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Convolution of Two Weighted Orlicz Spaces on Hypergroups

Convolución de dos espacios de Orlicz con pesos sobre hipergrupos

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Abstract. Let K be a locally compact hypergroup. In this paper, among other results we give a sufficient condition for the inclusion $L_w^{\Phi_1}(K) * L_w^{\Phi_2}(K) \subseteq L_w^{\Phi_1}(K)$ to hold. Also, as an application, we provide a new sufficient condition for the weighted Orlicz space $L_w^{\Phi}(K)$ to be a convolution Banach algebra.

Key words and phrases. Locally compact group, locally compact hypergroup, weighted Orlicz space, Young function, convolution, Banach algebra.

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Resumen. Sea K un hipergrupo localmente compacto. En este artículo, entre otros resultados, damos una condición suficiente para que se cumpla la inclusión $L_w^{\Phi_1}(K) * L_w^{\Phi_2}(K) \subseteq L_w^{\Phi_1}(K)$. Como una aplicación de este resultado, logramos dar una nueva condición suficiente para que el espacio de Orlicz con pesos $L_w^{\Phi}(K)$ sea un álgebra de Banach bajo la convolución.

Palabras y frases clave. Grupos localmente compactos, hipergrupos localmente compactos, espacios de Orlicz con pesos, función de Young, convolución, álgebras de Banach.

1. Introduction

For each locally compact group G with a sub-multiplicative mapping w, $L^1(G)$ and $L_w^1(G)$ are convolution Banach algebras. If $1 < p < \infty$, then it is well-known that the Lebesgue space $L^p(G)$ is a convolution Banach algebra if and only if G is compact [15]. But the situation for weighted ones is different: a weighted Lebesgue space $L_w^p(G)$, $p > 1$, can be a convolution algebra for some non-compact groups G. Yu. N. Kuznetsova (see [8, 9]) gave some conditions under which the weighted Lebesgue space $L^p_w(G)$ is a convolution Banach algebra (see also [1]). In [16], this topic was studied for weighted Lebesgue spaces on locally compact hypergroups. Since Orlicz spaces are extensions of Lebesgue spaces, it is natural to study the same question about Orlicz spaces and it has been the motivation for many papers. For example, H. Hudzik, A. Kamiska and J. Musielak in [5, Theorem 2] proved that if Φ is a Young function with $\Phi \in \Delta_2$, then $L^\Phi(G)$ is a Banach algebra under convolution if and only if $\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$ or G is compact.

In [17] some necessary and sufficient condition is given for an Orlicz space $L^\Phi(G)$ to be a convolution Banach algebra, where G is a compactly generated second countable abelian group and Φ belongs to a special class of Young functions.

A. Osançlıol and S. Öztop in [10] introduced and studied the weighted Orlicz algebras on locally compact groups, and proved that, if $L^\Phi_w(G) \subseteq L^1_w(G)$, then $L^\Phi_w(G)$ is a convolution Banach algebra. In [11], these results were extended to the hypergroup case. T.S. Quek and L.Y.H. Yap in the nice paper [12] showed that the generalized L^p-conjecture is true for locally compact abelian groups. More precisely, they proved that if G is a locally compact abelian group and $1 < p, q < \infty$, then the inclusion

$$L^p(G) * L^q(G) \subseteq L^p(G) \tag{1}$$

holds if and only if G is compact [12, Corollary 1.4]. In this paper we give a characterization for the existence a constant $k > 0$ such that $\|f * g\|_{\Phi_1,w} \leq k\|f\|_{\Phi_1,w}\, \|g\|_{\Phi_2,w}$ for all $f \in L^{\Phi_1}_w(K)$ and $g \in L^{\Phi_2}_w(K)$, where K is a locally compact hypergroup, Φ_1 and Φ_2 are Young functions with $\Phi_1 \in \Delta_2$, and w is a weight function on K. Since hypergroups are extensions of locally compact groups and Orlicz spaces are in turn extensions of Lebesgue spaces, this result is a weighted version of the mentioned fact [12, Corollary 1.4], to a much larger setting. As an application, we give a sufficient condition for the inclusion $L^{\Phi_1}_w(K) * L^{\Phi_2}_w(K) \subseteq L^{\Phi_1}_w(K)$ to hold. In particular, whenever $\Phi_1 = \Phi_2 = \Phi$, we provide a condition for the weighted Orlicz space $L^\Phi_w(K)$ to be a Banach algebra. This paper is organized as follows. In section 2, we recall some definitions, notations and facts about locally compact hypergroups and Orlicz spaces, which we will need in the sequel. The main results are in section 3.

2. Preliminaries

2.1. Locally Compact Hypergroups

Let K be a locally compact Hausdorff space. We denote the space of all bounded Radon measures on K by $\mathcal{M}(K)$, and the set of non-negative elements of $\mathcal{M}(K)$ is denoted by $\mathcal{M}^+(K)$. The support of each measure $\mu \in \mathcal{M}(K)$ and the Dirac measure at the point $x \in K$ are denoted by suppμ and δ_x, respectively.

Definition 2.1. Let K be a locally compact Hausdorff space with the following property:

(1) there is a mapping $* : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathcal{M}(K)$ (called *convolution*) such that $(\mathcal{M}(K), *, +)$ is a complex Banach algebra;

(2) for each $\mu, \nu \in \mathcal{M}^+(K)$, $\mu * \nu$ is a non-negative measure in $\mathcal{M}(K)$ and the mapping $(\mu, \nu) \mapsto \mu * \nu$ from $\mathcal{M}^+(K) \times \mathcal{M}^+(K)$ into $\mathcal{M}^+(K)$ is continuous, where $\mathcal{M}^+(K)$ is equipped with the cone topology;

(3) for all $x, y \in K$, $\delta_x * \delta_y$ is a compact supported probability measure;

(4) the mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of all non-empty compact subsets of K, equipped with the Michael topology, is continuous;

(5) there is an element e (called the *identity*) such that for each $x \in K$, $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$;

(6) there is a homeomorphism $x \mapsto x^-$ from K onto K (called *involution*) such that for each $x, y \in K$ we have $(x^-)^- = x$ and $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$, where for each $\mu \in \mathcal{M}(K)$ and Borel set $E \subseteq K$, $\mu^-(E) := \mu(\{x^- : x \in E\})$;

(7) for each $x, y \in K$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$.

Then, $K \equiv (K, *, .^-, e)$ is called a *(locally compact) hypergroup*.

A non-zero non-negative Radon measure m on a hypergroup K is called a left Haar measure if for each $x \in K$, $\delta_x * m = m$. Throughout, we assume that $(K, *, \cdot^-, e)$ is a locally compact hypergroup with a left Haar measure m. Also, the integrals without any specified measure are considered with the given left Haar measure m. Any locally compact group equipped with the usual convolution and the inverse mapping as involution is a hypergroup. In general, a topological space can be a hypergroup without necessarily an action between its elements. See the book [2] and the papers [3, 6] and [4, 19] for more examples and details about this structure.

For each measurable function $f : K \to \mathbb{C}$ and $x, y \in K$ we put

$$f(x * y) := \int_K f(t)\, d(\delta_x * \delta_y)(t).$$

Also, we denote $L_x f(y) := f(x^- * y)$. By [6, Theorem 5.1D], for each measurable functions $f, g : K \to \mathbb{C}$, if either f or g is σ-finite with respect to m, then

$$\int_K f(y * x) g(x)\, dm(x) = \int_K f(x) g(y^- * x)\, dm(x). \tag{2}$$

The *convolution* of f and g is defined by

$$(f * g)(x) := \int_K f(y) g(y^- * x)\, dm(y), \quad (x \in K),$$

whenever this integral exists.

2.2. Orlicz Spaces

The main references for Orlicz spaces are the books [13, 14]. Before giving the definition of an Orlicz space one needs to introduce Young functions.

A convex even mapping $\Phi : \mathbb{R} \to [0, \infty)$ is called a *Young function* if $\Phi(0) = \lim_{x\to 0} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$.

The *complementary* of a Young function Φ is defined by

$$\Psi(x) := \sup\{y|x| - \Phi(y) : \, y \geq 0\}, \quad (x \in \mathbb{R}).$$

In this case, (Φ, Ψ) is called a *complementary pair*.

We say that a Young function Φ satisfies Δ_2-*condition* (and write $\Phi \in \Delta_2$) if for some constants $c > 0$ and $x_0 \geq 0$,

$$\Phi(2x) \leq c\, \Phi(x), \quad (x \geq x_0).$$

Let Φ be a Young function with the complementary Ψ. The set of all Borel measurable functions $f : K \to \mathbb{C}$ such that for some $\alpha > 0$,

$$\int_K \Phi(\alpha |f(x)|)\, dm(x) < \infty,$$

is denoted by $L^\Phi(K)$. We identify two elements of $L^\Phi(K)$ if they are equal a.e. For each $f \in L^\Phi(K)$ we define

$$\|f\|_\Phi := \sup \left\{ \int_K |fv|\, dm : \int_K \Psi(|v|)\, dm \leq 1 \right\}.$$

Then, $\|\cdot\|_\Phi$ is a complete norm on $L^\Phi(K)$ and the pair $(L^\Phi(K), \|\cdot\|_\Phi)$ is called an *Orlicz space*; see [13, Chapter III, Proposition 11].

For each $1 < p < \infty$, the function Φ_p defined by $\Phi_p(x) := |x|^p$ for all $x \in \mathbb{R}$, is a Young function and the Orlicz space $L^{\Phi_p}(K)$ is the same as the usual Lebesgue space $L^p(K)$. Orlicz spaces, as extensions of Lebesgue spaces, have been studied in several recent decades; see for example [7], a recent paper about Orlicz spaces on locally compact hypergroups.

Set

$$\mathcal{N}_\Phi(f) := \inf \left\{ \lambda > 0 : \int_K \Phi(\frac{1}{\lambda} |f(x)|)\, dm(x) \leq 1 \right\}, \quad (f \in L^\Phi(K)).$$

Then, $\mathcal{N}_\Phi(\cdot)$ is also a norm on $L^\Phi(K)$ and for each $f \in L^\Phi(K)$,

$$\mathcal{N}_\Phi(f) \le \|f\|_\Phi \le 2\mathcal{N}_\Phi(f).$$

If $f \in L^\Phi(K)$ and $g \in L^\Psi(K)$, then by [13, Page 58] we have

$$\int_K |f(x)g(x)|\, dm(x) \le 2\|f\|_\Phi\, \|g\|_\Psi, \tag{3}$$

which is the *Hölder's inequality* for Orlicz spaces.

Any continuous function $w : K \to (0, \infty)$ is called a *weight*, and we write $w^{-1} := \frac{1}{w}$. The space of all measurable functions f on K such that $wf \in L^\Phi(K)$ is called the *weighted Orlicz space* and is denoted by $L^\Phi_w(K)$. For each $f \in L^\Phi_w(K)$ we put $\|f\|_{\Phi,w} := \|wf\|_\Phi$. The space $(L^\Phi_w(K), \|\cdot\|_{\Phi,w})$ is a Banach space. The weighted Orlicz space $L^\Phi_w(K)$ is called a *convolution Banach algebra* if there exists a constant $c > 0$ such that for all $f, g \in L^\Phi_w(K)$, $f * g \in L^\Phi_w(K)$ and

$$\|f * g\|_{\Phi,w} \le c\, \|f\|_{\Phi,w}\, \|g\|_{\Phi,w}.$$

If $\Phi \in \Delta_2$, then just as in the non-weighted case [13, Page 111] (see also [10]), the dual of the Banach space $L^\Phi_w(K)$ equals $L^\Psi_{w^{-1}}(K)$ via the duality formula

$$\langle f, g\rangle = \int_K f(x)g(x)\, dm(x). \tag{4}$$

3. Main Results

For each pair $x, y \in K$ we define

$$\Omega(x, y) := \frac{w(x * y)}{w(x)w(y)}.$$

Also, for each Borel measurable functions $f, g : K \to \mathbb{C}$ we denote

$$\Lambda_w(f, g) := \int_K f(x)g(x)w(x)dm(x),$$

whenever this integral exists.

In the sequel, sometimes we specify the position of a variable by $(\cdot)$. We define

$$\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right)(x) := \Lambda_w\left(g, L_x(\frac{f}{w})\right), \quad (x \in K). \tag{5}$$

Comparing with [12, Corollary 1.4] (see the inclusions (1) and (12)), the following theorem and its corollary can be considered as a positive solution for the generalized weighted L^p-conjecture for Orlicz spaces, although it is stated on the more general framework of hypergroups.

Theorem 3.1. *Let K be a locally compact hypergroup, and Φ_1 and Φ_2 be Young functions with $\Phi_1, \Psi_1 \in \Delta_2$, where Ψ_1 is the complementary of Φ_1. Let w be a weight function on K. Then, the following are equivalent:*

(1) There is a constant $k > 0$ such that for each $f \in L_w^{\Phi_1}(K)$ and $g \in L_w^{\Phi_2}(K)$, $\|f * g\|_{\Phi_1,w} \le k\|f\|_{\Phi_1,w}\, \|g\|_{\Phi_2,w}$.

(2) There is a constant $c > 0$ such that for each $f \in L^{\Phi_2}(K)$ and $g \in L^{\Psi_1}(K)$, we have $\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right) \in L^{\Psi_1}_{w^{-1}}(K)$ and

$$\left\|\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right)\right\|_{\Psi_1,w^{-1}} \le c\|f\|_{\Phi_2}\|g\|_{\Psi_1}.$$

Proof. (1)⇒(2): Note that in this proof, we will use the identity $\left(L_w^{\Phi_1}(K)\right)^* \cong L^{\Psi_1}_{w^{-1}}(K)$. For this step, we use the assumption that $\Phi_1 \in \Delta_2$, but we do not require the assumption for Φ_2.

Suppose that there is a constant $k > 0$ such that for each $h_1 \in L_w^{\Phi_1}(K)$ and $h_2 \in L_w^{\Phi_2}(K)$,

$$\|h_1 * h_2\|_{\Phi_1,w} \le k\|h_1\|_{\Phi_1,w}\, \|h_2\|_{\Phi_2,w}. \tag{6}$$

Let $f \in L^{\Phi_2}(K)$ and $g \in L^{\Psi_1}(K)$. Then by (1), for each $h \in L^{\Phi_1}(K)$ we have

$$\left\|(\frac{h}{w} * \frac{f}{w})w\right\|_{\Phi_1} = \left\|\frac{h}{w} * \frac{f}{w}\right\|_{\Phi_1,w} \le k\left\|\frac{h}{w}\right\|_{\Phi_1,w}\left\|\frac{f}{w}\right\|_{\Phi_2,w} < \infty \tag{7}$$

since $\frac{h}{w} \in L_w^{\Phi_1}(K)$ and $\frac{f}{w} \in L_w^{\Phi_2}(K)$. This implies that $(\frac{h}{w} * \frac{f}{w})w \in L^{\Phi_1}(K)$.

We define the mapping T on $L_w^{\Phi_1}(K)$ by

$$T(\varphi) := \int_K \Lambda_w\left(g, L_y(\frac{f}{w})\right)\varphi(y)\, dm(y), \quad (\varphi \in L_w^{\Phi_1}(K)). \tag{8}$$

Then, by the Fubini's Theorem and the Hölder's inequality (3), for each $h \in L^{\Phi_1}(K)$ we have

$$\begin{aligned}
\left|T(\frac{h}{w})\right| &= \left|\int_K \int_K g(x)(\frac{f}{w})(y^- * x)w(x)\frac{h(y)}{w(y)}\,dm(x)dm(y)\right| \\
&= \left|\int_K w(x)g(x)\left(\int_K (\frac{h}{w})(y)(\frac{f}{w})(y^- * x)\,dm(y)\right)dm(x)\right| \\
&= \left|\int_K (\frac{h}{w} * \frac{f}{w})(x)w(x)g(x)\,dm(x)\right| \\
&\le \int_K \left|(\frac{h}{w} * \frac{f}{w})(x)w(x)g(x)\right|\,dm(x) \\
&\le 2\left\|(\frac{h}{w} * \frac{f}{w})w\right\|_{\Phi_1} \|g\|_{\Psi_1} \\
&= 2\left\|\frac{h}{w} * \frac{f}{w}\right\|_{\Phi_1,w} \|g\|_{\Psi_1} \\
&\le 2k\left\|\frac{h}{w}\right\|_{\Phi_1,w} \left\|\frac{f}{w}\right\|_{\Phi_2,w} \|g\|_{\Psi_1} \\
&= 2k\|h\|_{\Phi_1}\|f\|_{\Phi_2}\|g\|_{\Psi_1}.
\end{aligned}$$

In other words, for each $\varphi \in L_w^{\Phi_1}(K)$ we have

$$|T(\varphi)| \le 2k\|\varphi w\|_{\Phi_1}\|f\|_{\Phi_2}\|g\|_{\Psi_1} = 2k\|\varphi\|_{\Phi_1,w}\|f\|_{\Phi_2}\|g\|_{\Psi_1}. \tag{9}$$

This relation shows that $T \in \left(L_w^{\Phi_1}(K)\right)^* \cong L_{w^{-1}}^{\Psi_1}(K)$. So, there is a unique function u such that

$$T(\varphi) = \int_K u(y)\varphi(y)\,dm(y)$$

for all $\varphi \in L_w^{\Phi_1}(K)$, $u \in L_{w^{-1}}^{\Psi_1}(K)$ and $\|T\| = \|u\|_{\Psi_1,w^{-1}}$. By (8) we have

$$\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right) = u \quad a.e.$$

Therefore, $\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right) \in L_{w^{-1}}^{\Psi_1}(K)$, and by the relation (9),

$$\left\|\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right)\right\|_{\Psi_1,w^{-1}} = \|T\| \le 2k\|f\|_{\Phi_2}\|g\|_{\Psi_1}.$$

(2)⇒(1): Suppose that there is a constant $c > 0$ such that for each $f \in L^{\Phi_2}(K)$ and $g \in L^{\Psi_1}(K)$,

$$\left\|\Lambda_w\left(g, L_{(\cdot)}(\frac{f}{w})\right)\right\|_{\Psi_1,w^{-1}} \le c\|f\|_{\Phi_2}\|g\|_{\Psi_1}.$$

Let $h \in L_w^{\Phi_1}(K)$ and $f \in L_w^{\Phi_2}(K)$. We define

$$S(\varphi) := \int_K (h * f)(x)\varphi(x)\, dm(x) \tag{10}$$

for all $\varphi \in L_{w^{-1}}^{\Psi_1}(K)$. Then, for each $g \in L^{\Psi_1}(K)$ we have

$$\begin{aligned}
|S(gw)| &= \left| \int_K (h * f)(x) g(x) w(x)\, dm(x) \right| \\
&= \left| \int_K \int_K h(y) L_y f(x) g(x) w(x)\, dm(y) dm(x) \right| \\
&= \left| \int_K h(y) \left(\int_K L_y f(x) g(x) w(x)\, dm(x) \right) dm(y) \right| \\
&= \left| \left\langle \frac{1}{w} \Lambda_w \left(g, L_{(\cdot)} f\right), wh \right\rangle \right| \\
&\le 2 \left\| \Lambda_w \left(g, L_{(\cdot)}(\frac{fw}{w}) \right) \right\|_{\Psi_1, w^{-1}} \|wh\|_{\Phi_1} \\
&\le 2c \|h\|_{\Phi_1, w} \|fw\|_{\Phi_2} \|g\|_{\Psi_1} \\
&= 2c \|h\|_{\Phi_1, w} \|f\|_{\Phi_2, w} \|gw\|_{\Psi_1, w^{-1}}.
\end{aligned}$$

In other words, for each $\varphi \in L_{w^{-1}}^{\Psi_1}(K)$ we have

$$|S(\varphi)| \le 2c \|h\|_{\Phi_1, w} \|f\|_{\Phi_2, w} \|\varphi\|_{\Psi_1, w^{-1}}. \tag{11}$$

Hence, $S \in \left(L_{w^{-1}}^{\Psi_1}(K) \right)^*$. This implies that there is a unique function u such that

$$S(\varphi) = \int_K u(x)\varphi(x)\, dm(x) \quad (\varphi \in L_{w^{-1}}^{\Psi_1}(K)),$$

$u \in L_w^{\Phi_1}(K)$ and $\|S\| = \|u\|_{\Phi_1, w}$. So, by definition of S we have $u = h * f$ a.e., and therefore by (11),

$$\|h * f\|_{\Phi_1, w} = \|S\| \le 2c \|h\|_{\Phi_1, w} \|f\|_{\Phi_2, w},$$

and the proof is completed. ☑

Corollary 3.2. *Let K be a locally compact hypergroup, and Φ_1 and Φ_2 be Young functions with $\Phi_1, \Psi_1 \in \Delta_2$, where Ψ_1 is the complementary of Φ_1. Let w be a weight function on K. Suppose that there is a constant $c > 0$ such that for each $f \in L^{\Phi_2}(K)$ and $g \in L^{\Psi_1}(K)$,*

$$\left\| \Lambda_w \left(g, L_{(\cdot)}(\frac{f}{w}) \right) \right\|_{\Psi_1, w^{-1}} \le c \|f\|_{\Phi_2} \|g\|_{\Psi_1},$$

where $\Lambda_w \left(g, L_{(\cdot)}(\frac{f}{w}) \right)$ is defined as in (5). Then,

$$L_w^{\Phi_1}(K) * L_w^{\Phi_2}(K) \subseteq L_w^{\Phi_1}(K). \tag{12}$$

Proof. The assumption is same as the condition (2) in Theorem 3.1. So, by the equivalence given in Theorem 3.1, for each $f \in L_w^{\Phi_1}(K)$ and $g \in L_w^{\Phi_2}(K)$, we have $\|f * g\|_{\Phi_1,w} < \infty$, i.e. $f * g \in L_w^{\Phi_1}(K)$, and the inclusion (12) follows. ☑

Corollary 3.3. *Let K be a locally compact hypergroup with a weight function w, and (Φ, Ψ) be a complementary pair with $\Phi, \Psi \in \Delta_2$. Then, $L_w^{\Phi}(K)$ is a convolution Banach algebra if and only if there is a constant $c > 0$ such that for each $f \in L^{\Phi}(K)$ and $g \in L^{\Psi}(K)$,*

$$\left\| \Lambda_w \left(g, L_{(\cdot)}(\frac{f}{w}) \right) \right\|_{\Psi, w^{-1}} \leq c \|f\|_{\Phi} \|g\|_{\Psi}.$$

If K is a locally compact group, then using a change of variables we have

$$\begin{aligned}
\Lambda_w \left(g, L_{(\cdot)}(\frac{f}{w}) \right)(x) &= \Lambda_w \left(g, L_x(\frac{f}{w}) \right) \\
&= \int_K g(y) L_x(\frac{f}{w})(y) w(y)\, dm(y) \\
&= \int_K g(y) \frac{f(x^{-1}y)}{w(x^{-1}y)} w(y)\, dm(y) \\
&= \int_K f(y) g(xy) \frac{w(xy)}{w(y)}\, dm(y).
\end{aligned}$$

With this relation, one can easily conclude the following two facts from the above results in the context of locally compact groups.

Corollary 3.4. *Let K be a locally compact group with a weight function w, and Φ_1 and Φ_2 be Young functions with $\Phi_1, \Psi_1 \in \Delta_2$, where Ψ_1 is the complementary of Φ_1. Then, the followings are equivalent:*

(i) There is a constant $k > 0$ such that for each $f \in L_w^{\Phi_1}(K)$ and $g \in L_w^{\Phi_2}(K)$, $\|f * g\|_{\Phi_1,w} \leq k \|f\|_{\Phi_1,w} \|g\|_{\Phi_2,w}$.

(ii) There is a constant $c > 0$ such that for each $f \in L^{\Phi_2}(K)$ and $g \in L^{\Psi_1}(K)$,

$$\left\| \int_K f(y) g(\cdot y) \Omega(\cdot, y)\, dm(y) \right\|_{\Psi_1} \leq c \|f\|_{\Phi_2} \|g\|_{\Psi_1}.$$

We can also conclude Theorem 3.1 of [18] from Corollary 3.3.

Corollary 3.5. *Let K be a locally compact group with a weight function w, and (Φ, Ψ) be a complementary pair with $\Phi, \Psi \in \Delta_2$. Then, $L_w^{\Phi}(K)$ is a convolution Banach algebra if and only if there is a constant $c > 0$ such that for each $f \in L^{\Phi}(K)$ and $g \in L^{\Psi}(K)$,*

$$\left\| \int_K f(y) g(\cdot\, y) \Omega(\cdot, y)\, dm(y) \right\|_{\Psi} \leq c \|f\|_{\Phi} \|g\|_{\Psi}.$$

Thanks to Corollary 3.3, in the following result we find a sufficient condition for a weighted Orlicz space on a hypergroup to be a convolution algebra. For each $g \in L^\Psi(K)$ and $x, y \in K$ we put

$$H^g(x,y) := \frac{(gw)(y * x)}{w(x)w(y)}.$$

Also, we denote the set of all measurable functions v with $\int_K \Phi(|v|) \leq 1$ by Γ_Φ.

Theorem 3.6. *Let K be a hypergroup with a weight function w, and (Φ, Ψ) be a complementary pair with $\Phi, \Psi \in \Delta_2$. Suppose that*

$$\sup\{\left\| \left\| H^g_{(\cdot)} \right\|_\Psi \right\|_\Psi : \|g\|_\Psi \leq 1\} < \infty, \tag{13}$$

where $H^g_y(x) := H^g(x,y)$ for all $x, y \in K$, and

$$\|H^g_{(\cdot)}\|_\Psi(y) := \|H^g_y\|_\Psi, \quad (y \in K).$$

Then, $L^\Phi_w(K)$ is a convolution Banach algebra.

Proof. If we put

$$M := \sup\{\left\| \left\| H^g_{(\cdot)} \right\|_\Psi \right\|_\Psi : \|g\|_\Psi \leq 1\},$$

then for each $g \in L^\Psi(K)$ and $v \in \Gamma_\Phi$,

$$\int_K \|H^g_y\|_\Psi \, |v(y)| \, dm(y) \leq M\|g\|_\Psi.$$

This implies that for almost every $y \in K$, $H^g_y \in L^\Psi(K)$. By the duality relation (4),

$$\|H^g_y\|_\Psi = \sup_{\|f\|_\Phi \leq 1} \left| \int H^g_y(x) f(x) \, dm(x) \right|.$$

Hence, for each $v \in \Gamma_\Phi$ we have

$$\int_K \sup_{\|f\|_\Phi \leq 1} \left| \int_K H^g(x,y) f(x) \, dm(x) \right| \, |v(y)| \, dm(y) \leq M\|g\|_\Psi.$$

Thus,

$$\int_K \left| \int_K \frac{(gw)(y * x)}{w(x)w(y)} f(x) \, dm(x) \right| \, |v(y)| \, dm(y) \leq M\|f\|_\Phi \|g\|_\Psi$$

for all $f \in L^\Phi(K)$. Now, applying the relation (2), for each $v \in \Gamma_\Phi$ we have

$$\begin{aligned}
\int_K \left| \int_K \frac{g(x)}{w(y)} \, (\frac{f}{w})(y^- * x) w(x) \, dm(x) \right| |v(y)| \, dm(y) \\
&= \int_K \frac{1}{w(y)} \left| \int_K (gw)(x) \, (\frac{f}{w})(y^- * x) \, dm(x) \right| \, |v(y)| \, dm(y) \\
&= \int_K \frac{1}{w(y)} \left| \int_K (gw)(y * x) \, (\frac{f}{w})(x) \, dm(x) \right| \, |v(y)| \, dm(y) \\
&= \int_K \left| \int_K \frac{(gw)(y * x)}{w(x) w(y)} f(x) \, dm(x) \right| \, |v(y)| \, dm(y) \\
&\leq M \|f\|_\Phi \|g\|_\Psi .
\end{aligned}$$

So,

$$\left\| \Lambda_w \left(g, L_{(\cdot)}(\frac{f}{w}) \right) \right\|_{\Psi, w^{-1}} \leq c \|f\|_\Phi \|g\|_\Psi$$

and, thanks to Corollary 3.3, this inequality completes the proof. ☑

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