

Ineffable limits of weakly compact cardinals and similar results

Límites inefables de cardinales débilmente compactos

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ABSTRACT. It is proved that if an uncountable cardinal κ has an ineffable subset of weakly compact cardinals, then κ is a weakly compact cardinal, and if κ has an ineffable subset of Ramsey (Rowbottom, Jónsson, ineffable or subtle) cardinals, then κ is a Ramsey (Rowbottom, Jónsson, ineffable or subtle) cardinal.

Key words and phrases. Weakly compact cardinal, subtle cardinal, ineffable cardinal, ineffable set, Jónsson cardinal, Rowbottom cardinal, Ramsey cardinal.

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RESUMEN. Se prueba que si un cardinal no contable κ tiene un subconjunto casi inefable de cardinales débilmente compactos entonces κ es un cardinal débilmente compacto. Y si κ tiene un conjunto inefable de cardinales de Ramsey (Rowbottom, Jónsson, inefables o sutiles) entonces κ es cardinal de Ramsey (Rowbottom, Jónsson, inefable o sutil).

Palabras y frases clave. Cardinal débilmente compacto, cardinal sutil, cardinal inefable, conjunto inefable, cardinal Jónsson, cardinal Rowbottom, cardinal Ramsey.

Large cardinals imply the existence of stationary subsets of smaller large cardinals. For instance weakly compact cardinals have a stationary subset of Mahlo cardinals, measurable cardinals imply the set of Ramsey cardinals below the measurable cardinal κ has measure 1 and Ramsey cardinals imply the set of weakly compact cardinals below the Ramsey cardinal κ is a stationary subset of κ .

In addition there are cases in which if κ has enough large cardinals below it, κ turns out to be a larger cardinal like in Menas [6]: if κ is a measurable cardinal limit of strongly compact cardinals then κ is a strongly compact cardinal.

This article aims to find similar results, more precisely to determine how big a subset of a certain kind of large cardinal below κ should be in order to become a larger cardinal.

In this paper, the correct notion of being a big set, at least for cardinals, corresponds to being ineffable. The definition is a combinatorial property:

Definition 1. Let κ be a regular cardinal. $R \subseteq \kappa$ is an ineffable subset of κ if for every sequence $\langle S_\alpha \mid \alpha \in R \rangle$ such that $S_\alpha \subseteq \alpha$ for $\alpha \in R$ there exists $T \subseteq R$ a stationary subset of κ such that for every $\alpha, \beta \in T$, $\alpha < \beta$, $S_\alpha = \alpha \cap S_\beta$. If $R = \kappa$ we say κ is an ineffable cardinal. $R \subseteq \kappa$ is an almost ineffable subset of κ if for every sequence $\langle S_\alpha \mid \alpha \in R \rangle$ such that $S_\alpha \subseteq \alpha$ for $\alpha \in R$ there exists $T \subseteq R$ unbounded in κ such that for every $\alpha, \beta \in T$, $\alpha < \beta$, $S_\alpha = \alpha \cap S_\beta$. If $R = \kappa$ we say κ is an almost ineffable cardinal.

It is clear that every ineffable subset of κ is an almost ineffable subset of κ . Thus every ineffable cardinal is an almost ineffable cardinal. It is also the case that every almost ineffable subset of κ is a stationary subset of κ .

Also every almost ineffable cardinal is a weakly compact cardinal. And if κ is an ineffable cardinal, the set of almost ineffable cardinals below κ is an ineffable subset of κ (so not every almost ineffable subset of κ is an ineffable subset of κ). If κ is an almost ineffable cardinal, the set of weakly compact cardinals below κ is an almost ineffable subset of κ (see [1]).

Notice that if X is an almost ineffable subset of κ then κ itself is an almost ineffable cardinal: let X be an almost ineffable subset of κ and let $\langle S_\alpha \mid \alpha \in \kappa \rangle$ be a sequence such that $S_\alpha \subseteq \alpha$ for $\alpha \in \kappa$, then for the sequence $\langle S_\alpha \mid \alpha \in X \rangle$ there exists $T \subseteq X$ unbounded in κ such that for every $\alpha, \beta \in T$, $\alpha < \beta$, $S_\alpha = \alpha \cap S_\beta$. In particular $T \subseteq \kappa$, so κ is an almost ineffable subset of κ .

In [1], Theorem 4.1, it is proved that if X is an almost ineffable subset of κ , then the set $\{\alpha \in X : \alpha \text{ is a } \Pi_n^1\text{-indescribable cardinal}\}$ for every $n < \omega$, is also an almost ineffable subset of κ . So the main theorem (Theorem 3) also follows from the existence of an almost ineffable subset of κ (if X is an almost ineffable subset of κ then κ is an almost ineffable cardinal and every almost ineffable cardinal is a weakly compact cardinal and every weakly compact cardinal is a Π_1^1 -indescribable cardinal, see [2]).

Thus the results are much more interesting when the large cardinal is not directly implied by the ineffability or almost ineffability of the subset of large cardinals below κ . Every Ramsey cardinal is a Rowbottom cardinal, and every Rowbottom cardinal is a Jónsson cardinal, only if κ is a completely Ramsey cardinal then κ is an ineffable cardinal, see [3].

If κ is a measurable cardinal it is also true that every subset in a normal ultrafilter on κ is an ineffable subset of κ . In particular if κ is a measurable cardinal the set of weakly compact (or Ramsey) cardinals below κ is in a normal measure on κ , and it is an ineffable subset of κ , see [7].

Definition 2. Let κ be an uncountable cardinal. κ is a weakly compact cardinal if and only if $\kappa \rightarrow (\kappa)_2^2$ *i.e.*, for every $f : [\kappa]^2 \rightarrow \{0, 1\}$ there exists $H \in [\kappa]^\kappa$ such that $|f''[H]^2| = 1$. H is said to be a homogeneous set for f .

In this paper it is proved that if the set of weakly compact cardinals is an almost ineffable subset of κ and κ is an uncountable cardinal, then κ becomes a weakly compact cardinal. If the set of weakly compact cardinals below κ is only a stationary subset of κ , κ is not necessarily a weakly compact cardinal (η_ω , the first Erdős cardinal has a stationary subset of weakly compact cardinals, but it is not weakly a compact cardinal since it is Π_1^1 -describable, see [4]).

A similar result is also proved for ineffable subsets of Ramsey cardinals, Rowbottom cardinals, Jónsson cardinals or even ineffable cardinals.

All these cardinals have in common that they are also defined in terms of combinatorial properties that imply the existence of homogeneous subsets. Based on the ineffability of sets it is possible to find a coherent sequence of small homogeneous subsets in order to build such a homogeneous subset from the small homogeneous subsets. The subset of κ is not necessarily ineffable in every case, it is possible to relax the condition to be almost ineffable subset for Ramsey, Rowbottom, Jónsson and ineffable cardinals or to a stationary subset for subtle cardinals.

Theorem 3 (Main theorem). *If κ is an uncountable cardinal such that the set of weakly compact cardinals below κ is an almost ineffable set, then κ is a weakly compact cardinal.*

Proof. Observe κ is a Mahlo cardinal since the set of inaccessible cardinals below κ is a stationary subset of κ , so κ is an inaccessible cardinal. Let $f : [\kappa]^2 \rightarrow 2$ be a function and $I = \{\lambda < \kappa \mid \lambda \text{ is a weakly compact cardinal}\}$. Since $f \upharpoonright [\lambda]^2 : [\lambda]^2 \rightarrow 2$ for $\lambda \in I$, there exists $H_\lambda \in [\lambda]^\lambda$ such that $|(f \upharpoonright [\lambda]^2)''[H_\lambda]^2| = 1$. Define now the sequence $S_\alpha = H_\alpha$ for $\alpha \in I$, using the hypothesis that I is an almost ineffable subset of κ , then take $T \subseteq I$ unbounded in κ such that for every $\alpha < \beta \in T$, $S_\alpha = \alpha \cap S_\beta$. Take $H := \bigcup_{\alpha \in T} H_\alpha$. We now prove that H is a homogeneous set for f (since κ is regular, $H \in [\kappa]^\kappa$). If false, there exists $(\eta, \mu), (\xi, \nu) \in [H]^2$ such that $f(\eta, \mu) = 0$ and $f(\xi, \nu) = 1$. However there would be $\lambda \in T$, such that $(\eta, \mu), (\xi, \nu) \in [H_\lambda]^2$ with $f(\eta, \mu) = 0$ and $f(\xi, \nu) = 1$. Contradiction. \square

For the case of weakly compact cardinals we could have used the definition of weakly compact cardinals in terms of the tree property (*i.e.* κ is a weakly

compact cardinal if and only if κ is an inaccessible cardinal and every κ -tree T has a cofinal branch) and find a cofinal branch instead of the homogeneous set H for f . (Specifically, if $(T, <)$ is a κ -tree we can suppose $< \subseteq \kappa \times \kappa$, and for $\alpha \in A = \{\alpha < \kappa : \alpha \text{ is a weakly compact cardinal}\}$, take $T_\alpha := T \upharpoonright \alpha$. The set T_α is an α -tree and we can find a cofinal branch $B_\alpha \subseteq \alpha$. So for the sequence $\langle B_\alpha : \alpha \in A \rangle$, there exists a $D \subseteq A$, an unbounded subset of κ such that $\langle B_\alpha : \alpha \in D \rangle$ is coherent, so $\bigcup_{\alpha \in D} B_\alpha$ is a cofinal branch in T).

Definition 4. Let κ be an uncountable cardinal. κ is a Jónsson cardinal if and only if $\kappa \rightarrow [\kappa]_\kappa^{<\omega}$ i.e. for every $f : [\kappa]^{<\omega} \rightarrow \kappa$ there exists $H \in [\kappa]^\kappa$ such that $f''[H]^{<\omega} \neq \kappa$.

Theorem 5. If κ is an inaccessible cardinal such that the set of Jónsson cardinals below κ is an ineffable subset of κ , then κ is a Jónsson cardinal.

Proof. Let $f : [\kappa]^{<\omega} \rightarrow \kappa$ be a function and $I = \{\lambda < \kappa \mid \lambda \text{ is a Jónsson cardinal}\}$. The set $C = \{\beta < \kappa \mid (f \upharpoonright [\beta]^{<\omega})''[\beta]^{<\omega} \subseteq \beta\}$ is a club subset of κ . Therefore for $\lambda \in I \cap C$, there exists $H_\lambda \in [\lambda]^\lambda$ such that $(f \upharpoonright [\lambda]^{<\omega})''[H_\lambda]^{<\omega} \neq \lambda$. Now we define the sequence $S_\alpha \subseteq \alpha$ for $\alpha \in I$ as follows: for $\alpha \in I \cap C$, $S_\alpha := H_\alpha$, otherwise $S_\alpha = \alpha$. Since I is an ineffable subset of κ , there exists a stationary $T \subseteq I$, such that for every $\alpha < \beta \in I$, $S_\alpha = \alpha \cap S_\beta$. Since $T \cap C$ is also a stationary subset of κ , take $H := \bigcup_{\alpha \in T \cap C} H_\alpha$, so $H \in [\kappa]^\kappa$. We now prove $f''[H]^{<\omega} \neq \kappa$. It is also true that the set

$$D := \{\delta < \kappa \mid (V_\delta, \in, f \upharpoonright [\delta]^{<\omega}, T \cap C \cap \delta, H \cap \delta) \prec (V_\kappa, \in, f, T \cap C, H)\}$$

is a club subset of κ since κ is an inaccessible cardinal. Therefore there is $\lambda \in T \cap C \cap D$, such that $(V_\lambda, \in, f \upharpoonright [\lambda]^{<\omega}, T \cap C \cap \lambda, H \cap \lambda) \prec (V_\kappa, \in, f, T \cap C, H)$, $H \cap \lambda = H_\lambda$, and $(f \upharpoonright [\lambda]^{<\omega})''[H_\lambda]^{<\omega} \neq \lambda$.

By elementarity $f''[H]^{<\omega} \neq \kappa$. □

Definition 6. For $\omega < \nu < \kappa$, ν, κ cardinals, κ is a ν -Rowbottom cardinal if and only if for every $\lambda < \kappa$, $\kappa \rightarrow [\kappa]_{\lambda, <\nu}^{<\omega}$ i.e. for every $f : [\kappa]^{<\omega} \rightarrow \lambda$ there exists $H \in [\kappa]^\kappa$ such that $|f''[H]^{<\omega}| < \nu$. κ is Rowbottom if and only if κ is ω_1 -Rowbottom.

Theorem 7. If κ is an uncountable regular cardinal such that the set of Rowbottom cardinals below κ is an almost ineffable subset of κ , then κ is a Rowbottom cardinal.

Proof. Let $\mu < \kappa$ be an uncountable cardinal and let $f : [\kappa]^{<\omega} \rightarrow \mu$. Since the set $R = \{\lambda < \kappa \mid \lambda \text{ is a Rowbottom cardinal}\}$ is an almost ineffable subset of κ , it is unbounded in κ . Hence there exists a Rowbottom cardinal $\lambda > \mu$ less than κ and there is an $H_\lambda \in [\lambda]^\lambda$ such that for $f \upharpoonright [\lambda]^{<\omega} : [\lambda]^{<\omega} \rightarrow \mu$, $|(f \upharpoonright [\lambda]^{<\omega})''[H_\lambda]^{<\omega}| < \omega_1$. In fact, for every $\lambda > \mu$ in R such H_λ exists. Let $\langle H_\lambda : \lambda \in R \cap (\mu, \kappa) \rangle$ be such that each H_λ is homogeneous for $f \upharpoonright [\lambda]^{<\omega}$. Since

$R \cap (\mu, \kappa)$ is also an almost ineffable subset of κ , there exists an $S \subseteq R \cap (\mu, \kappa)$ unbounded in κ such that for every $\lambda < \eta \in S$, $H_\lambda = \lambda \cap H_\eta$. By the regularity of κ , $H := \bigcup_{\lambda \in S} H_\lambda$ has cardinality κ . We claim that $|f''[H]^{<\omega}| < \omega_1$; otherwise there would be a sequence $\beta_i \in [H]^{<\omega}$ for $i < \omega_1$ such that $\{f(\beta_i) : i \in \omega_1\}$ is uncountable, but the sequence already exists in $[H_\lambda]^{<\omega}$ for some λ Rowbottom. This is a contradiction. \checkmark

Definition 8. Let κ be an uncountable cardinal. κ is a Ramsey cardinal if and only if $\kappa \rightarrow (\kappa)_2^{<\omega}$ i.e. for every $f : [\kappa]^{<\omega} \rightarrow 2$ there exists $H \in [\kappa]^\kappa$ such that $|f''[H]^{<\omega}| = 1$. Such a set H is said to be a homogeneous set for f .

Theorem 9. *If κ is an uncountable regular cardinal such that the set of Ramsey cardinals below κ is an almost ineffable set, then κ is a Ramsey cardinal.*

Proof. The proof of Theorem 9 is the same as the proof of Theorem 3, with exponent 2 replaced by exponent $< \omega$. \checkmark

Theorem 10. *If κ is a cardinal such that the set of ineffable cardinals below κ is an ineffable subset of κ , then κ is an ineffable cardinal.*

Proof. We use in this case that κ is an ineffable cardinal if and only if for every $f : [\kappa]^2 \rightarrow 2$, there exists H , a stationary subset of κ such that $|f''[H]^2| = 1$ (see [2] VII, Theorem 2.1). So let $f : [\kappa]^2 \rightarrow 2$ be a partition and $B = \{\lambda < \kappa \mid \lambda \text{ is an ineffable cardinal}\}$. Then for every $\lambda \in B$ there is H_λ , a stationary subset of λ , such that $|(f \upharpoonright [\lambda]^2)''[H_\lambda]^2| = 1$. Since B is an ineffable subset of κ , for the sequence $\langle H_\lambda \mid \lambda \in B \rangle$ there exists $X \subseteq B$ a stationary subset of κ such that for every $\lambda < \eta \in X$, $H_\lambda = \lambda \cap H_\eta$. We show $H := \bigcup_{\lambda \in X} H_\lambda$ is a stationary subset of κ and is such that $|f''[H]^2| = 1$. To see this, let $C \subseteq \kappa$ be a club in κ and let \bar{C} be the set of its limit points. The set \bar{C} is also a club subset of C . Since B is a stationary subset of κ , there exists $\lambda < \kappa$ an ineffable cardinal such that $\lambda \in \bar{C} \subseteq C$, and $\lambda \cap \bar{C}$ is club subset of λ . Hence $C \cap H_\lambda \neq \emptyset$, so H is a stationary subset of κ . The fact H is homogeneous for f now follows as in the proof of Theorem 3. \checkmark

In the next theorem a cardinal becomes subtle only having a stationary subset of subtle cardinals:

Definition 11. Let κ be a regular cardinal. κ is a subtle cardinal if and only if for every sequence $\langle S_\alpha \mid \alpha \in \kappa \rangle$ such that $S_\alpha \subseteq \alpha$ for $\alpha \in \kappa$ and for every $C \subseteq \kappa$ a club set in κ , there exists $\alpha, \beta \in C$, $\alpha < \beta$ such that $S_\alpha = \alpha \cap S_\beta$.

Theorem 12. *If κ is a regular cardinal such that the set of subtle cardinals below κ is a stationary set, then κ is a subtle cardinal.*

Proof. Let $\langle S_\alpha \mid \alpha < \kappa \rangle$ be a sequence such that $S_\alpha \subseteq \alpha$ for $\alpha < \kappa$ and let $C \subseteq \kappa$ be a club subset. Therefore there exists $\lambda \in \bar{C} \subseteq C$ a subtle cardinal

since the set of subtle cardinals below κ is stationary. In addition, because $\lambda \cap \bar{C}$ is club in λ there exist $\alpha < \beta$ in $\lambda \cap \bar{C}$ such that $S_\alpha = \alpha \cap S_\beta$. \square

Remark 13. Kunen [5] has shown that there is a model in which every Jónsson cardinal is a Ramsey cardinal. So, it is possible to have κ a limit of Jónsson cardinals that is not a Jónsson cardinal. For the same reason a limit of Rowbottom or Ramsey cardinals is not necessarily a Rowbottom or Ramsey cardinal.

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