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# A Glivenko-Cantelli Bootstrap Theorem for the Foster-Greer-Thorbecke Poverty Index

## Un Teorema Glivenko-Cantelli Bootstrap para la Medida de Pobreza de Foster-Greer-Thorbecke

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Abstract. We assume the Foster-Greer-Thorbecke (FGT) poverty index as an empirical process indexed by a particular Glivenko-Cantelli class or collection of functions and define this poverty index as a functional empirical process of the bootstrap type, to show that the outer almost sure convergence of the FGT empirical process is a necessary and sufficient condition for the outer almost sure convergence of the FGT bootstrap empirical process; that is: both processes are asymptotically equivalent respect to this type of convergence.

*Key words and phrases.* Foster-Greer-Thorbecke poverty index, convergence of empirical processes, Glivenko-Cantelli classes, bootstrap empirical processes.

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Resumen. Asumimos el indicador de pobreza de Foster-Greer-Thorbecke (FGT) como un proceso empírico indexado por una particular clase o colección de funciones Glivenko-Cantelli y definimos este indicador de pobreza como un proceso empírico funcional del tipo *bootstrap*, para probar que la convergencia casi segura exterior del proceso empírico FGT es una condición necesaria y suficiente para la convergencia casi segura exterior del proceso empírico bootstrap FGT; esto es: ambos procesos son asintóticamente equivalentes respecto de este tipo de convergencia.

*Palabras y frases clave.* Indicador de pobreza de Foster-Greer-Thorbecke, convergencia de procesos empíricos, clases Glivenko-Cantelli, procesos empíricos bootstrap.

## 1. Introduction

The problem of estimating one-dimensional poverty measures is theoretically addressed in this paper, developing a *characterization in law of large numbers*, in the framework of *bootstrap empirical processes*. To achieve this goal, first we introduce some basics elements: let $N$ be a *statistical universe of individuals* (let us say households), such that for each one of them it is possible to determine its *level of income* following *e.g.* [12, 17], for a random sample of $n$ individuals withdrawn from this population, a *measure or classic index of poverty* is a function $\mathcal{P} : \mathbb{R}_+^{n+1} \to [0,1]$, where the value of $\mathcal{P}(y,z)$ indicates the *degree or level of poverty* associated with the *vector of incomes* $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$ and the *fixed poverty line* $z \in \mathbb{R}_+$, such that any $j$-th individual of the random sample is considered *poor* if $y_j < z$.

This type of measures is commonly denominated *one-dimensional poverty indices* because in their construction only one *economic dimension* is considered. With the research published by Sen in 1976 about the *first axioms or properties of the axiomatic method of poverty* (see [18]), the idea of studying this problem as a phenomenon that depends only on the income acquires greater *mathematical rigor* within the *economic theory*, and various measures of poverty begin to be proposed, all of which are supported in the *Sen's axiomatic definition*. In this approach, one of the most important measures is the Foster-Greer-Thorbecke (FGT) poverty index (1984, [7]):

$$FGT(y,z,\alpha) = \frac{1}{n}\sum_{j=1}^{q}\left(\frac{z-y_j}{z}\right)^{\alpha}, \tag{1}$$

that emphasizes the *degree of aversion to poverty* by including the parameter $\alpha \geq 0$, where according with [12], [17] and [22] among others, the sum in (1) is only over $q$: *the number of poor individuals for the random sample*[1].

On the other hand, the *bootstrap technique* was introduced by Efron in 1979 and 1982 [5, 6], as a method to estimate the sample distribution of a *statistics*. In general, let $Y_1, Y_2, \ldots, Y_n$ be a *finite collection of i.i.d. random variables with law of probability* $\mathbb{P}$, if $\theta := \theta(\mathbb{P})$ is a *parameter of interest*, $\theta_n := \theta_n(Y_1, Y_2, \ldots Y_n; \mathbb{P})$ an *estimator of* $\theta$, and $\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n$ an *i.i.d. random sample with replacement of the empirical probability measure* $\mathbb{P}_n$. Then, the "bootstrap principle" consists in estimating the unknown distribution of $\theta_n$ through $\hat{\theta}_n := \theta_n(\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n; \mathbb{P}_n)$.

In 2009, Lo and Seck found that the FGT poverty index defined in (1) understood as an empirical process satisfies a very particular law of large numbers (see [14]). Now, we found that it is possible to establish an important

[1]For example, if $\alpha = 0$ the index is interpreted as the *incidence of poverty*, while for $\alpha = 1$ and $\alpha = 2$ is interpreted as the *intensity or severity of poverty* and the *depth or inequality among the poor*, respectively. For a detailed discussion about the axiomatic method and all the one-dimensional poverty indices proposed in the literature, see *e.g.* [22].

convergence relationship between the FGT empirical process of Lo and Seck and another one of the bootstrap type defined below. The statements of our main result presented in the paper are inspired (among others) in the theorems 3.3 of [21] and 10.15 of [13], that succinctly tell the reader: *if one wants to obtain a uniform bootstrap approximation one should check if a certain class is Glivenko-Cantelli*. Indeed, the theoretical proposal presented here is a particular contribution over the literature: it formally states that under certain conditions, the FGT empirical process considered as an *estimator of the average poverty level* (statistics) converges almost surely to the *real and unknown average poverty level* (parameter) reflected in the mean function of the corresponding process, if and only if the FGT bootstrap empirical process considered as a *bootstrap estimator of the average poverty level* (bootstrap statistics) converges almost surely to the correspondent estimator, for a random sample of incomes statistically large and representative of a statistical universe of households.

The article is structured in four Sections, including this introduction. In Section 2, we present the problem statement. Consequently, Section 3 presents the main result, and finally, Section 4 contains all the tools required for its development.

## 2. The problem

Consider the *product probability space* $(\Omega, \Sigma, P) := (\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}})$. In this framework $\mathcal{X}^{\mathbb{N}}$ is the *sample space of all infinite-numerable sequences of incomes*, such that for any infinite-numerable sequence $\omega := (y_1, y_2, \ldots) \in \mathcal{X}^{\mathbb{N}}$, we can define a *function or coordinate projection* $Y : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ such that $Y(\omega) = y \in \mathcal{X}$, with *probability distribution function* $\mathbb{F}(z) = \mathbb{P}(Y \leq z)$ for $z \in \mathbb{R}_+$ fixed. Moreover, according with [3] and [9] among others, we can define a *finite collection of functions* $Y_1, Y_2, \ldots, Y_n$ *i.i.d.*$\sim \mathbb{P}$, so that for each $j \in \mathbb{N}$, $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ is the *j-th coordinate projection on* $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}})$, such that for all $\omega := (y_1, y_2, \ldots) \in \mathcal{X}^{\mathbb{N}}$, $Y_j(\omega) = y_j \in \mathcal{X}$, with *empirical distribution function*:

$$\mathbb{F}_n(z) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\{Y_j < z\} = \frac{\sharp\{Y_j < z : 1 \leq j \leq n\}}{n}, \tag{2}$$

for $z \in \mathbb{R}_+$ fixed, where $q = n\mathbb{F}_n(z)$. Particularly, the i.i.d. collection $\{Y_j\}_{j=1}^n$ is an *empirical process of* $n$ *random variables*, where each projection $Y_j$ on the product probability space $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}})$ represents the *observed level of income for the* $j$*-th statistical individual of the random sample of size* $n$ in the probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$.

Let $\mathbb{P}_n : \mathcal{X}^{\mathbb{N}} \times \mathcal{A} \to [0,1]$ be the *empirical measure* associated with this sequence of random variables, where:

$$\mathbb{P}_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{Y_j}. \tag{3}$$

Lo and Seck (2009, [14]) define the *class or collection of functions* $\mathcal{F}_\Gamma := \{f_\alpha, \alpha \geq 0\}$:

$$f_\alpha(x) = \left|\frac{z-x}{z}\right|^\alpha \cdot \mathbb{I}\{x < z\}, \tag{4}$$

where the second term of (4) is an indicator function. In this setting, following the criterion frequently used in *probability theory*, if we "omit" the dependence on $\omega$ in the next notation, then (with some abuse of notation) for each $\omega := (y_1, y_2, \ldots) \in \mathcal{X}^{\mathbb{N}}$ fixed as infinite-numerable sequence of sample points, are obtained the *trajectories or realizations*:

$$\begin{aligned} f_\alpha \mapsto \mathbb{P}_n(f_\alpha) &= \frac{1}{n}\sum_{j=1}^{n} f_\alpha(Y_j) = \frac{1}{n}\sum_{j=1}^{n}\left|\frac{z-Y_j}{z}\right|^\alpha \cdot \mathbb{I}\{Y_j < z\} \\ &= \int_{\mathcal{X}} f_\alpha(y_j) d\mathbb{P}_n(y_j) = \int_0^z \left|\frac{z-y_j}{z}\right|^\alpha d\mathbb{F}_n(y_j) = \mathbb{E}_{\mathbb{F}_n}(f_\alpha(Y_j)), \end{aligned} \tag{5}$$

which allows to describe the FGT poverty index defined in (1) like the *functional or $\mathcal{F}_\Gamma$-indexed empirical process* $\{\mathbb{P}_n(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$, with $\mathcal{F}_\Gamma \subset L_1(\mathcal{X}, \mathcal{A}, \mathbb{P})$ and the compositions $f_\alpha(Y_j) \equiv f_\alpha \circ Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X} \to \mathbb{R}$, for all $\alpha \geq 0$, $j = 1, 2, \ldots, n$.

Let

$$\hat{\mathbb{P}}_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{\hat{Y}_j} \tag{6}$$

be the *classical Efron nonparametric bootstrap empirical measure*, where it is possible to consider *n bootstrap samples* $\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n$ of a determined collection of i.i.d.$\sim \mathbb{P}$ functions or coordinate projections $Y_1, Y_2, \ldots, Y_n$. Following *e.g.* [2, 15, 21], we can consider a *triangular array of exchangeable random variables* $\mathbf{W} := \{W_{nj} : j = 1, 2, \ldots, n, n = 1, 2, \ldots\}$ defined on $(\mathcal{W}, \mathcal{D}, \mathbb{P}_W)$, such that these random variables can be interpreted as *random weights*, in the sense that each component $W_{nj}$ reflects the number of times that $Y_j$ is selected for the $n$ trials of the *bootstrap sample with replacement*, where:

$$\hat{\mathbb{P}}_n^W := \frac{1}{n}\sum_{j=1}^{n} W_{nj}\delta_{Y_j}, \tag{7}$$

is just the *exchangeably weighted bootstrap empirical measure*, such that the classical measure $\hat{\mathbb{P}}_n$ defined above is a special case of $\hat{\mathbb{P}}_n^W$ obtained by taking $(W_{n1}, W_{n2}, \ldots, W_{nn})' = \underline{W}_n = \underline{M}_n$, with $\underline{M}_n = (M_{n1}, M_{n2}, \ldots, M_{nn})' \sim Mult_n(n, (1/n, 1/n, \ldots, 1/n))$. Consequently, we can define the *functional or $\mathcal{F}_\Gamma$-indexed* FGT *bootstrap empirical process* $\{\hat{\mathbb{P}}_n^W(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma, \alpha \geq 0\}$[2].

[2]We consider a *exchangeably weighted version of the bootstrap* in this paper, because under the hypothesis of "exchangeability", we can "emulate" the Strobl's lemma 4.9 of Section 4 for

In [14], Lo and Seck show that the class $\mathcal{F}_\Gamma$ is *strong* $\mathbb{P}$*-Glivenko-Cantelli*; that is:

$$\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} = \sup_{f_\alpha \in \mathcal{F}_\Gamma} |\mathbb{P}_n(f_\alpha) - \mathbb{P}(f_\alpha)| \xrightarrow{a.s.^*} 0, \tag{8}$$

as $n \to \infty$, where

$$\mathbb{P}(f_\alpha) = \int_{\mathcal{X}} f_\alpha(y) d\mathbb{P}_Y(y) = \int_0^z \left|\frac{z-y}{z}\right|^\alpha d\mathbb{F}(y) = \mathbb{E}_{\mathbb{F}}(f_\alpha(Y)), \tag{9}$$

is the correspondent *mean function* of the FGT empirical process $\mathbb{P}_n(f_\alpha)$ defined above. Now, we find that under the G.-C. hypothesis, the trajectories or realizations of $\mathbb{P}_n^W$ get uniformly closer to $\mathbb{P}_n$ as $n \to \infty$, and that the reciprocal is also true.

## 3. The Main Result

Specifically, we suppose in this paper that $\underline{W}_n = (W_{n1}, W_{n2}, \ldots, W_{nn})'$ satisfies the following conditions:

**A1.** $\underline{W}_n = (W_{n1}, W_{n2}, \ldots, W_{nn})'$ is *exchangeable* for all $n = 1, 2, \ldots$, that is, for any permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ of $(1,2,\ldots,n)$, the joint distribution of $\pi(\underline{W}_n) = (W_{n\pi(1)}, W_{n\pi(2)}, \ldots, W_{n\pi(n)})'$ is the same as that of $\underline{W}_n$.

**A2.** $W_{nj} \geq 0$ for all $n, j$, and $\sum_{j=1}^n W_{nj} = n$, for all $n$.

Using the last condition like in [21], page 598:

$$\hat{\mathbb{P}}_n^W - \mathbb{P}_n = \frac{1}{n}\sum_{j=1}^n (W_{nj} - 1)(\delta_{Y_j} - \mathbb{P}) =: \frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j. \tag{10}$$

That is, we have a *multiplier process* with $\xi_{nj} := W_{nj} - 1$ and $Z_j := \delta_{Y_j} - \mathbb{P}$, respectively, such that for any $f_\alpha \in \mathcal{F}_\Gamma$: $f_\alpha \mapsto (\hat{\mathbb{P}}_n^W - \mathbb{P}_n)(f_\alpha) = \frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j(f_\alpha)$, with $Z_j(f_\alpha) = f_\alpha(Y_j) - \mathbb{P}(f_\alpha)$. Particularly, we redefine **A1** and consider a couple of additional conditions for the weights $\xi_{nj}$:

**B1.** $\underline{\xi}_n = (\xi_{n1}, \xi_{n2}, \ldots, \xi_{nn})'$ is *exchangeable* for all $n$.

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the correspondent bootstrap processes in our main result, respect to the $\Sigma_n$-measurability and the backward submartingale property required in the Strobl's result, with $\Sigma_n$ being the filtration defined in remark 4.7 of this Section. Additionally, following *e.g.* [21], our main result developed in Section 3 will allow to present in future papers, at least one Glivenko-Cantelli theorem for the classical nonparametric bootstrap empirical measure $\hat{\mathbb{P}}_n := \frac{1}{n}\sum_{j=1}^n M_{nj}\delta_{Y_j}$ as a direct consequence of this main result (see *e.g.* Theorem 3.3 for the exchangeable bootstrap and Theorem 3.2 for Efron's bootstrap, that follows as a corollary of the first mentioned here in [21]).

**B2.** The norm $L_{2,1}$ of $\xi_{n1}$ is *finite*; that is,

$$\|\xi_{n1}\|_{2,1} = \int_0^{\infty} \sqrt{\mathbb{P}_W(|\xi_{n1}| \geq t)}dt \leq k < \infty.$$

**B3.** $\xi_{n1}$ satisfies the *weak second-moment condition*:

$$\lim_{t\to\infty} \limsup_{n\to\infty} t^2 \mathbb{P}_W(|\xi_{n1}| \geq t) = 0.$$

**Theorem 3.1.** *Let $\{Z_j\}_{j=1}^n$ be an i.i.d. empirical process, where $Z_j := \delta_{Y_j} - \mathbb{P}$, with $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ i.i.d. $\sim \mathbb{P}$. Let $\{\xi_{nj}\}_{j=1}^n$ be an i.i.d. collection of random weights independent of the collection $\{Z_j\}_{j=1}^n$, with mean $\mathbb{E}(\xi_{nj}) = \mu$ and that satisfies* **B1-B3***, such that $\xi_{nj} := W_{nj} - 1$ with $\{W_{nj}\}_{j=1}^n$ that satisfies* **A1-A2**. *Let the class or collection $\mathcal{F}_\Gamma := \{f_\alpha, \alpha \geq 0\}$ with measurable cover function $F^* : \mathcal{X} \to \mathbb{R}$ defined by $F^* := (\|f_\alpha\|_{\mathcal{F}_\Gamma})^* \in L_1(\mathcal{X}, \mathcal{A}, \mathbb{P})$ and*

$$f_\alpha(y_j) = \left|\frac{z - y_j}{z}\right|^\alpha \cdot \mathbb{I}\{y_j < z\},$$

*for all $y_j \in \mathcal{X}$ and the fixed poverty line $z \in \mathbb{R}_+$. Then, the following are equivalent:*

*(i) $\mathcal{F}_\Gamma$ is strong $\mathbb{P}$-Glivenko-Cantelli.*

*(ii) $\|\hat{\mathbb{P}}_n^W - \mathbb{P}_n\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} 0$, as $n \to \infty$.*

***Proof***. (i)⇒(ii): We know that $\hat{\mathbb{P}}_n^W - \mathbb{P}_n = \frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j$, by (10). It follows:

$$\mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma} \leq 2n_0 \mathbb{E}^*\|Z_1\|_{\mathcal{F}_\Gamma} \cdot \mathbb{E}\left(\max_{1\leq j\leq n} \frac{|\xi_{nj}|}{n}\right)$$
$$+ 4\frac{\|\xi_{n1}\|_{2,1}}{\sqrt{n}} \cdot \max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^k \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\},$$

for any $1 \leq n_0 < n$. This is the right side in the *lemma of inequalities for the bootstrap process* (Lemma 4.12). We point out a few properties related to this upper bound: (a) Since $\mathcal{F}_\Gamma$ is strong $\mathbb{P}$-Glivenko-Cantelli by hypothesis; *i.e.*, $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} 0$, as $n \to \infty$, then this implies $\mathbb{E}^*\|Z_1\|_{\mathcal{F}_\Gamma} = \mathbb{P}^*\|f_\alpha(Y_1) - \mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma} < \infty$, by the *lemma of the necessity of integrability of the envelope function* (Lemma 4.6). (b) $\mathbb{E}\left(\max_{1\leq j\leq n} \frac{|\xi_{nj}|}{n}\right) \to 0$, follows from the *lemma of convergence in mean for the maximum of the exchangeable weights* (lemma 4.13), under the conditions **B2** and **B3**. (c) $\|\xi_{n1}\|_{2,1} < \infty$, by **B2**. (d) Let $F^*$

the *measurable cover function* of $\mathcal{F}_\Gamma$ defined above, by the inequality (19) of the *Strobl's theorem for backward submartingales* (Lemma 4.9), it follows that

$$\begin{aligned}\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma} &= \left(\sup_{f_\alpha\in\mathcal{F}_\Gamma}\left|\frac{1}{n}\sum_{j=1}^{n} f_\alpha(Y_j)\right|\right)^* \\ &\le \frac{1}{n}\sum_{j=1}^{n} F^*(Y_j) \\ &= \mathbb{P}_n(F^*) < \infty.\end{aligned}$$

Emulating the Strobl's theorem for $\{\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}\}_{n\in\mathbb{N}}$, we can conclude that this process is a *backward submartingale* respect to $\{\Sigma_n\}_{n\in\mathbb{N}}$, the $\sigma$-algebra defined in Remark 4.7. Since $\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}\cdot\mathbb{I}\{\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma} > M\} \le \mathbb{P}_n(F^*)\cdot\mathbb{I}\{\mathbb{P}_n(F^*) > M\}$ for $M > 0$, then $\lim_{M\to\infty}\mathbb{E}(\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}\cdot\mathbb{I}\{\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma} > M\}) = 0$; that is, $\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}$ is *uniformly integrable*, and consequently $L_1$*-bounded*. By the *Doob's theorem of convergence* (Lemma 4.2), it follows that $\|\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}$ converges almost surely to a finite limit, and by the G.-C. hypothesis and the continuity of the uniform norm, $\|\mathbb{P}_n\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} \|\mathbb{P}\|_{\mathcal{F}_\Gamma}$. Then, applying the *theorem of convergence in mean for backward submartingales* (Lemma 4.3), $\mathbb{E}^*\|\mathbb{P}_n-\mathbb{P}\|_{\mathcal{F}_\Gamma}\to 0$, as $n\to\infty$, and by the lower bound of (24) in the *lemma of Rademacher symmetrization* (Lemma 4.10), we have:

$$\lim_{n\to\infty}\mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^{n}\epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma} = 0,$$

and this concludes (d). Therefore, by (a)-(b)-(c)-(d):

$$\lim_{n\to\infty}\mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma} = 0, \tag{11}$$

that is, $\mathbb{E}^*\|\hat{\mathbb{P}}^W_n-\mathbb{P}_n\|_{\mathcal{F}_\Gamma}\to 0$, as $n\to\infty$. This implies that $\|\hat{\mathbb{P}}^W_n-\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}\xrightarrow{\mathbb{P}} 0$, because the *convergence in outer mean* implies *convergence in outer probability*.

Let $\dot{F}^* := (\|f_\alpha-\mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma})^* \in L_1\ (\mathcal{X},\mathcal{A},\mathbb{P})$ the *measurable cover function* of the class $\dot{\mathcal{F}}_\Gamma := \{f_\alpha-\mathbb{P}(f_\alpha) : f_\alpha\in\mathcal{F}_\Gamma\}$, then for $f_\alpha\in\mathcal{F}_\Gamma$ fixed:

$$
\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}[f_\alpha-\mathbb{P}(f_\alpha)]\right| &\leq \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}[f_\alpha-\mathbb{P}(f_\alpha)]|\\
&= \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}|\cdot|f_\alpha-\mathbb{P}(f_\alpha)|\\
&\leq \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}|\cdot\|g_\alpha-\mathbb{P}(g_\alpha)\|^*_{\mathcal{F}_\Gamma},\\
\Rightarrow \left|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}[f_\alpha-\mathbb{P}(f_\alpha)]\right| &\leq \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}|\cdot\|g_\alpha-\mathbb{P}(g_\alpha)\|^*_{\mathcal{F}_\Gamma}\\
&= \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}|\cdot\|f_\alpha-\mathbb{P}(f_\alpha)\|^*_{\mathcal{F}_\Gamma}, \quad \text{for all } f_\alpha\in\mathcal{F}_\Gamma\\
\Rightarrow \left\|\frac{1}{n}\sum_{j=1}^{n}\xi_{nj}[f_\alpha-\mathbb{P}(f_\alpha)]\right\|^*_{\mathcal{F}_\Gamma} &\leq \frac{1}{n}\sum_{j=1}^{n}|\xi_{nj}|\cdot\dot{F}^*, \qquad (12)
\end{aligned}
$$

where the right side of (12) is integrable by remark 4.5. Using this fact and the condition of exchangeability **B1** for the random weights $\xi_{nj}$, we can emulate again the Strobl's theorem to see that $\{\|\hat{\mathbb{P}}_n^W-\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma},\Sigma_n\}_{n\in\mathbb{N}}$ is a *backward submartingale*. By inequality (12), it is clear that $\sup_{n\in\mathbb{N}}\mathbb{E}(\|\hat{\mathbb{P}}_n^W-\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma})<\infty$, and from Doob's theorem applied above, it follows that $\|\hat{\mathbb{P}}_n^W-\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}$ converges almost surely to a finite limit, but $\|\hat{\mathbb{P}}_n^W-\mathbb{P}_n\|^*_{\mathcal{F}_\Gamma}\xrightarrow{\mathbb{P}}0$, thus this limit must be equal to zero and then $\|\hat{\mathbb{P}}_n^W-\mathbb{P}_n\|_{\mathcal{F}_\Gamma}\xrightarrow{a.s.^*}0$.

(ii)⇒(i): Let $F^*:=(\|f_\alpha\|_{\mathcal{F}_\Gamma})^*\in L_1(\mathcal{X},\mathcal{A},\mathbb{P})$,

$$\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}=(\sup_{f_\alpha\in\mathcal{F}_\Gamma}|\hat{\mathbb{P}}_n^W(f_\alpha)|)^*\leq\hat{\mathbb{P}}_n^W(F^*)<\infty.$$

Under the condition of exchangeability **A1**, it is clear that $\{\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma},\Sigma_n\}_{n\in\mathbb{N}}$ is a *backward submartingale* too by Strobl's result. Since

$$\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}\cdot\mathbb{I}\{\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}>M\}\leq\hat{\mathbb{P}}_n^W(F^*)\cdot\mathbb{I}\{\hat{\mathbb{P}}_n^W(F^*)>M\}$$

for $M>0$, then $\lim_{M\to\infty}\mathbb{E}(\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}\cdot\mathbb{I}\{\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}>M\})=0$; that is, $\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}$ is *uniformly integrable*, and in fact $L_1$-*bounded*. Applying again the Doob's theorem 4.2, $\|\hat{\mathbb{P}}_n^W\|^*_{\mathcal{F}_\Gamma}$ converges almost surely to a finite limit, and by the hypothesis (ii) and the continuity of $\|\cdot\|_{\mathcal{F}_\Gamma}$, we have $\|\hat{\mathbb{P}}_n^W\|_{\mathcal{F}_\Gamma}\xrightarrow{a.s.^*}\|\mathbb{P}_n\|_{\mathcal{F}_\Gamma}$, and therefore, by the Theorem 4.3 (theorem of convergence in mean 4.3), it follows $\mathbb{E}^*\|\hat{\mathbb{P}}_n^W - \mathbb{P}_n\|_{\mathcal{F}_\Gamma} \to 0$, when $n \to \infty$. Since

$$\frac{1}{2}\|\xi_{n1} - \mu\|_1 \cdot \mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^{n} \epsilon_j Z_j \right\|_{\mathcal{F}_\Gamma} \leq \mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^{n} \xi_{nj} Z_j \right\|_{\mathcal{F}_\Gamma},$$

is the left side in (26) of the Lemma 4.12 of inequalities for the bootstrap process, where $\|\xi_{n1} - \mu\|_1 < \infty$ under **B2** by Remark 4.11, it is clear that

$$\lim_{n\to\infty} \mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^{n} \epsilon_j Z_j \right\|_{\mathcal{F}_\Gamma} = 0.$$

From the upper bound of (24) in the Lemma 4.10 of Rademacher symmetrization, it follows:

$$\lim_{n\to\infty} \mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^{n} Z_j \right\|_{\mathcal{F}_\Gamma} = 0, \tag{13}$$

that is, $\mathbb{E}^*\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \to 0$, when $n \to \infty$. This implies $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \xrightarrow{\mathbb{P}} 0$ using the convergence argument discussed above, we can conclude that $\mathcal{F}_\Gamma$ is *weak* $\mathbb{P}$-*Glivenko-Cantelli.*

To finish the proof, the process $\{\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}, \Sigma_n\}_{n\in\mathbb{N}}$ is a *backward submartingale* by Theorem 4.9 (Strobl's theorem) and $\mathbb{E}(\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}) \leq 2\mathbb{P}(F^*) < \infty$, by inequality (20), for each $n \in \mathbb{N}$. Consequently, $\sup_{n\in\mathbb{N}} \mathbb{E}(\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}) < \infty$, and from Doob's theorem, it follows that $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}$ converges almost surely to a finite limit, but $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \xrightarrow{\mathbb{P}} 0$, thus this limit must be equal to zero, and then $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} 0$. ☑

***Note* 1.** Following page 15 in [8], if we assume measurability for the collection $\{Y_j\}_{j=1}^n$, then the index "$*$" can be removed in the notation, and all the results presented here also hold (see *e.g.* the definition 4.4 in the next Section). Lemma 4.6 is similar to Lemma 8.13, page 141 in [13], applied now to the classes of functions $\mathcal{F}_\Gamma$ and $\dot{\mathcal{F}}_\Gamma$, respectively. Lemma 4.9 is basically the original Theorem 1.1, pages 826–829 in [19], for the class $\mathcal{F}_\Gamma$. Lemma 4.10 corresponds to Lemma 11.2.12, pages 343–344 in [4], applied to the class $\mathcal{F}_\Gamma$. The Lemma 4.12 developed for this collection, is similar to Lemma 2.9.1, pages 177–179 in [20]; or Lemma 2.2, pages 595–596 in [21]. Lemma 4.13 is similar to Lemma 4.7, page 2071 in [15], considering the random weights $\xi_{nj}$. For details about all the proofs see [11], and for a detailed discussion about the bootstrap see *e.g.* [10].

## 4. Tools Required for the Main Result

**Definition 4.1.** Let $Y_n : \Omega \to \mathbb{R}$ be a sequence of random variables and $\{\Sigma_n\}_{n\in\mathbb{N}}$ be a *filtration* on $(\Omega, \Sigma, P)$; that is, a decrescent sequence of sub-$\sigma$-algebras of $\Sigma$,

$$\Sigma \supset \Sigma_n \supset \Sigma_{n+1},$$

for each $n \in \mathbb{N}$ such that:

(1) $\{Y_n\}_{n\in\mathbb{N}}$ is adapted to the filtration $\{\Sigma_n\}_{n\in\mathbb{N}}$ (or more generally, adapted to the $P$-completion of $\{\Sigma_n\}_{n\in\mathbb{N}}$). That is, $Y_n$ is $\Sigma_n$-measurable for each $n \in \mathbb{N}$ (or more generally, measurable for the $P$-completion of $\{\Sigma_n\}_{n\in\mathbb{N}}$);

(2) $\mathbb{E}(|Y_n|) < \infty$, for each $n \in \mathbb{N}$;

(3) $\mathbb{E}(Y_{n+1}|\Sigma_n) \geq Y_{n+1}$ a.s., for each $n \in \mathbb{N}$.

Then $\{Y_n, \Sigma_n\}_{n\in\mathbb{N}}$ is said to be a *reversed or backward submartingale.*

**Lemma 4.2** (**Doob's theorem of convergence**). *Let* $\{Y_n\}_{n\in\mathbb{N}}$ *be a backward submartingale on* $(\Omega, \Sigma, P)$ *such that* $\sup_{n\in\mathbb{N}} \mathbb{E}(|Y_n|) < \infty$. *Then there is an integrable random variable* $Y$ *such that:*

$$\lim_{n\to\infty} Y_n = Y \ a.s. \tag{14}$$

***Proof.*** See [1], chapter 13, section 13.3, pages 417–419; or [16], chapter 7, section 7.9, pages 219–224. ☑

**Lemma 4.3** (**Theorem of convergence in mean**). *Let* $\{Y_n\}_{n\in\mathbb{N}}$ *be a backward submartingale on* $(\Omega, \Sigma, P)$ *uniformly integrable; that is,*

$$\lim_{M\to\infty} \mathbb{E}(|Y_n| \cdot \mathbb{I}\{|Y_n| > M\}) = \lim_{M\to\infty} \int_{(|Y_n|>M)} |Y_n| dP = 0.$$

*Then this process is convergent in mean; that is:*

$$\lim_{n\to\infty} \mathbb{E}(|Y_n - Y|) = 0. \tag{15}$$

***Proof.*** See [1], Theorem 13.3.5, page 420; or [16], Proposition 7.7, pages 227–228. ☑

**Definition 4.4.** Let $f : \Omega \to \overline{\mathbb{R}}$ be an arbitrary function not necessarily measurable on a probability space $(\Omega, \Sigma, P)$, where $\overline{\mathbb{R}} \equiv [-\infty, \infty]$ is the set of extended real numbers. The *outer expectation or outer integral of* $f$ *with respect to* $P$, is defined as:

$$\mathbb{E}^*_P(f) := \int^* f dP = \inf\{\mathbb{E}_P(g) : g \geq f,\ g : \Omega \to \overline{\mathbb{R}} \text{ measurable and } \mathbb{E}_P(g) \text{ exists}\}, \tag{16}$$

where $\mathbb{E}_P(g) = \int g dP = \int g^+ dP + \int g^- dP = \mathbb{E}_P(g^+) + \mathbb{E}_P(g^-)$, such that $\mathbb{E}_P(g)$ exist, if at least $\mathbb{E}_P(g^+)$ or $\mathbb{E}_P(g^-)$ is finite.

If $f$ is a *measurable function quasi-integrable*, then $\mathbb{E}^*_P(f) = \mathbb{E}_P(f)$. Moreover, if $\mathbb{E}^*_P(f) < \infty$, then $\mathbb{E}^*_P(f) = \mathbb{E}_P(f^*)$,where $f^* : \Omega \to \overline{\mathbb{R}}$ is the *minimal measurable majorant or smallest measurable function above* $f$, that satisfies:

(1) $f^*(\omega) \geq f(\omega)$ for each $\omega \in \Omega$;

(2) For any measurable function $g : \Omega \to \overline{\mathbb{R}}$ with $g \geq f$ a.s., $f^* \leq g$ a.s.

***Remark* 4.5.** Consider the class $\mathcal{F}_\Gamma := \{f_\alpha, \alpha \geq 0\}$, and let $F : \mathcal{X} \to \mathbb{R}$ be the *envelope function* of this collection, with $F := \|f_\alpha\|_{\mathcal{F}_\Gamma}$ such that $|f_\alpha(y)| \leq F(y) = \sup_{f_\alpha \in \mathcal{F}_\Gamma} |f_\alpha(y)|$ for each $y \in \mathcal{X}$ and $f_\alpha \in \mathcal{F}_\Gamma$. Now we can define the class $\dot{\mathcal{F}}_\Gamma := \{f_\alpha - \mathbb{P}(f_\alpha) : f_\alpha \in \mathcal{F}_\Gamma\}$, with $\dot{F} : \mathcal{X} \to \mathbb{R}$ defined by $\dot{F} := \|f_\alpha - \mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma}$, such that $|f_\alpha - \mathbb{P}(f_\alpha)| \leq \dot{F}$. According to the above definition, under integrability, the outer expectation of an envelope function for a determined class or collection is equal to the expected value of the measurable cover function respect to this envelope. In other terms, if $\mathbb{P}^*(F) < \infty$ and $\mathbb{P}^*(\dot{F}) < \infty$, then is true that $\mathbb{P}^*(F) = \mathbb{P}(F^*)$ and $\mathbb{P}^*(\dot{F}) = \mathbb{P}(\dot{F}^*)$, where $F^* := (\|f_\alpha\|_{\mathcal{F}_\Gamma})^*$ and $\dot{F}^* := (\|f_\alpha - \mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma})^*$ are the *measurable cover functions* for $F$ and $\dot{F}$ of the classes $\mathcal{F}_\Gamma$ and $\dot{\mathcal{F}}_\Gamma$, respectively. In this setting, $F \leq F^*$, $F^*$ is measurable, and $F^* \leq h$ $P$-a.s. for all measurable function $h \geq F$. The same analysis follows for $\dot{F}$ and $\dot{F}^*$.

**Lemma 4.6** (**Necessity of integrability of the envelope function**). *If the class of functions $\mathcal{F}_\Gamma$ is strong $\mathbb{P}$-Glivenko-Cantelli; i.e., $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \xrightarrow{a.s.^*} 0$, as $n \to \infty$. Then this implies that $\mathbb{P}^*\|f_\alpha - \mathbb{P}(f_\alpha)\|_{\mathcal{F}_\Gamma} < \infty$. Consequently, if $\mathcal{F}_\Gamma$ is $L_1(\mathbb{P})$-bounded; i.e., $\|\mathbb{P}\|_{\mathcal{F}_\Gamma} = \sup_{f_\alpha \in \mathcal{F}_\Gamma} |\mathbb{P}(f_\alpha)| < \infty$, then $\mathbb{P}^*(F) < \infty$ for an envelope function $F$.*

***Proof*.** See [11], Lemma 3.1, pages 83–84. ☑

***Remark* 4.7.** Let $\mathbb{P}_n$ be the $n$-th empirical measure on $(\Omega, \Sigma, P) := (\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}})$, it follows that the class $\mathcal{C}$ of all sets invariant under permutations of the $n$ first coordinates $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$ is a $\sigma$-algebra[3]. Specifically, we can define $\Sigma_n$ as the smallest $\sigma$-algebra that contains all the sets

$$\{A \in \Sigma : \mathbb{I}_A(y) = \mathbb{I}_A(\pi y)\};$$

that is, invariants under any permutation $\pi \in S(n)$ of the first $n$ coordinates $Y_j : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}$, such that $\Sigma_n \supset \Sigma_{n+1}$, for each $n \in \mathbb{N}$.

[3] See *e.g.* [3], Lemma A.2.8, pages 127–128. In general, for a non empty set $X$, a *permutation of* $X$ is a bijection $f_\pi : X \to X$, such that if $(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) \in X$, then any permutation $\pi$ of the first $n$ terms of this sequence; that is, $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1}, x_{n+2}, \ldots)$ also belongs to $X$, and it is part of the *symmetric group of* $n$ denoted by $S(n)$, where:

$$S(n) := \{\pi : \pi(j) = j^{'} \text{ if and only if } \pi(j^{'}) = j, \text{ for all } j, j^{'} \leq n \in \mathbb{N}\}.$$

***Remark* 4.8.** Let $A \in \Sigma_n$. For $1 \leq j < j^{'} \leq n$,

$$\mathbb{E}[f_\alpha(Y_j)\big|\Sigma_n] = \mathbb{E}[f_\alpha(Y_j^{'})\big|\Sigma_n]. \tag{17}$$

To see this, consider the two sides of (17) are $\Sigma_n$-measurable by properties of the conditional expectation. Now, for any $A \in \Sigma_n$, $\pi \in S(n)$ the symmetric group of order $n$:

$$(y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots) \in A \Leftrightarrow (y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)}, y_{n+1}, y_{n+2}, \ldots) \in A.$$

Hence, if $y \in Y_j(A)$, then for some $\{z_m\} \in A : z_j = y$, but now we have $(z_{\pi(1)}, z_{\pi(2)}, \ldots, z_{\pi(n)}, z_{n+1}, z_{n+2}, \ldots) \in A$, where $\pi \in S(n) : \pi(j^{'}) = j$, so

$$Y_{j'}((z_{\pi(1)}, z_{\pi(2)}, \ldots, z_{\pi(n)}, z_{n+1}, z_{n+2}, \ldots)) = z_{\pi(j')} = z_j = y \Rightarrow y \in Y_{j'}(A).$$

Applying the same argument as above, it is clear that if $y \in Y_{j'}(A)$, then $y \in Y_j(A)$. Therefore $Y_j(A) = Y_{j'}(A)$, and it follows that:

$$\begin{aligned}
\mathbb{E}[f_\alpha(Y_j)\cdot \mathbb{I}_C] &= \int_\Omega f_\alpha(Y_j(\{y_m\}_{m\in\mathbb{N}}))\cdot \mathbb{I}_A(\{y_m\}_{m\in\mathbb{N}}) d\mathbb{P}^{\mathbb{N}}(\{y_m\}_{m\in\mathbb{N}}) \\
&= \int_A f_\alpha(Y_j(\{y_m\}_{m\in\mathbb{N}})) d\mathbb{P}^{\mathbb{N}}(\{y_m\}_{m\in\mathbb{N}}) \\
&= \int_{Y_j(A)} f_\alpha(y_j) d(\mathbb{P}^{\mathbb{N}} \circ Y_j^{-1})(y_j) = \int_{Y_j(A)} f_\alpha(y) d\mathbb{P}(y) \\
&= \int_{Y_{j'}(A)} f_\alpha(y) d\mathbb{P}(y) = \int_{Y_{j'}(A)} f_\alpha(y_{j'}) d(\mathbb{P}^{\mathbb{N}} \circ Y_{j'}^{-1})(y_{j'}) \\
&= \int_A f_\alpha(Y_{j'}(\{y_m\}_{m\in\mathbb{N}})) d\mathbb{P}^{\mathbb{N}}(\{y_m\}_{m\in\mathbb{N}}) \\
&= \int_\Omega f_\alpha(Y_{j'}(\{y_m\}_{m\in\mathbb{N}}))\cdot \mathbb{I}_A(\{y_m\}_{m\in\mathbb{N}}) d\mathbb{P}^{\mathbb{N}}(\{y_m\}_{m\in\mathbb{N}}) \\
&= \mathbb{E}[f_\alpha(Y_{j'})\cdot \mathbb{I}_C].
\end{aligned}$$

**Lemma 4.9 (Strobl's theorem).** *Let* $(\Omega, \Sigma, P) := (\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}})$*, and consider the collection of functions* $\mathcal{F}_\Gamma \subset L_1(\mathcal{X}, \mathcal{A}, \mathbb{P})$*. If* $\mathcal{F}_\Gamma$ *has a measurable cover function* $F^* \in L_1(\mathcal{X}, \mathcal{A}, \mathbb{P})$*, then* $\{\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}, \Sigma_n\}_{n\in\mathbb{N}}$ *is a backward submartingale; that is,* $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}$ *is* $\Sigma_n$*-measurable,* $P$*-integrable, and*

$$\|\mathbb{P}_{n+1} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \leq \mathbb{E}(\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}\big|\Sigma_{n+1}) \ \ P\text{-a.s.} \tag{18}$$

*for each* $n \in \mathbb{N}$.

***Proof*.** First, we show the integrability of $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}$. Let

$$\begin{aligned}\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} &= \left( \sup_{f_\alpha \in \mathcal{F}_\Gamma} \left| \frac{1}{n} \sum_{j=1}^{n} f_\alpha(Y_j) - \int_{\mathcal{X}} f_\alpha(y) d\mathbb{P}_Y(y) \right| \right)^* \\ &\leq \frac{1}{n} \sum_{j=1}^{n} F^*(Y_j) + \int_{\mathcal{X}} F^*(y) d\mathbb{P}_Y(y) \\ &= \mathbb{P}_n(F^*) + \mathbb{P}(F^*). \end{aligned} \tag{19}$$

The right side of (19) is $\Sigma$-measurable, $P$-integrable and real-valued. Hence we have that $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \in L_1(\Omega, \Sigma, P)$ for each $n \in \mathbb{N}$. In fact, respect to the integrability of the outer empirical discrepancy, by (19),

$$\mathbb{E}(\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}) \leq \mathbb{E}[\mathbb{P}_n(F^*) + \mathbb{P}(F^*)] < 2\mathbb{P}(F^*) < \infty, \tag{20}$$

for each $n \in \mathbb{N}$.

Now, let $g := \|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} : \mathcal{X}^{\mathbb{N}} \to \mathbb{R}$. We define $f_\pi : \mathcal{X}^{\mathbb{N}} \to \mathcal{X}^{\mathbb{N}}$ for a permutation $\pi \in S(n)$ by

$$f_\pi(y_1, y_2, \ldots) := (y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(n)}, y_{n+1}, y_{n+2}, \ldots).$$

Since $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma}$ is invariant under all permutations of the first $n$ coordinates,

$$g \circ f_\pi \geq \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma} \circ f_\pi = \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma},$$

for all $\pi \in S(n)$ and then,

$$\min_{\pi \in S(n)} g \circ f_\pi \geq \|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}_\Gamma},$$

where the left side of the inequality is a $\Sigma$-measurable function. Therefore:

$$\min_{\pi \in S(n)} g \circ f_\pi \geq \|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} = g \;\; P\text{-a.s.},$$

by the definition of measurable cover functions. Thus,

$$\min_{\pi \in S(n)} g \circ f_\pi = g \;\; P\text{-a.s.}$$

For each rational $q \geq 0$, let $A_q := A_{q,n}$ the sets where $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \geq q$; i.e.

$$A_q := A_{q,n} := \{\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \geq q\},$$

we have that

$$\{g \geq q\} = \{\min_{\pi \in S(n)} g \circ f_\pi \geq q\}, \tag{21}$$

with $\{\min_{\pi \in S(n)} g \circ f_\pi \geq q\} \in \Sigma_n$. Consequently, $\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}$ is $\Sigma_n$-measurable.

Finally, for $j = 1, 2, \dots, n+1$, let

$$\mathbb{P}_{n,j} := \frac{1}{n} \sum_{j'=1, j' \neq j}^{n+1} \delta_{Y_{j'}},$$

then $\mathbb{P}_{n,n+1} \equiv \mathbb{P}_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{Y_j}$, and for $j \neq n+1$, $\mathbb{P}_{n,j}$ has the same properties as $\mathbb{P}_n$. Therefore,

$$\begin{aligned}
\|\mathbb{P}_{n+1} - \mathbb{P}\|_{\mathcal{F}_\Gamma} &= \left\| \frac{1}{n+1} \left( \sum_{l=1}^{n+1} \delta_{Y_l} \right) - \mathbb{P} \right\|_{\mathcal{F}_\Gamma} \\
&= \frac{1}{n+1} \left\| \frac{1}{n} \left( \sum_{l=1}^{n+1} n\delta_{Y_l} \right) - (n+1)\mathbb{P} \right\|_{\mathcal{F}_\Gamma} \\
&= \frac{1}{n+1} \left\| \left( \sum_{j=1}^{n+1} \mathbb{P}_{n,j} \right) - (n+1)\mathbb{P} \right\|_{\mathcal{F}_\Gamma} \\
&= \frac{1}{n+1} \left\| \sum_{j=1}^{n+1} \left( \frac{1}{n} \sum_{j'=1, j' \neq j}^{n+1} \delta_{Y_{j'}} - \mathbb{P} \right) \right\|_{\mathcal{F}_\Gamma} \\
&\leq \frac{1}{n+1} \sum_{j=1}^{n+1} \left\| \frac{1}{n} \sum_{j'=1, j' \neq j}^{n+1} \delta_{Y_{j'}} - \mathbb{P} \right\|_{\mathcal{F}_\Gamma} \\
&= \frac{1}{n+1} \sum_{j=1}^{n+1} \|\mathbb{P}_{n,j} - \mathbb{P}\|_{\mathcal{F}_\Gamma} \\
&\leq \frac{1}{n+1} \sum_{j=1}^{n+1} \|\mathbb{P}_{n,j} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}. \qquad (22)
\end{aligned}$$

The right side of (22) is $\Sigma$-measurable, so it is an upper bound of the outer empirical discrepancy $\|\mathbb{P}_{n+1} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma}$ $P$-a.s., too. Therefore,

$$\begin{aligned}
\|\mathbb{P}_{n+1} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} &= \mathbb{E}(\|\mathbb{P}_{n+1} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \big| \Sigma_{n+1}) \\
&\leq \frac{1}{n+1} \sum_{j=1}^{n+1} \mathbb{E}(\|\mathbb{P}_{n,j} - \mathbb{P}\|^* \big| \Sigma_{n+1}) \;\; P\text{-a.s.}
\end{aligned}$$

Thus, it is enough to prove that for each $1 \leq j \leq n+1$,

$$\mathbb{E}(\|\mathbb{P}_{n,j} - \mathbb{P}\|^* \big| \Sigma_{n+1}) = \mathbb{E}(\|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} \big| \Sigma_{n+1}) \;\; P\text{-a.s.} \qquad (23)$$

For $m \in \mathbb{N}$ fix but arbitrary, let $h : \mathcal{X} \to \mathbb{R}$ such that

$$h(y_1, y_2, \dots y_m) = \left( \sup_{f_\alpha \in \mathcal{F}_\Gamma} \left| \frac{1}{m} \sum_{j=1}^{m} f_\alpha(y_j) - \mathbb{P}(f_\alpha) \right| \right)^*,$$

if

$$\|\mathbb{P}_{n,n+1} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} = \|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} = h \circ (Y_1, Y_2, \dots, Y_{j-1}, Y_j, Y_{j+1}, Y_{j+2}, \dots, Y_n),$$

we have

$$\|\mathbb{P}_{n,j} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} = h \circ (Y_1, Y_2, \dots, Y_{j-1}, Y_{j+1}, Y_{j+2}, \dots, Y_{n+1}).$$

However

$$(Y_1, \dots, Y_{j-1}, Y_j, Y_{j+1}, \dots, Y_n)(A) = (Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_{n+1})(A),$$

for all $A \in \Sigma_{n+1}$. Thus, we can use the argument developed in Remark 4.8 to verify (17), and obtain:

$$\int_A \|\mathbb{P}_n - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} d\mathbb{P}^{\mathbb{N}} = \int_A \|\mathbb{P}_{n,j} - \mathbb{P}\|^*_{\mathcal{F}_\Gamma} d\mathbb{P}^{\mathbb{N}},$$

using the fact that the sets $A$ in $\Sigma_{n+1}$ are invariants under all permutations of the first $n+1$ coordinates. Hence, the equality (23) is true, and consequently the backward submartingale property (18) is satisfied. ☑

**Lemma 4.10** (**Rademacher symmetrization).** *Given the basic product probability space* $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}}) \times (\mathcal{Z}, \mathcal{C}, \mathbb{P}_\epsilon)$. *Let* $\{Y_j\}_{j=1}^n$ *be an i.i.d.*$\sim$ $\mathbb{P}$ *empirical process, independent of the i.i.d. Rademacher collection* $\{\epsilon_j\}_{j=1}^n$*. Then,*

$$\frac{1}{2}\mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^n \epsilon_j (\delta_{Y_j} - \mathbb{P}) \right\|_{\mathcal{F}_\Gamma} \leq \mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^n (\delta_{Y_j} - \mathbb{P}) \right\|_{\mathcal{F}_\Gamma} \leq 2\mathbb{E}^* \left\| \frac{1}{n} \sum_{j=1}^n \epsilon_j (\delta_{Y_j} - \mathbb{P}) \right\|_{\mathcal{F}_\Gamma}, \tag{24}$$

*where* (24) *it is also true if* $\mathbb{P}(f_\alpha)$ *is deleted.*

***Proof***. See [11], Lemma 3.3, pages 93–94. ☑

***Remark*** **4.11.** If $\|\xi_{n1}\|_{2,1} < \infty$, then this implies $\|\xi_{n1}\|_2 := \mathbb{E}(|\xi_{n1}|^2) < \infty$. Indeed,

$$
\begin{aligned}
\mathbb{E}(|\xi_{n1}|^2) &= \int_0^\infty \mathbb{P}_W(|\xi_{n1}|^2 > t)dt \\
&= \int_0^\infty \mathbb{P}_W(|\xi_{n1}| > \sqrt{t})dt \\
&= \int_0^\infty \mathbb{P}_W(|\xi_{n1}| > u)2udu \;\; (u = \sqrt{t} \Rightarrow u^2 = t, \text{ y } 2udu = dt) \\
&= \int_0^\infty 2u\sqrt{\mathbb{P}_W(|\xi_{n1}| > u)}\sqrt{\mathbb{P}_W(|\xi_{n1}| > u)}du \\
&\leq 2\int_0^\infty u\sqrt{\frac{1}{u^2}\mathbb{E}(|\xi_{n1}|^2)}\sqrt{\mathbb{P}_W(|\xi_{n1}| > u)}du \text{ (by Markov's inequality)} \\
&= 2\sqrt{\mathbb{E}(|\xi_{n1}|^2)}\int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| > u)}du \\
\Rightarrow \sqrt{\mathbb{E}(|\xi_{n1}|^2)} &\leq 2\int_0^\infty \sqrt{\mathbb{P}_W(|\xi_{n1}| > u)}du \\
\Rightarrow \sqrt{\mathbb{E}(|\xi_{n1}|^2)} &\leq 2\|\xi_{n1}\|_{2,1}. \qquad (25)
\end{aligned}
$$

Consequently $\|\xi_{n1}\|_2$ is finite, and therefore $\|\xi_{n1}\|_1 := \mathbb{E}(|\xi_{n1}|) < \infty$. Furthermore, $\mathbb{E}(\xi_{n1})$ is defined and finite if and only if $\mathbb{E}(|\xi_{n1}|) < \infty$. Hence, condition **B2** for $\|\xi_{n1}\|_{2,1}$ ensures that all these moments are finite.

**Lemma 4.12 (Inequalities for the bootstrap process with exchangeable multiplier random weights).** *Given $(\mathcal{X}^{\mathbb{N}}, \mathcal{A}^{\mathbb{N}}, \mathbb{P}^{\mathbb{N}}) \times (\mathcal{W}, \mathcal{D}, \mathbb{P}_W) \times (\mathcal{Z}, \mathcal{C}, \mathbb{P}_\epsilon)$, the basic product probability space. Let $\{Z_j\}_{j=1}^n$ be an i.i.d. empirical process such that $\mathbb{E}^*\|Z_j\|_{\mathcal{F}_\Gamma} < \infty$ for each $j \leq n$, independent of the i.i.d. Rademacher collection $\{\epsilon_j\}_{j=1}^n$. Then, for a collection of i.i.d. exchangeable random weights $\{\xi_{nj}\}_{j=1}^n$ with $\|\xi_{n1}\|_{2,1} < \infty$ and $\mathbb{E}(\xi_{nj}) = \mu$, independent of the collection $\{Z_j\}_{j=1}^n$ and any $1 \leq n_0 < n$,*

$$
\begin{aligned}
\frac{1}{2}\|\xi_{n1}-\mu\|_1 \cdot \mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^n \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma} &\leq \mathbb{E}^*\left\|\frac{1}{n}\sum_{j=1}^n \xi_{nj} Z_j\right\|_{\mathcal{F}_\Gamma} \\
&\leq 2n_0\mathbb{E}^*\|Z_1\|_{\mathcal{F}_\Gamma} \cdot \mathbb{E}\left(\max_{1\leq j\leq n}\frac{|\xi_{nj}|}{n}\right) \\
&\quad + 4\frac{\|\xi_{n1}\|_{2,1}}{\sqrt{n}} \cdot \max_{n_0<k\leq n}\left\{\mathbb{E}^*\left\|\frac{1}{\sqrt{k}}\sum_{j=n_0+1}^k \epsilon_j Z_j\right\|_{\mathcal{F}_\Gamma}\right\}. \qquad (26)
\end{aligned}
$$

*For symmetrically distributed variables $\xi_{nj}$ around $\mu$, the constants $1/2$, $2$ and $4$ can all be replaced by $1$, and $\mu$ in the left-hand side of (26) is equal to zero.*

***Proof***. See [11], Lemma 3.4, pages 97–102. ☑

**Lemma 4.13 (Uniformity square-integrable for the second moment, and convergence in mean for the maximum of the exchangeable multiplier weights).** *Let* $\xi := \{|\xi_{nj}| : j = 1, 2, \ldots, n, n = 1, 2, \ldots\}$ *be a triangular array of non-negative and exchangeable random variables, defined on the probability space* $(\mathcal{W}, \mathcal{D}, \mathbb{P}_W)$*. If* $\xi$ *satisfies conditions* **B2** *and* **B3***, this implies that the sequence* $\{|\xi_{n1}|\}_{n\in\mathbb{N}}$ *is uniformly square-integrable; that is,*

$$\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{E}(|\xi_{n1}|^2 \cdot \mathbb{I}\{|\xi_{n1}| \geq t\}) = 0. \tag{27}$$

*Furthermore,* **B2** *and* **B3** *also imply that*

$$\mathbb{E}\left( \max_{1\leq j\leq n} \frac{|\xi_{nj}|}{n} \right) \to 0. \tag{28}$$

***Proof.*** See [11], Lemma 3.5, pages 102–104. ☑

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