers is assumed. The problem of finding reasonable conditions for the L^2-boundedness of spectral pseudo-multipliers, was proposed by S. Bagchi and S. Thangavelu in [1]. To solve this problem, it was considered in [16], the following Hörmander conditions,

$$\|m\|_{l.u.,H^s} := \sup_{r>0,\, y\in\mathbb{R}^n} r^{(s-\frac{n}{2})} \|\langle x\rangle^s \mathscr{F}[m(y,\cdot)\psi(r^{-1}|\cdot|)](x)\|_{L^2(\mathbb{R}^n_x)} < \infty, \tag{10}$$

$$\|m\|_{l.u.,\mathcal{H}^s} := \sup_{k>0} \sup_{y\in\mathbb{R}^n} 2^{k(s-\frac{n}{2})} \|\langle x\rangle^s \mathscr{F}_H^{-1}[m(y,\cdot)\psi(2^{-k}|\cdot|)](x)\|_{L^2(\mathbb{R}^n_x)} < \infty, \tag{11}$$

defined by the Fourier transform $\mathscr{F}$ and the inverse Fourier-Hermite transform $\mathscr{F}_H^{-1}$. More precisely, the Hörmander condition (10) of order $s > \frac{3n}{2}$, uniformly in $y \in \mathbb{R}^n$, or the condition (11) for $s > \frac{3n}{2} - \frac{1}{12}$, uniformly in $y \in \mathbb{R}^n$, guarantee

[3]Which consists of those functions f such that $\|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mathrm{meas}(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty$.

the L^2- boundedness of the pseudo-multiplier (21). As it was pointed out in [16], in (10) we consider functions m on $\mathbb{R}^n\times\mathbb{R}^n$, but to these functions we associate a pseudo-multiplier with symbol $\{m(x,\nu)\}_{x\in\mathbb{R}^n,\nu\in\mathbb{N}_0^n}$. On the other hand (see Corollary 2.3 of [16]), if we assume the condition,

$$|\Delta_\nu^\alpha m(x,\nu)|\leq C_\alpha(1+|\nu|)^{-|\alpha|},\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\leq\rho, \tag{12}$$

for $\rho=[3n/2]+1$, then the pseudo-multiplier in (21) extends to a bounded operator on $L^2(\mathbb{R}^n)$, and for $\rho=2n+1$ we have its $L^p(\mathbb{R}^n)$-boundedness for all $1<p<\infty$. Now, we record the main theorem of [16]:

Theorem 1.1. *Let us assume that $2\leq p<\infty$. If $A=T_m$ is a pseudo-multiplier with symbol m satisfying* (10)*, then under one of the following conditions,*

- $n\geq 2$, $2\leq p<\frac{2(n+3)}{n+1}$, *and* $s>s_{n,p}:=\frac{3n}{2}+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})$,
- $n\geq 2$, $p=\frac{2(n+3)}{n+1}$, *and* $s>s_{n,p}:=\frac{3n}{2}+\frac{n-1}{2(n+3)}$,
- $n\geq 2$, $\frac{2(n+3)}{n+1}<p\leq\frac{2n}{n-2}$, *and* $s>s_{n,p}:=\frac{3n}{2}-\frac{1}{6}+\frac{2n}{3}(\frac{1}{2}-\frac{1}{p})$,
- $n\geq 2$, $\frac{2n}{n-2}\leq p<\infty$, *and* $s>s_{n,p}:=\frac{3n-1}{2}+n(\frac{1}{2}-\frac{1}{p})$,
- $n=1$, $2\leq p<4$, $s>s_{1,p}:=\frac{3}{2}$,
- $n=1$, $p=4$, $s>s_{1,4}:=2$,
- $n=1$, $4<p<\infty$, $s>s_{1,p}:=\frac{4}{3}+\frac{2}{3}(\frac{1}{2}-\frac{1}{p})$,

the operator T_m extends to a bounded operator on $L^p(\mathbb{R}^n)$. For $1<p\leq 2$, under one of the following conditions

- $n\geq 2$, $\frac{2(n+3)}{n+5}\leq p\leq 2$, *and* $s>s_{n,p}:=\frac{3n}{2}+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})$,
- $n\geq 2$, $\frac{2n}{n+2}\leq p\leq\frac{2(n+3)}{n+5}$, *and* $s>s_{n,p}:=\frac{3n}{2}-\frac{1}{6}+\frac{2n}{3}(\frac{1}{2}-\frac{1}{p})$,
- $n\geq 2$, $1<p\leq\frac{2n}{n+2}$, *and* $s>s_{n,p}:=\frac{3n-1}{2}+n(\frac{1}{2}-\frac{1}{p})$,
- $n=1$, $\frac{4}{3}\leq p<2$, $s>s_{1,p}:=\frac{3}{2}$,
- $n=1$, $1<p<\frac{4}{3}$, $s>s_{1,p}:=\frac{4}{3}+\frac{2}{3}(\frac{1}{2}-\frac{1}{p})$,

the operator T_m extends to a bounded operator on $L^p(\mathbb{R}^n)$. However, in general:

- *for every $\frac{4}{3}<p<4$ and every n, the condition $s>\frac{3n}{2}$ implies the L^p-boundedness of T_m.*

If the symbol m of the pseudo-multiplier T_m satisfies the Hörmander condition (11)*, in order to guarantee the L^p-boundedness of T_m, in every case above we can take $s > s_{n,p} - \frac{1}{12}$. Moreover, the condition $s > \frac{3n}{2} - \frac{1}{12}$ implies the L^p-boundedness of T_m for all $\frac{4}{3} < p < 4$.*

Now we present our main result. We will provide a version of Theorem 1.1 for the critical case $p = \infty$. Because in harmonic analysis the John-Nirenberg class BMO (see [15]) is a good substitute of L^∞, we will investigate the boundedness of pseudo-multipliers from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.2. *Let $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be a continuous linear operator such that its symbol $m = \{m(x,\nu)\}_{x\in G,\, \nu\in\mathbb{N}_0^n}$* (see (20)) *satisfies one of the following conditions,*

(CI): *m satisfies the Hörmander-Mihlin condition*

$$\|m\|_{l.u.,H^s} := \sup_{r>0,\, y\in\mathbb{R}^n} r^{(s-\frac{n}{2})} \|\langle x\rangle^s \mathscr{F}[m(y,\cdot)\psi(r^{-1}|\cdot|)](x)\|_{L^2(\mathbb{R}^n_x)} < \infty, \tag{13}$$

where $s > \max\{\frac{7n}{4} + \varkappa, \frac{n}{2}\}$, and $\varkappa$ is defined as in (43)*,*

(CII): *m satisfies the Marcinkiewicz type condition,*

$$|\Delta_\nu^\alpha m(x,\nu)| \leq C_\alpha (1+|\nu|)^{-|\alpha|},\ |\alpha| \leq [7n/4 - 1/12] + 1. \tag{14}$$

Then the operator $A = T_m$ extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$.

Now, we will discuss some consequences of our main result.

***Remark* 1.3.** In relation with the results of Epperson [9] and Bagchi and Thangavelu [1] mentioned above, Theorem 1.2 implies that under one of the following conditions,

- $n = 1,\ |\Delta_\nu^\gamma m(x, 2\nu+1)| \leq C_\gamma (2\nu+1)^{-|\gamma|},\ 0 \leq \gamma \leq 2,$
- $n \geq 2,\ |\Delta_\nu^\gamma m(x, 2|\nu|+n)| \leq C_\gamma (2|\nu|+n)^{-|\gamma|},\ 0 \leq |\gamma| \leq [7n/4 - 1/12] + 1,$

the spectral pseudo-multiplier

$$Ef(x) = \sum_{\nu\in\mathbb{N}_0^n} m(x, 2|\nu|+n) \widehat{f}(\phi_\nu)\phi_\nu(x),\ f \in \mathrm{Dom}(E) \tag{15}$$

extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$.

***Remark* 1.4.** For $n = 1$, Theorem 1.1 implies that the symbol inequalities

$$|\Delta_\nu^\gamma m(x,\nu)| \leq C_\gamma(1+\nu)^{-\alpha},\ \ 0 \leq \gamma \leq 2, \tag{16}$$

are sufficient conditions for the $L^p(\mathbb{R})$-boundedness of pseudo-multipliers with $\frac{4}{3} < p < 4$, and also under the estimates

$$|\Delta_\nu^\gamma m(x,\nu)| \leq C_\gamma(1+\nu)^{-\alpha},\ \ 0 \leq \gamma \leq 3, \tag{17}$$

we obtain the $L^p(\mathbb{R})$-boundedness of T_m for all $p \in (1, 4/3) \cup (4, \infty)$. However, we can improve the conditions on the number of derivatives imposed in (17) to discrete derivatives up to order 2 in order to assure the $L^p(\mathbb{R})$-boundedness of T_m for all $4/3 \leq p < \infty$. Indeed, from Theorem 1.2, the hypothesis (16) implies the boundedness of T_m from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R})$ and also its $L^p(\mathbb{R})$-boundedness for $4/3 \leq p < \infty$, in view of the Stein-Fefferman interpolation theorem applied to the L^2-L^2 and L^∞-BMO boundedness results.

***Remark* 1.5.** Let us consider a multiplier T_m of the harmonic oscillator. Theorem 1.2 assures that under one of the following conditions,

(CI)': m satisfies the Hörmander-Mihlin condition

$$\|m\|_{l.u.,H^s} := \sup_{r>0} r^{(s-\frac{n}{2})} \|\langle x\rangle^s \mathscr{F}[m(\cdot)\psi(r^{-1}|\cdot|)](x)\|_{L^2(\mathbb{R}^n_x)} < \infty, \tag{18}$$

where $s > \max\{\frac{7n}{4} + \varkappa, \frac{n}{2}\}$, and $\varkappa$ is defined as in (43),

(CII)': m satisfies the Marcinkiewicz type condition,

$$|\Delta_\nu^\alpha m(\nu)| \leq C_\alpha(1+|\nu|)^{-|\alpha|},\ |\alpha| \leq [7n/4 - 1/12] + 1, \tag{19}$$

the operator T_m extends to a bounded operator from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}(\mathbb{R}^n)$. Moreover, the duality argument shows the boundedness of T_m from $\mathrm{H}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

For certain spectral aspects and applications to PDE of the theory of pseudo-multipliers we refer the reader to the works [6, 3, 4, 5] and [19]. This paper is organised as follows. Section 2 introduces the necessary background of harmonic analysis that we will use throughout this work. Finally, in Section 3 we prove our main theorem.

2. Preliminaries

2.1. Pseudo-multipliers of the harmonic oscillator

To motivate the definition of pseudo-multipliers we will prove that these operators arise, for example, as bounded linear operators on the Schwartz class $\mathscr{S}(\mathbb{R}^n)$.

Theorem 2.1. *Let us consider the set* $G := \{z \in \mathbb{R}^n : \phi_\nu(z) \neq 0, \text{ for all } \nu\}$, *and let* $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *be a continuous linear operator. Then, the function* $m : G \times \mathbb{N}_0^n \to \mathbb{C}$,[4] *defined by*

$$m(x,\nu) := \phi_\nu(x)^{-1} A\phi_\nu(x), \;\; x \in G, \nu \in \mathbb{N}_0^n, \tag{20}$$

satisfies the property

$$Af(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x,\nu)\widehat{f}(\phi_\nu)\phi_\nu(x), \;\; x \in G, \, f \in \mathscr{S}(\mathbb{R}^n). \tag{21}$$

Proof. Let us assume that A is a continuous linear operator $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Because, for every $\nu \in \mathbb{N}_0^n$, $\phi_\nu \in \mathscr{S}(\mathbb{R}^n) = \mathrm{Dom}(A)$, define for every $x \in G$, and $\nu \in \mathbb{N}_0^n$, the function

$$m(x,\nu) := \phi_\nu(x)^{-1} A\phi_\nu(x). \tag{22}$$

Let $f \in \mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and let us consider its Hermite series

$$f = \sum_{\nu \in \mathbb{N}_0^n} \widehat{f}(\phi_\nu)\phi_\nu. \tag{23}$$

Because $\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_\nu |\widehat{f}(\phi_\nu)\|^2 < \infty$, by Simon Theorem (see Theorem 1 of B. Simon [18]), the series

$$f_N = \sum_{|\nu| \leq N} \widehat{f}(\phi_\nu)\phi_\nu, \;\; N \in \mathbb{N} \tag{24}$$

converges to f in the topology of the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. Because, $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, is a continuous linear operator, we have that Af_n converges to Af in the topology of $\mathscr{S}(\mathbb{R}^n)$. Consequently, we have proved that

$$Af = \sum_{\nu \in \mathbb{N}_0^n} \widehat{f}(\phi_\nu)A\phi_\nu. \tag{25}$$

By observing that $m(x,\nu) := \phi_\nu(x)^{-1} A\phi_\nu(x)$, we obtain the identity,

$$Af(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x,\nu)\widehat{f}(\phi_\nu)\phi_\nu(x), \;\; x \in G, \, f \in \mathscr{S}(\mathbb{R}^n).$$

So, we end the proof. ☑

[4]The symbol m is defined *a.e.* $(x,\nu) \in \mathbb{R}^n \times \mathbb{N}_0^n$. Indeed, note that $D = \{z : \phi_\nu(z) = 0 \text{ for some } \nu\}$ is a countable set, has zero measure and that m is defined on $G \times \mathbb{N}_0^n$, where $G = \mathbb{R}^n - D$.

***Remark* 2.2.** It is a well known fact that several classes of pseudo-differential operators

$$T_\sigma f(x) = \int\limits_{\mathbb{R}^n} e^{i2\pi x\xi}\sigma(x,\xi)\widehat{f}(\xi)d\xi, \ \ f \in C_0^\infty(\mathbb{R}^n), \tag{26}$$

are continuous linear operators on the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. For example, if σ is a tempered and smooth function (*i.e.* that $\sigma \in C^\infty(\mathbb{R}^{2n})$ satisfies $\int |\sigma(x,\xi)|(1+|x|+|\xi|)^{-\kappa}dxd\xi < \infty$ for some $\kappa > 0$) then $T_\sigma : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, extends to a continuous linear operator. More interesting cases arise with pseudo-differential operators with symbols σ in the Hörmander classes, or with more generality, in the Weyl-Hörmander classes (see L. Hörmander [13, 14]). From Theorem 2.1 we have that continuous pseudo-differential operators on $\mathscr{S}(\mathbb{R}^n)$ also can be understood as pseudo-multipliers of the harmonic oscillator.

2.2. Functions of bounded mean oscillation BMO.

We will consider in the following two subsections the necessary notions for introducing the BMO and H^1 spaces. For this, we will follow Fefferman and Stein [11]. Let f be a locally integrable function on $\mathbb{R}^n$. Then f is of bounded mean oscillation (abreviated as $f \in \mathrm{BMO}(\mathbb{R}^n)$), if

$$\sup_Q \frac{1}{|Q|}\int\limits_Q |f(x) - f_Q|dx := \|f\|_* < \infty, \tag{27}$$

where the supremum ranges over all finite cubes Q in $\mathbb{R}^n$, $|Q|$ is the Lebesgue measure of Q, and f_Q denote the mean value of f over Q, $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$. It is a well known fact that $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}$. Moreover $\ln(|x|) \in \mathrm{BMO}$. The class of functions of bounded mean oscillation, modulo constants, is a Banach space with the norm $\|\cdot\|_*$, defined above. According to the John-Nirenberg inequality, $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if the inequality

$$|\{x \in Q : |f(x) - f_Q| > \alpha\}| \leq e^{-\frac{C_\alpha}{\|f\|_*}}|Q|, \tag{28}$$

holds true for every $\alpha > 0$. For understanding the behaviour of a function $f \in \mathrm{BMO}(\mathbb{R}^n)$, it can be checked that

$$\int\limits_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}dx < \infty. \tag{29}$$

Moreover, a function $f \in \mathrm{BMO}(\mathbb{R}^n)$, if and only if (29) holds and

$$\iint\limits_{|x-x_0|<\delta; 0<t<\delta} t|\nabla u(x,t)|^2 dxdt \lesssim \delta^n, \tag{30}$$

for all $x_0 \in \mathbb{R}^n$ and $\delta > 0$. Here, $u(x,t)$ is the Poisson integral of f defined on $\mathbb{R}^n \times (0,\infty)$ by (see Fefferman [10]),

$$u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)dy, \;\; P_t(x) := \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}. \tag{31}$$

2.3. The space H^1

The Hardy spaces $H^p(\mathbb{D})$, $0 < p < \infty$, were first studied as part of complex analysis by G. H. Hardy [12]. An analytic function F on the disk $\mathbb{D}$ is in $H^p(\mathbb{D})$, if

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta < \infty. \tag{32}$$

For $1 < p < \infty$, we can identify $H^p(\mathbb{D})$, with $L^p(\mathbb{T})$, where $\mathbb{T}$ is the circle. This identification does not hold, however, for $p \leq 1$. Unfortunately, these results cannot be extended to higher dimensions using the theory of functions of several complex variables. So, let us introduce the Hardy space $\mathrm{H}^1(\mathbb{R}^n)$. Let $R_1, \cdots, R_n$, be the Riesz transform on $\mathbb{R}^n$,

$$R_j f(x) = \lim_{\varepsilon\to 0} \int_{|\xi|>\varepsilon} e^{i2\pi x\cdot\xi} \xi_j/|\xi| \widehat{f}(\xi), \;\; f \in \mathrm{Dom}(R_j), \tag{33}$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x\cdot\xi} f(x)dx$, is the Fourier transform of f at ξ. Then, $\mathrm{H}^1(\mathbb{R}^n)$ consists of those functions f on $\mathbb{R}^n$, satisfying,

$$\|f\|_{\mathrm{H}^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)}. \tag{34}$$

The main remark in this subsection is that the dual of $\mathrm{H}^1(\mathbb{R}^n)$ is $\mathrm{BMO}(\mathbb{R}^n)$ (see Fefferman and Stein [11]). This can be understood in the following sense:

(a) If $\phi \in \mathrm{BMO}(\mathbb{R}^n)$, then $\Phi : f \mapsto \int_{\mathbb{R}^n} f(x)\phi(x)dx$, admits a bounded extension on $\mathrm{H}^1(\mathbb{R}^n)$.

(b) Conversely, every continuous linear functional Φ on $\mathrm{H}^1(\mathbb{R}^n)$ arises as in (a) with a unique element $\phi \in \mathrm{BMO}(\mathbb{R}^n)$.

The norm of ϕ as a linear functional on $\mathrm{H}^1(\mathbb{R}^n)$ is equivalent to the BMO norm. Important properties of the BMO and the H^1 norm are the following,

$$\|f\|_* = \sup_{\|g\|_{\mathrm{H}^1}=1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|, \quad \|g\|_{\mathrm{H}^1} = \sup_{\|f\|_{\mathrm{BMO}}=1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|. \tag{35}$$

For our further analysis we will use the following fact (see Fefferman and Stein [11, pag. 183]): if $f\in \mathrm{H}^1(\mathbb{R}^n)$, and $\phi\in\mathscr{S}(\mathbb{R}^n)$ satisfies $\int\phi(x)dx=1$, let us define

$$u^{+,f}(x):=\sup_{t>0}|\phi_t*f(x)|=\sup_{t>0}\left|\int\limits_{\mathbb{R}^n}\phi_t(x-y)f(y)dy\right|,\ \ \phi_t(x)=t^{-n}\phi(\frac{x}{t}). \tag{36}$$

Then, $u^{+,f}\in L^1(\mathbb{R}^n)$, $f(x)=\lim_{t\to 0}\phi_t*f(x)$, $a.e.x$, and there exist positive constants A and B satisfying

$$A\|f\|_{\mathrm{H}^1}\leq\|u^{+,f}\|_{L^1}\leq B\|f\|_{\mathrm{H}^1}. \tag{37}$$

The duals of the $H^p(\mathbb{R}^n)$ spaces, $0<p<1$, are Lipschitz spaces. This is due to P. Duren, B. Romberg and A. Shields [8] on the unit circle, and to T. Walsh [23] in $\mathbb{R}^n$.

2.4. The Hörmander-Mihlin condition for pseudo-multipliers

As we mentioned in the introduction, if m is a function on $\mathbb{R}^n$, we say that m satisfies the Hörmander condition of order $s>0$, if

$$\|m\|_{l.u.H^s}:=\sup_{r>0}\|m(r\cdot)\eta(|\cdot|)\|_{H^s(\mathbb{R}^n)}=\sup_{r>0}r^{s-\frac{n}{2}}\|m(\cdot)\eta(r^{-1}|\cdot|)\|_{H^s(\mathbb{R}^n)}<\infty, \tag{38}$$

where $H^s(\mathbb{R}^n)$ is the usual Sobolev space of order s. Indeed, we also can use the following formulation for the Hörmander-Mihlin condition,

$$\|m\|_{l.u.H^s}:=\sup_{j\in\mathbb{Z}}\|m(2^j|\cdot|)\eta(\cdot)\|_{H^s(\mathbb{R}^n)}=\sup_{j\in\mathbb{Z}}2^{j(s-\frac{n}{2})}\|m(\cdot)\eta(2^{-j}|\cdot|)\|_{H^s(\mathbb{R}^n)}<\infty. \tag{39}$$

In particular, if we choose $\eta\in\mathscr{D}(0,\infty)$ with compact support in $[1/2,2]$, and we assume that m has support in $\{\xi:|\xi|>2\}$, we have that $m(\cdot)\eta(2^{-j}|\cdot|)=0$ for $j\leq 0$. So, for a such symbol m, we have

$$\|m\|_{l.u.H^s}:=\sup_{j\geq 1}\|m(2^j|\cdot|)\eta(\cdot)\|_{H^s(\mathbb{R}^n)}=\sup_{j\geq 1}2^{j(s-\frac{n}{2})}\|m(\cdot)\eta(2^{-j}|\cdot|)\|_{H^s(\mathbb{R}^n)}<\infty. \tag{40}$$

Because we define multipliers by associating to T_m the restriction of m to $\mathbb{N}_0^n$, we always can split $T_m=T_0+S_m$, where T_0 has symbol supported in $\{\nu:|\nu|\leq 2\}$ and the pseudo-multiplier S_m has symbol supported in $\{\nu:|\nu|>2\}$. We will apply the Hörmander condition to S_m in order to assure its L^∞-BMO boundedness, and later we will conclude that T_m is L^∞-BMO bounded, by observing that the L^∞-BMO boundedness of T_0 is trivial. This analysis will be developed in detail in the next section, in the context of pseudo-multipliers by employing the Hörmander type condition

$$\|m\|_{l.u.H^s}:=\sup_{j\geq 1,x\in\mathbb{R}^n}2^{j(s-\frac{n}{2})}\|m(x,\cdot)\eta(2^{-j}|\cdot|)\|_{H^s(\mathbb{R}^n)}<\infty, \tag{41}$$

for s large enough which follows from (13).

3. L^∞-BMO continuity for pseudo-multipliers

In this section we present the proof of our main result. The main strategy in the proof of Theorem 1.2 will be a suitable Littlewood-Paley decomposition of the symbol together with some suitable estimates for the operator norm of pseudo-multipliers associated to each part of this decomposition. Our starting point is the following lemma. We use the symbol $X \lesssim Y$ to denote that there exists a universal constant C such that $X \leq CY$.

Lemma 3.1. *Let ϕ_ν, $\nu \in \mathbb{N}_0^n$ be a Hermite function. Then, there exists $\varkappa \leq -1/12$, such that*

$$\|\phi_\nu\|_{\mathrm{BMO}} \lesssim |\nu|^{\varkappa}. \tag{42}$$

***Proof*.** By using that $L^\infty \subset \mathrm{BMO}$, we have $\|\phi_\nu\|_{\mathrm{BMO}} \lesssim \|\phi_\nu\|_{L^\infty}$. Now, from Remark 2.5 of [16] we can estimate $\|\phi_\nu\|_{L^\infty} \lesssim |\nu|^{-1/12}$ which implies the desired estimate. Indeed, if

$$\varkappa := \inf\{\omega \in \mathbb{R} : \|\phi_\nu\|_{\mathrm{BMO}} \lesssim |\nu|^{\omega}\}, \tag{43}$$

we have that $\varkappa \leq -1/12$. ☑

Proof of Theorem 1.2. We will prove that if m satisfies the condition (CI), then $A = T_m$ can be extended to a bounded operator from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. Let us consider the operator

$$\mathcal{R} := \frac{1}{2}(H - n), \tag{44}$$

where H is the harmonic oscillator on $\mathbb{R}^n$, and let us fix a dyadic decomposition of its spectrum: we choose a function $\psi_0 \in C_0^\infty(\mathbb{R})$, $\psi_0(\lambda) = 1$, if $|\lambda| \leq 1$, and $\psi(\lambda) = 0$, for $|\lambda| \geq 2$. For every $j \geq 1$, let us define $\psi_j(\lambda) = \psi_0(2^{-j}\lambda) - \psi_0(2^{-j+1}\lambda)$. Then we have

$$\sum_{l\in\mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0. \tag{45}$$

Let us consider $f \in L^\infty(\mathbb{R}^n)$. We will decompose the symbol m as

$$m(x,\nu) = m(x,\nu)(\psi_0(|\nu|)+\psi_1(|\nu|))+\sum_{k=2}^{\infty} m_k(x,\nu), \quad m_k(x,\nu) := m(x,\nu)\cdot\psi_k(|\nu|). \tag{46}$$

Let us define the sequence of pseudo-multipliers $T_{m(j)}$, $j \in \mathbb{N}$, associated to every symbol m_j, for $j \geq 2$, and by T_0 the operator with symbol $\sigma \equiv m(x,\nu)(\psi_0 + \psi_1)$. Then we want to show that the operator series

$$T_0 + S_m,\ S_m := \sum_k T_{m(k)}, \tag{47}$$

satisfies

$$\|T_m\|_{\mathscr{B}(L^\infty,\mathrm{BMO})} \leq \|T_0\|_{\mathscr{B}(L^\infty,\mathrm{BMO})} + \sum_k \|T_{m(k)}\|_{\mathscr{B}(L^\infty,\mathrm{BMO})}, \tag{48}$$

where the series in the right hand side converges. Because, $f \in L^\infty(\mathbb{R}^n)$ and for every j, $T_{m(j)}$ has symbol with compact support, $T_{m(j)} : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is bounded, and consequently $T_{m(j)}f \in L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. Now, because $T_{m(j)}f \in \mathrm{BMO}(\mathbb{R}^n)$, we will estimate its BMO norm $\|T_{m(j)}f\|_*$. By using that every symbol m_k has variable ν supported in $\{\nu : 2^{k-1} \leq |\nu| \leq 2^{k+1}\}$, we have

$$T_{m(k)}f(x) = \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} m_k(x,\nu)\phi_\nu(x)\widehat{f}(\phi_\nu), \ x \in \mathbb{R}^n.$$

Consequently,

$$\|T_{m(k)}f\|_* \leq \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} \|m_k(\cdot,\nu)\phi_\nu(\cdot)\|_*|\widehat{f}(\phi_\nu)|. \tag{49}$$

From (35) and by using the Fourier inversion formula we have,

$$\begin{aligned}\|m_k(\cdot,\nu)\phi_\nu(\cdot)\|_* &= \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \left| \int_{\mathbb{R}^n} m_k(x,\nu)\phi_\nu(x)\Omega(x)dx \right| \\ &= \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i2\pi\nu\cdot\xi}\widehat{m}_k(x,\xi)d\xi\, \phi_\nu(x)\Omega(x)dx \right| \\ &\leq \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{m}_k(x,\xi)|d\xi \times \int_{\mathbb{R}^n} |\phi_\nu(x)||\Omega(x)|dx.\end{aligned}$$

By the Cauchy-Schwarz inequality, and the condition $s > n/2$, we have

$$\int_{\mathbb{R}^n} |\widehat{m}_k(x,\xi)|d\xi \leq \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}|\widehat{m}_k(x,\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-2s}d\xi \right)^{\frac{1}{2}}. \tag{50}$$

Consequently, we claim that

$$\int_{\mathbb{R}^n} |\widehat{m}_k(x,\xi)|d\xi \leq C\|m\|_{l.u.H^s} \times 2^{-k(s-\frac{n}{2})}. \tag{51}$$

Indeed, if $\tilde{\psi}(\lambda) := \psi_0(\lambda) - \psi_0(2\lambda)$, then $\tilde{\psi} \in \mathscr{D}(\mathbb{R})$ and,

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{m}_k(x,\xi)|d\xi &\lesssim \|m_k(x,\cdot)\|_{H^s(\mathbb{R}^n)} = \|m(x,\cdot)\tilde{\psi}(2^{-k}|\cdot|)\|_{H^s(\mathbb{R}^n)} \\ &\lesssim \|m(x,\cdot)\|_{l.u.H^s} \times 2^{-k(s-\frac{n}{2})} \leq \|m\|_{l.u.,H^s} \times 2^{-k(s-\frac{n}{2})}.\end{aligned}$$

So, we obtain

$$\begin{aligned}\|m_k(\cdot,\nu)\phi_\nu(\cdot)\|_* &\leq \|m\|_{l.u.,H^s} \times 2^{-k(s-\frac{n}{2})} \times \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \int_{\mathbb{R}^n} |\phi_\nu(x)||\Omega(x)|dx \\ &= \|m\|_{l.u.,H^s} \times 2^{-k(s-\frac{n}{2})} \times \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \int_{\mathbb{R}^n} \mathrm{sig}(\Omega(x))|\phi_\nu(x)|\Omega(x)dx,\end{aligned}$$

where $\mathrm{sig}(\Omega(x)) = -1$, if $\Omega(x) < 0$, and $\mathrm{sig}(\Omega(x)) = 1$, if $\Omega(x) \geq 0$. By the duality relation (35) and by using that

$$\|\,\mathrm{sig}(\Omega(x))|\phi_\nu(x)|\|_{\mathrm{BMO}} \leq 2\|\,|\mathrm{sig}(\Omega(x))|\phi_\nu(x)|\,\|_{\mathrm{BMO}} = 2\|\,|\phi_\nu(x)|\,\|_{\mathrm{BMO}}, \tag{52}$$

we conclude that

$$\|m_k(\cdot,\nu)\phi_\nu(\cdot)\|_* \lesssim \|m\|_{l.u.,H^s} 2^{-k(s-\frac{n}{2})} \sup_{\|\Omega\|_{\mathrm{H}^1}=1} \|\phi_\nu\|_{\mathrm{BMO}} \|\Omega\|_{\mathrm{H}^1}.$$

Returning to the estimate (49), we can write

$$\begin{aligned}\|T_{m(k)}f\|_* &\leq \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} \|m\|_{l.u.,H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\mathrm{BMO}} |\widehat{f}(\phi_\nu)| \\ &\leq \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} \|m\|_{l.u.,H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\mathrm{BMO}} \|\phi_\nu\|_{L^1} \|f\|_{L^\infty}.\end{aligned}$$

Thus, the analysis above implies the following estimate for the operator norm of $T_{m(k)}$, for all $k \geq 2$,

$$\|T_{m(k)}\|_{\mathscr{B}(L^\infty(\mathbb{R}^n),\mathrm{BMO}(\mathbb{R}^n))} \lesssim \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} \|m\|_{l.u.,H^s} 2^{-k(s-\frac{n}{2})} \|\phi_\nu\|_{\mathrm{BMO}} \|\phi_\nu\|_{L^1}.$$

By using Lemma 2.2 of [16] we have $\|\phi_\nu\|_{L^1(\mathbb{R}^n)} \lesssim |\nu|^{\frac{n}{4}}$. Additionally, the inequality (42)

$$\|\phi_\nu\|_{\mathrm{BMO}} \lesssim |\nu|^{\varkappa}$$

implies that

$$\begin{aligned}\|T_{m(k)}\|_{\mathscr{B}(L^\infty(\mathbb{R}^n),\mathrm{BMO}(\mathbb{R}^n))} &\lesssim \sum_{2^{k-1}\leq|\nu|\leq 2^{k+1}} 2^{k(\frac{n}{4}+\varkappa)} \times \|m\|_{l.u.H^s} \times 2^{-k(s-\frac{n}{2})} \\ &\asymp 2^{kn} \times 2^{k(\frac{n}{4}+\varkappa)} \times \|m\|_{l.u.H^s} \times 2^{-k(s-\frac{n}{2})}.\end{aligned}$$

Now, by using that T_0 is a pseudo-multiplier whose symbol has compact support in the ν-variables, we conclude that T_0 is bounded from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and

$$\|T_0\|_{\mathscr{B}(L^\infty(\mathbb{R}^n)),\mathrm{BMO}(\mathbb{R}^n)} \leq C\|m\|_{L^\infty}.$$

This analysis allows us to estimate the operator norm of T_m as follows,

$$\begin{aligned}\|T_m\|_{\mathscr{B}(L^\infty(\mathbb{R}^n),\mathrm{BMO}(\mathbb{R}^n))} &\leq \|T_0\|_{\mathscr{B}(L^\infty(\mathbb{R}^n)),\mathrm{BMO}(\mathbb{R}^n)}+\sum_k \|T_{m(k)}\|_{\mathscr{B}(L^\infty(\mathbb{R}^n)),\mathrm{BMO}(\mathbb{R}^n)}\\ &\lesssim \|m\|_{L^\infty}+\sum_{k=1}^{\infty} 2^{-k(s-\frac{7n}{4}-\varkappa)}\|m\|_{l.u.H^s}\\ &\leq C(\|m\|_{L^\infty}+\|m\|_{l.u.H^s})<\infty,\end{aligned}$$

provided that $s>\frac{7n}{4}+\varkappa$, for some $\varkappa\leq -1/12$. So, we have proved the L^∞-BMO boundedness of T_m. In order to end the proof we only need to prove that, under the condition (CII), the operator T_m is bounded from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. But, if m satisfies (CII), then it also does to satisfy (CI), in view of the inequality,

$$\|m\|_{l.u.H^s}\lesssim \sup_{|\alpha|\leq[7n/4-1/12]+1}(1+|\nu|)^{|\alpha|}\sup_{x,\nu}|\Delta^\alpha m(x,\nu)|, \tag{53}$$

for $s>0$ satisfying, $\frac{7n}{4}-\frac{1}{12}<s<[7n/4-1/12]+1$, (see Eq. (2.29) of [16]).

***Remark* 3.2.** According to the proof of Theorem 1.2, if T_m satisfies the condition (CI), then we have

$$\|T_m\|_{\mathscr{B}(L^\infty(\mathbb{R}^n),\mathrm{BMO}(\mathbb{R}^n))}\leq C(\|m\|_{L^\infty}+\|m\|_{l.u.H^s}). \tag{54}$$

On the other hand, if we assume (CII), the operator norm of T_m satisfies

$$\|T_m\|_{\mathscr{B}(L^\infty(\mathbb{R}^n),\mathrm{BMO}(\mathbb{R}^n))}\leq C\sup_{|\alpha|\leq[7n/4-1/12]+1}(1+|\nu|)^{|\alpha|}\sup_{x,\nu}|\Delta^\alpha m(x,\nu)|. \tag{55}$$

☑

Acknowledgements. I would like to thanks Professor Michael Ruzhansky for several discussions on the subject.

References

[1] S. Bagchi and S. Thangavelu, *On Hermite pseudo-multipliers*, J. Funct. Anal. **268** (2015), no. 1, 140–170.

[2] S. Blunck, *A Hörmander-type spectral multiplier theorem for operators without heat kernel*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **5** (2003), no. 2, 449–459.

[3] D. Cardona, *A brief description of operators associated to the quantum harmonic oscillator on Schatten-von Neumann classes*, Rev. Integr. Temas Mat. **36** (2018), no. 1, 49–57.

[4] ______, *L^p-estimates for a Schrödinger equation associated with the harmonic oscillator.*, Electron. J. Differential Equations (2019), no. 20, 1–10.

[5] ______, *Sharp estimates for the Schrödinger equation associated to the twisted Laplacian*, Rep. Math. Phys. (to appear), arXiv:1810.02940.

[6] D. Cardona and E. S. Barraza, *On nuclear lp multipliers associated to the harmonic oscillator*, Perspectives from Developing Countries, Springer Proceedings in Mathematics & Statistics, Springer, Imperial College London, UK, 2016. M. Ruzhansky and J. Delgado (Eds), 2019.

[7] P. Chen, E. M. Ouhabaz, A. Sikora, and L. Yan, *Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means*, J. Anal. Math. **129** (2016), 219–283.

[8] P. Duren, B. Romberg, and A. Shields, *Linear functionals on h^p spaces with* $0 < p < 1$, J. Reine Angew. Math. **238** (1969), 32–60.

[9] J. Epperson, *Hermite multipliers and pseudo-multipliers*, Proc. Amer. Math. Soc. **124** (1996), no. 7, 2061–2068.

[10] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588.

[11] C. Fefferman and E. Stein, *h^p-spaces of several variables*, Acta Math **129** (1972), 137–193.

[12] G. H. Hardy, *The mean value of the modulus of an analytic function*, Proc. London Math. Soc. **14** (1914), 269–277.

[13] L. Hörmander, *Pseudo-differential operators and hypo-elliptic equations*, Proc. Symposium on Singular Integrals, Amer. Math. Soc. **10** (1967), 138–183.

[14] ______, *The analysis of the linear partial differential operators vol. iii.*, Springer-Verlag, 1985.

[15] F. John and L. Nirenberg, *On Functions of Bounded Mean Oscillation*, Comm. Pure Appl. Math. (1961), 415–426.

[16] M. Ruzhansky and D. Cardona, *Hörmander condition for pseudo-multipliers associated to the harmonic oscillator*, arXiv:1810.01260.

[17] M. Ruzhansky and N. Tokmagambetov, *Nonharmonic analysis of boundary value problems without WZ condition*, Math. Model. Nat. Phenom. **12** (2017), 115–140.

[18] B. Simon, *Distributions and their hermite expansions*, J. Math. Phys. **12** (1971), 140–148.

[19] S. Thangavelu, *Multipliers for hermite expansions*, Revist. Mat. Ibero. **3** (1987), 1–24.

[20] ______, *Lectures on Hermite and Laguerre Expansions*, Math. Notes, vol. 42, Princeton University Press, Princeton, 1993.

[21] ______, *Hermite and special Hermite expansions revisited*, Duke Math. J. **94** (1998), no. 2, 257–278.

[22] N. Tokmagambetov and M. Ruzhansky, *Nonharmonic analysis of boundary value problems*, Int. Math. Res. Notices **12** (2016), 3548–3615.

[23] T. Walsh, *The dual of* $\mathbb{H}^p(\mathbb{R}^{n+1}_+)$ *for* $p < 1$, Can. J. Math. **25** (1973), 567–577.

(Recibido en septiembre de 2019. Aceptado en enero de 2020)

Department of Mathematics: Analysis, Logic, and Discrete Mathematics
Ghent University
Krijgslaan 281,
Ghent, Belgium
e-mail: duvanc306@gmail.com, duvan.cardonasanchez@ugent.be