Revista Colombiana de Matemáticas
Volumen 54(2020)2, páginas 109-116

Certain Properties of Square Matrices over Fields with Applications to Rings

Algunas propiedades de matrices cuadradas sobre cuerpos con aplicaciones a anillos

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Abstract. We prove that any square nilpotent matrix over a field is a difference of two idempotent matrices as well as that any square matrix over an algebraically closed field is a sum of a nilpotent square-zero matrix and a diagonalizable matrix. We further apply these two assertions to a variation of π-regular rings. These results somewhat improve on establishments due to Breaz from Linear Algebra & Appl. (2018) and Abyzov from Siberian Math. J. (2019) as well as they also refine two recent achievements due to the present author, published in Vest. St. Petersburg Univ. - Ser. Math., Mech. & Astr. (2019) and Chebyshevskii Sb. (2019), respectively.

Key words and phrases. Nilpotent matrices, idempotent matrices, Jordan canonical form, algebraically closed fields, super π-regular rings.

2020 Mathematics Subject Classification. 16U99; 16E50; 16W10; 13B99.

Resumen. Probamos que toda matriz cuadrada nilpotente sobre un cuerpo es igual a la resta de dos matrices idempotentes, también probamos que toda matriz cuadrada con coeficientes en un cuerpo algebraicamente cerrado es la suma de una matriz nilpotente cuyo cuadrado es nulo y una matriz diagonalizable. También aplicamos estos resultados en una variante de anillos π-regulares. Estos resultados mejoran los resultados presentados por Breaz en Linear Algebra & Appl. (2018) y aquellos de Abyzov presentados en Siberian Math. J. (2019) al igual que aquellos publicados por el autor del presente artículo en Vest. St. Petersburg Univ. - Ser. Math., Mech. & Astr. (2019) y en Chebyshevskii Sb. (2019), respectivamente.

Palabras y frases clave. Matrices nilpotentes, matrices idempotentes, forma canónica de Jordan, cuerpos algebraicamente cerrados, anillos π-regulares.

1. Introduction and Fundamentals

All rings R are assumed here to be associative, containing the identity element 1 which differs from the zero element 0 of R. Recall that a ring R is said to be *π-regular*, provided every element r is π-regular, that is, there exists $n \in \mathbb{N}$ depending on r such that $r^n \in r^nRr^n$, and we call a ring R *super π-regular*, provided that $r^m \in r^mRr^m$ for each $m \in \mathbb{N}$ – thus it is pretty clear that super π-regularity implies π-regularity, an irreversible implication. An important class of super π-regular rings is the class of *von Neumann regular* rings in the sense that they are π-regular with $n = 1$ for all r. On another vein, the class of π-regular rings was considerably extended in [9] to the class of so-called *regularly nil clean* rings, that are rings R for which, for any $a \in R$, there is an idempotent $e \in aR$ such that $(1 - e)a$ is nilpotent (equivalently, there is an idempotent $f \in Ra$ with $(1 - f)a$ nilpotent).

The presentation of a matrix over a ring as a sum/difference of some special elements like units, nilpotents, idempotents, potents, etc., always plays a central role in matrix ring theory. A brief collection of principally known historical facts in this branch are as follows: In [6] was shown that any square matrix over the finite two elements field $\mathbb{Z}_2$ is a sum of a nilpotent matrix and an idempotent matrix; thereby the full matrix $n \times n$ ring $\mathbb{M}_n(\mathbb{Z}_2)$ is called *nil-clean.* This important fact was strengthened in [23] by showing that, for any $n \in \mathbb{N}$ and for every $n \times n$ matrix A over $\mathbb{Z}_2$, there exists an idempotent matrix E such that $(A - E)^4 = 0$, while over the finite indecomposable ring $\mathbb{Z}_4$ consisting of four elements this relation is precisely $(A - E)^8 = 0$ (see [2] and [22] for some further generalizations and specifications, too). In [23] is showed also that the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_2)$ is both nil-clean and von Neumann regular but *not* strongly π-regular, whereas the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_4)$ is both nil-clean and regularly nil clean but *not* π-regular (see [9], as well). Likewise, in [10] was established that the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(K)$ over an algebraically closed field K is regularly nil clean even in a more thin setting by viewing that the required nilpotent is of exponent 2.

Moreover, a rather actual question is the following one: *Is every matrix over each field presentable as the direct sum of a nilpotent and a potent*? In that aspect, it was proved in [5] that every $n \times n$ matrix M over a field of odd cardinality q has a decomposition of the form $M = P + N$, where $P^q = P$ is q-potent and N is nilpotent with $N^3 = 0$ but $N^2 \neq 0$ in general (compare also with the results obtained in [1]).

We, however, conjecture that this is not always true; it is rather a sum of a non-singular matrix and a nilpotent matrix – see, *e.g.*, [19]. But over the four element field $\mathbb{F}_4$, which case is in sharp contrast to the aforementioned result from [5], this surely implies that it is a sum of a potent and a nilpotent, not knowing what are the exact degrees neither of the potent nor the nilpotent, however. In this way, a rather eluding question is *whether or not for all $n \in \mathbb{N}$*

each element from the matrix ring $\mathbb{M}_n(\mathbb{Z}_4)$ *is the sum of a (square) nilpotent and a potent*? We just refer for the more general case of rings of the kind $\mathbb{Z}_{p^m}$, where p is an arbitrary prime and m is an arbitrary natural, to the good source [2] (Lemma 1 and Theorem 4), in which it is proved that any element from $\mathbb{M}_n(\mathbb{Z}_{p^m})$ is a sum of a nilpotent (not necessarily of order 2) and a potent.

So, we come to the following basic and intriguing problem, whose complete resolution seems to be extremely difficult:

Conjecture. Every square matrix A over a field F with at least four elements can be represented as $A = D + Q$ with $Q^2 = 0$ and D being diagonalizable over F.

It is worthwhile noticing that, for fields of three elements (*i.e.*, over $\mathbb{F}_3 = \mathbb{Z}_3$), the conjecture fails as illustrated in [5, Example 6]. Nevertheless, concerning the fields with $|F| = 3$, we are believing that the same conjecture holds, but only for matrices A such that the exceptional 3×3 matrix from [5] does not appear as a rational normal form block of A, $A + I$ and $A - I$, where I stands for the standard matrix identity, and also it may be the case that the matrices with such a block have to require index three nilpotents instead of these in the stated above conjecture.

The aim of this short article is to settle this conjecture in the case of algebraically closed fields. This will be successfully done in the sequel (compare with [12] and [8] as well). Further eventual applications of such decompositions could be realized in coding theory and, in particular, in noncommutative coding theory (*cf.* [4], [7], [14], [16], [15], [17] and [20]).

2. Results and Questions

Before proceeding to prove our first chief results, we will show the validity of the next technicality, which sounds rather effectible and is of independent interest as well (to keep a record straight, we notice that it was originally established at first in [18, Proposition 1] as well as it somewhat also appeared in an unpublished draft [21] – however, our approach will be totally direct and rather more easy).

Lemma 2.1. Any square nilpotent matrix over a field is the difference of two idempotent matrices.

Proof. Take a nilpotent matrix N over a field F. Standardly, put N in Jordan form, possible because all its eigenvalues (0) lie in F. The only property we really need now of N is that it is strictly upper triangular with all its nonzero entries in the first super-diagonal. In fact, even much more weaker than that – nonzero entries have odd-parity of positions (i, j), meaning if i is even, then j is odd, and the other way round. Let E be the diagonal idempotent matrix with diagonal entries the sequence $1, 0, 1, 0, 1, \cdots$ as follows

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Observe at once that for a general matrix A, the matrix EAE retains the entries of A in the odd, odd positions and sets everything else to 0. Similarly, $(I-E)A(I-E)$ keeps the even, even elements and wipes out the rest, where I is the identity matrix. Hence, it is immediate that

$$ENE = (I-E)N(I-E) = 0.$$

Therefore,

$$\begin{aligned} N &= ENE + EN(I-E) + (I-E)NE + (I-E)N(I-E) \\ &= EN(I-E) + (I-E)NE \\ &= [E + EN(I-E)] - [E - (I-E)NE], \end{aligned}$$

which gives N as the difference of the two square-bracketed idempotents, as required. ☑

So, we are ready to establish the following statement, which is an acceptable reminiscent of [11, Proposition 2.1] and is also proved in [21] (note that the proof there depends entirely on Lemma 2.1, whereas we here will give a more conceptual proof which entirely relies on already known results).

Theorem 2.2. *Let R be a super π-regular ring. Then every nilpotent is the difference of two idempotents.*

Proof. In the context of $a \in R$ being a nilpotent element with a^s regular for each s, from [13, Corollary 2.10], it must be that a can be expressed as an orthogonal sum of Jordan blocks (each of them, a matrix of certain size $n \times n$ with 1's in its second diagonal).

Let A be one of those Jordan blocks: $A = \sum_k e_{k+1,k}$. Then $A = E - F$ with

$E = (e_{11} + e_{12}) + (e_{33} + e_{34}) + (e_{55} + e_{56}) + \dots$

$F = e_{11} + (-e_{23} + e_{33}) + (-e_{45} + e_{55}) + \dots$

where both E and F are idempotents, because they consist on 1×1 or 2×2 idempotent blocks.

Repeating this process for each of the Jordan blocks, we will obtain that the original element a can be expressed as the difference of two idempotents, as expected. ☑

We now come to our second main result, resolving the stated above "Conjecture" in the case of algebraically closed fields (which are necessarily *infinite* fields).

Theorem 2.3. *Any square matrix over an algebraically closed field is a sum of a nilpotent square-zero matrix and a diagonalizable matrix.*

Proof. Assume that the matrix is in Jordan canonical form, and so one can just work with (upper-triangular) Jordan blocks. From each such Jordan block A (of size at least 3×3) subtract a matrix B of the same size, having nonzero entries only in every other slot on the diagonal below the main diagonal, with those entries being distinct (and zeros elsewhere). Then B is clearly nilpotent of index 2 and, moreover, $A - B$ is diagonalizable.

Concretely, in the 3×3 case, the nilpotent matrix B would be as follows:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the 4×4 case, the nilpotent matrix B would be as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

And so on, by using the same technique. Specifically, in terms of unit matrices, we define

$$B = a_1 e_{21} + a_2 e_{43} + a_3 e_{65} + \cdots ,$$

where the elements $a_1, a_2, a_3, \ldots$ are distinct. That is, we chose algebraically closed fields, so that one could use the Jordan canonical form and have infinitely many distinct elements in the field.

Alternatively, if zeros and non-zeros alternate, then such a matrix B is indeed square-zero. At first look, it seems harder to see why $A - B$ is diagonalizable; however, it is easier to see the property of $A - B$ being diagonalizable if we just take B to be the bottom left matrix unit, which is obviously square-zero. ☑

We finish off our work with the following two challenging queries:

Problem 2.4. Extend the considered above property from Theorem 2.3 for any field F which is *not* necessarily algebraically closed.

Problem 2.5. Examine those rings R for which, for any $a \in R$, there exists an idempotent $e \in aRa$ such that $a(1 - e)a$ is a nilpotent.

As a valuable non-trivial example of such a class of rings we can visualize the class of strongly π-regular regular rings, that are rings R such that, for each $a \in R$, there is $n \in \mathbb{N}$ which depends on a and possesses the property $a^n \in a^{n+1}R \cap Ra^{n+1}$. In fact, in an equivalent form this means that $a^n = a^{2n}x = a^n x a^n$ for some $n \in \mathbb{N}$ and some $x \in R$ with $xa = ax$. Thus, $a^n x = e \in Id(R)$ and by squaring we deduce that $e = a^{2n}x^2 = a^n x^2 a^n \in aRa$. Furthermore, $(1-e)a = a(1-e) = a(1-a^n x) \in Nil(R)$ since one sees that

$$[a(1-e)]^n = [a(1-a^n x)]^n = a^n(1-a^n x)^n = a^n(1-a^n x)(1-a^n x)^{n-1} =$$

$$= (a^n - a^{2n}x)(1-a^n x)^{n-1} = 0.$$

So, one finds that $a(1-e)a = a^2(1-e) = [a(1-e)]^2 \in Nil(R)$, as promised.

On the other side, as already showed in [3], this lastly demonstrated assertion could also be attacked by virtue of both Theorems 2.2 and 2.3, since we may interpret the nilpotent and idempotent elements in such rings as their corresponding elements in matrix rings.

Acknowledgements. The author is very thankful to the referee for his/her valuable comments and suggestions made, which improved the presentation of the article, as well as to the handling editor, W. A. Zuñiga-Galindo, Professor at CINVESTAV, Mexico, for his professional management of the paper. The author also expresses his deep thanks to Professor Zachary Mesyan from United States of America, to Professor Yaroslav Shitov from Russia, and to Professors Esther García and Miguel Lozano from Spain, for the valuable communications on the present object.

Funding: The work of the author on this paper was partially supported by the Bulgarian National Science Fund under Grant KP-06 No 32/1 of December 07, 2019.

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(Recibido en abril de 2020. Aceptado en junio de 2020)

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