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MELNIKOV DEVIATIONS AND LIMIT CYCLES FOR LIENARD EQUATIONS(*)

by

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SUMMARY. In this paper we analyze the method of small parameter and we calculate explicitly the derivatives of order two and three with respect to the parameter in the case of a homoclinic orbit and in the case of periodic orbits. Also, we apply this method to the Liénard equation of degree five on the plane:

$$X^{f_a}: \begin{cases} \dot{x} = y - (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5) \\ \dot{y} = -x \end{cases}, a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$$

We conclude that there is a neighborhood $\mathcal{U}$ of the origin of $\mathbb{R}^5$, such that for all $a \in \mathcal{U} - C$ the vector field X^{f_a} has at most two limit cycles, where C is a cone containing the plane a_2, a_4 and which is tangent to it at the origin.

§0. Introduction

One important problem in qualitative theory of ordinary differential equations on the plane (vector fields) is to analyze the behavior by perturbation of non-stable critical elements, i.e. non hyperbolic singularities and periodic orbits. It is also interesting to understand the dynamic of the system on a neighborhood of a trajectory connecting two saddle points or connecting

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a saddle point to itself.

The Melnikov's method of small deviations, exposed on Chapter §1, offers a technic on detecting limit cycles which prevail by perturbation of a center and on studying how a periodic orbit or a saddle-self-connection is broken by perturbation. On these subjects there are several results in the mathematical literature (for example [1], [2], [6]). This method is concerned with the analysis of a function that measures the deviation of a closed orbit or a saddle-self-connection trajectory under a perturbation. This function, depends on the parameters of the perturbation; it cannot be, in general, explicitly calculated, but we can consider the first approximations. Our contribution to this method, included in Chapter §2, is the explicit computation of the first three derivatives of this function.

Chapter §3 is devoted to an application of the method to a Liénard equation of degree five on the plane. There are several results on the general Liénard equation. Qualitative theory on this equation has been developed by Lloyd [4], [5], Rychkov [7], Lins [3] and the mathematical Chinese school, [8], [9], [10]. The Melnikov's method allows the computation of the bifurcation curves on a neighborhood of zero in the parameter-space. Finally, we give a summary of the qualitative properties of this equation, by using compactification at infinity.

§1. Melnikov's Deviations

We consider a C^∞ family of vector fields on the plane, given by the equation $\dot{x} = X(x, \varepsilon)$, $x \in \mathbb{R}^2$, where $0 \leq \varepsilon < \tilde{\varepsilon}$ is a parameter. Let us suppose, that for $\varepsilon = 0$, $X(x, 0)$ has a saddle-connection γ, in $p \in \mathbb{R}^2$, fig.(a), or $X(x, 0)$ has a periodic orbit γ through $p \in \mathbb{R}^2$, fig.(b).

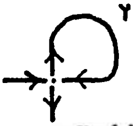
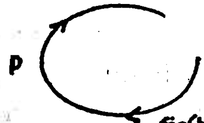


fig.(a)

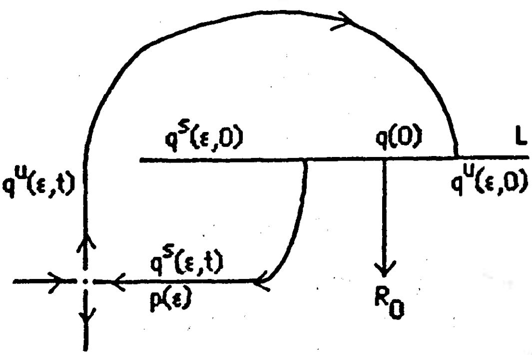


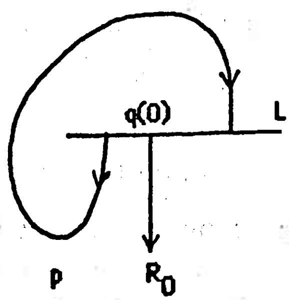
fig(b)

Let $q(0)$ be in γ, $q(0) \neq p$, $R_0 = X(q(0), 0)$, L an orthogonal segment to R_0 in $q(0)$, let $q^s(\varepsilon, t)$ and $q^u(\varepsilon, t)$ be orbits of $X(x, \varepsilon)$ which parametrize the stable and unstable manifolds of the hyperbolic saddle $p(\varepsilon)$, in the case (a), with $q^s(\varepsilon, 0) \in L$ and $q^u(\varepsilon, 0) \in L$; and in the case (b) $q^s(\varepsilon, t) = \varphi_\varepsilon(t + T^s_\varepsilon, p)$, $q^u(\varepsilon, t) = \varphi_\varepsilon(t + T^u_\varepsilon, p)$, where φ_ε is the flow of $X(x, \varepsilon)$, $T^s_\varepsilon = \sup\{t < 0 \,/\, \varphi_\varepsilon(t, p) \in L\}$ and $T^u_\varepsilon = \inf\{t > 0 \,/\, \varphi_\varepsilon(t, p) \in L\}$.

Let us remark that in case (a): $T^u_\varepsilon = +\infty, T^s_\varepsilon = -\infty$ for all small $\varepsilon \geq 0$ and that $p(\varepsilon) = q^u(\varepsilon, -T^u_\varepsilon) = q^s(\varepsilon, -T^s_\varepsilon)$ implies: $\partial^k q^u/\partial\varepsilon^k(\varepsilon, -T^u_0) = \partial^k q^s/\partial\varepsilon^k(\varepsilon, -T^s_0) = p^{(k)}(\varepsilon)$.

Under the assumptions (b), $q^u(\varepsilon, -T^u_\varepsilon) = q^s(\varepsilon, -T^s_\varepsilon) = p$ for all small $\varepsilon \geq 0$.





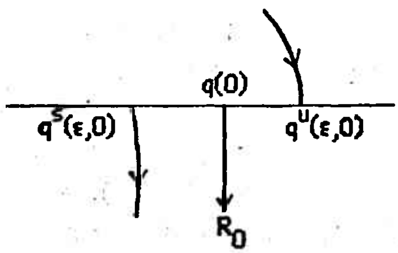
Let us call $q(t)$ the orbit of $X(x, 0)$, passing by $q(0)$ at $t = 0$. By the above definition:

$$q(t) = \begin{cases} q^s(0, t) & \text{for} \quad t \geq 0 \\ q^u(0, t) & \text{for} \quad t \leq 0. \end{cases}$$

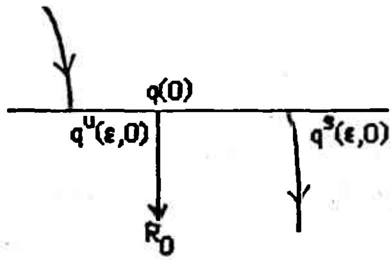
An ε - deviation of Melnikov at $q(0)$, is the number

$$d(\varepsilon) = R_0 \wedge [q^u(\varepsilon, 0) - q^s(\varepsilon, 0)]$$

where $\wedge$ indicates the determinant. If $d(\varepsilon) > 0$, the deviation is called a **repelling deviation** and if $d(\varepsilon) < 0$ it is an **attracting deviation.**

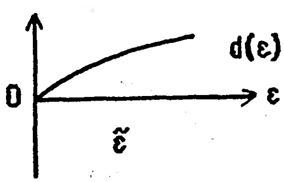


repelling ε-deviation

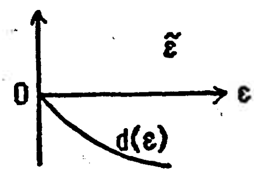


attracting ε-deviation

We say that the vector field $X(x, \varepsilon)$ has a **repelling rupture** (**resp. attracting rupture**) in $q(0)$ if there exists $\tilde{\varepsilon} > 0$ so that $d(\varepsilon) > 0$ (resp. $d(\varepsilon) < 0$) $\forall \varepsilon, 0 < \varepsilon < \tilde{\varepsilon}$.



repelling rupture



attracting rupture

The function $d(\varepsilon)$ is of C^∞-class (because $X(x, \varepsilon)$ is of C^∞-class), then $X(x, \varepsilon)$ has a repelling rupture (resp. attracting rupture) in $q(0)$ if $d'(0) > 0$ (resp. $d'(0) < 0$).

Note that if the ε_0-deviation is zero, then it prevails a saddle-connection or a periodic orbit by $q(0)$, for the vector field $X(x, \varepsilon_0)$. This function was introduced by V. K. Melnikov in [6], in studying stability of the center for time periodic perturbations.

§2. Derivatives of the deviations

Proposition 2.1. (Melnikov's integral). *On the above assumptions*:

$$d'(0) = \int_{-T_0^u}^{-T_0^s} \exp\left[\int_0^t \mathrm{div}X(q(\tau), 0)\, d\tau\right]\left[X(q(t), 0) \wedge \frac{\partial X}{\partial \varepsilon}(q(t), 0)\right] dt.$$

Proof. $d'(0) = R_0 \wedge \frac{\partial q^u}{\partial \varepsilon}(0, 0) - R_0 \wedge \frac{\partial q^s}{\partial \varepsilon}(0, 0)$. We define

$\Delta^u(t) = X(q(t), 0) \wedge \frac{\partial q^u}{\partial \varepsilon}(0, t)$, $-T_0^u \le t \le 0$; $\Delta^s(t) = X(q(t), 0) \wedge \frac{\partial q^s}{\partial \varepsilon}(0, t)$, $0 \le t \le -T_0^s$. We are interested in $\Delta^u(0) - \Delta^s(0)$. We have:

$$\frac{\partial}{\partial t}\Delta^u(t) = DX(q(t), 0)\cdot X(q(t), 0) \wedge \frac{\partial q^u}{\partial \varepsilon}(0, t) = X(q(t), 0) \wedge \frac{\partial^2 q^u}{\partial t \partial \varepsilon}(0, t),$$

where "D" denotes the differentiation with respect to x. But

$$\frac{\partial^2 q^u}{\partial t \partial \varepsilon}(0, t) = \frac{\partial}{\partial \varepsilon}\left[\frac{\partial q^u}{\partial t}(\varepsilon, t)\right]_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left[X(q^u(\varepsilon, t), \varepsilon)\right]_{\varepsilon=0}$$

$$= DX(q(t), 0)\frac{\partial q^u}{\partial \varepsilon}(0, t) + \frac{\partial X}{\partial \varepsilon}(q(t), 0)$$

then we obtain

$$\frac{d}{dt}\Delta^u(t) = DX(q(t), 0)\cdot X(q(t), 0) \wedge \frac{\partial q^u}{\partial \varepsilon}(0, t)$$

$$+ X(q(t), 0) \wedge DX(q(t), 0)\frac{\partial q^u}{\partial \varepsilon}(0, t) + X(q(t), 0) \wedge \frac{\partial X}{\partial \varepsilon}(q(t), 0).$$

By the vector-calculus rule: $Av \wedge w + v \wedge Aw = \mathrm{Tr}\, A(v \wedge w)$, we have

$$\frac{d}{dt}\Delta^u(t) = \mathrm{Tr}DX(q(t),0)\left[X(q(t),0)\wedge\frac{\partial q^u}{\partial\varepsilon}(0,t)\right] + X(q(t),0)\wedge\frac{\partial X}{\partial\varepsilon}(q(t),0),$$

that is, $\Delta^u(t)$, satisfies the first order linear equation

$$\mathring{\Delta}^u(t) = \mathrm{div}\, X(q(t),0)\cdot\Delta^u + b(t), \text{ with } b(t) = X(q(t),0)\wedge\frac{dX}{d\varepsilon}(q(t),0)$$

Hence $\Delta^u(t) =$

$$\exp\left[\int_0^t \mathrm{div}X(q(s),0)\,ds\right]\cdot\left[\Delta^u(0) + \int_0^t \exp\left\{-\int_0^s \mathrm{div}X(q(r),0)\,dr\right\}b(s)ds\right]$$

that is $\Delta^u(0) =$

$$\exp\left[\int_t^0 \mathrm{div}X(q(s),0)\,ds\right]\cdot\Delta^u(t) + \int_t^0 \exp\left\{-\int_0^s \mathrm{div}X(q(r),0)\,dr\right\}b(s)ds.$$

Now, since p is an hyperbolic saddle, if $-\mu < 0$ and $\eta > 0$ are the eigenvalues of $DX(p, 0)$ there are nonzero vectors $c \in \mathbb{R}^2$, $d \in \mathbb{R}^2$, such that

$$\lim_{t\to\infty} e^{\mu t}X(q(t),0) = c, \quad \lim_{t\to-\infty} e^{-\eta t}X(q(t),0) = d.$$

If $\sigma_0 = \mathrm{Tr}\, DX(p,0) = -\mu + \eta$ and $\beta(t) = \exp \int_t^0 \mathrm{div}\, X(q(s),0)ds$ then the following limits exist and are not zero: $\lim_{t\to\pm\infty} e^{\sigma_0 t}\beta(t) =: L^{\pm}$.

Therefore

$$\exp\left(\int_t^0 \mathrm{div}\, X(q(s),0)\,ds\right)\cdot\Delta^u(t) = \beta(t)X(q(t),0)\wedge\frac{\partial q^u}{\partial\varepsilon}(0,t)$$

$$= e^{\mu t}\left(e^{\sigma_0 t}\beta(t)\right)e^{-\eta t}X(q(t),0)\wedge\frac{\partial q^u}{\partial\varepsilon}(0,t).$$

Thus, $\exp\left(\int_t^0 \operatorname{div} X(q(s), 0)\, ds\right) \cdot \Delta^u(t)$ approaches $0 \cdot L^- \cdot d \wedge p'(0) = 0$

as $t \to -\infty$.

In the case (b):

$$\lim_{t \to T_0^u} \Delta^u(t) = X(p, 0) \wedge p'(0) = X(p, 0) \wedge 0 = 0$$

and

$$\exp\left(\int_t^0 \operatorname{div} X(q(s), 0)\, ds\right) < \infty \quad \text{for } t \geq -T_0^u > -\infty .$$

Hence, in both cases

$$\lim_{t \to -T_0^u} \exp\left(\int_t^0 \operatorname{div} X(q(s), 0)\, ds\right) \Delta^u(t) = 0.$$

Thus

$$\Delta^u(0) = \int_{-T_0^u}^{0} \exp\left\{-\int_0^s \operatorname{div} X(q(r), 0)\, dr\right\} b(s)\, ds.$$

Similarly we show that:

$$\Delta^s(0) = -\int_0^{-T_0^s} \exp\left\{-\int_0^s \operatorname{div} X(q(r), 0)\, dr\right\} b(s)\, ds.$$

Finally,

$$d'(0) = \Delta^u(0) - \Delta^s(0) = \int_{-T_0^u}^{-T_0^s} \exp\left\{-\int_0^s \operatorname{div} X(q(r), 0)\, dr\right\} b(s)\, ds. \blacksquare$$

On the following theorems we give formulae for the second and third derivatives of d. The proof of these propositions is similar to the previous one (2.1), taking care on the choice of the appropriate linear equation for $\partial q^u/\partial\varepsilon$ and $\partial^2 q^u/\partial\varepsilon^2$.

Proposition 2.2. *The second derivative of* d *in* 0 *is:*

$$d''(0) = \int_{-T_0^u}^{0} \exp\left\{-\int_0^t \operatorname{div} X(q(r), 0)\, dr\right\} \cdot b^u(s)\, ds$$

$$+ \int_{0}^{-T_0^s} \exp\left\{-\int_0^t \operatorname{div} X(q(r), 0)\, dr\right\} \cdot b^s(s)\, ds$$

where $b^\alpha(s) = X(q(s), 0) \wedge [\frac{\partial^2 X}{\partial\varepsilon^2}(q(s), 0) + D^2X(q(s), 0)\left(\frac{\partial q^\alpha}{\partial\varepsilon}(0, s)\right)^2$

$$+ 2\frac{\partial DX}{\partial\varepsilon}(q(s), 0)\, \frac{\partial q^\alpha}{\partial\varepsilon}(0,s)]$$

with $\alpha = u, s$. (*The notation* z^k *where* z *is in* $\mathbb{R}^2$ *corresponds to the* k-*vector* (z, z, ... , z)).

Proposition 2.3. *The third derivative of* d *in* 0 *is* :

$$d'''(0) = \int_{-T_0^u}^{0} \exp\left\{-\int_0^t \operatorname{div} X(q(r), 0)\, dr\right\} \cdot b^s(s)\, ds$$

$$+\int_0^{-T_0^s} \exp\left\{-\int_0^t \operatorname{div} X(q(r),0)\,dr\right\}\cdot b^u(s)\,ds$$

with $b^\alpha(s) = X(q(s),0) \wedge \Big[D^3X(q(s),0)\cdot\left(\frac{\partial q^\alpha}{\partial\varepsilon}(0,s)\right)^3 + \frac{\partial^3 X}{\partial\varepsilon^3}(q(s),0)$

$$+3D^2X(q(s),0)\left(\frac{\partial^2 q^\alpha}{\partial\varepsilon^2}(0,s);\frac{\partial q^\alpha}{\partial\varepsilon}(0,s)\right)+3\frac{\partial D^2X}{\partial\varepsilon}(q(s),0)\left(\frac{\partial q^\alpha}{\partial\varepsilon}(0,s)\right)^2$$

$$+3\frac{\partial DX}{\partial\varepsilon}(q(s),0)\frac{\partial^2 q^\alpha}{\partial\varepsilon^2}(0,s)+3\frac{\partial^2 DX}{\partial\varepsilon^2}(q(s),0)\cdot\frac{\partial q^\alpha}{\partial\varepsilon}(0,s)\Big].$$

§3. Application to the Liénard Equation of degree 5

3.1. Let us consider the Liénard equation of degree 5 on the plane:

$$X^{\varepsilon f_a}:\begin{cases}\dot{x} = y - \varepsilon f_a(x)\\ \dot{y} = -x\end{cases}$$

with $f_a(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ and $\varepsilon \geq 0$ a small parameter.

Let $p = (-h, 0)$ with $h > 0$, and $L = \{(x, 0) / x \geq 0\}$. It is clear that L is transversal to the flow of $X^{\varepsilon f_a}$, for $0 \leq \varepsilon < \tilde{\varepsilon}$, and small $\tilde{\varepsilon}$. Therefore we can define: $d: \mathbb{R} \times [0,\tilde{\varepsilon}) \to \mathbb{R}$, $(h, \varepsilon) \to d_h(\varepsilon)$ where $d_h(\varepsilon)$ is the ε-Melnikov's deviation:

$$d_h(\varepsilon) = X^0(h,0) \wedge \left[q^u(\varepsilon,0) - q^s(\varepsilon,0)\right] \quad \text{(as in 1.1).}$$

Being $X^{\varepsilon f_a}$ a C^∞-vector field, d is a C^∞-mapping. Since: $R_0 = X^0(h, 0)$

= (0, -h), $q(\theta)$ = (h cosθ, - h sinθ) and $X(q(\theta), 0)$ = (-h sinθ, - h cosθ), div $X(q(\theta), 0) = 0$, then, the first derivative of d with respect to ε is:

$$\frac{\partial d}{\partial \varepsilon}(h, 0) = -\int_{-\pi}^{\pi} \sum_{i=1}^{5} a_i(h\cos\theta)^{i+1} d\theta = -\pi h^2\left[a_1 + \frac{3a_3}{4}h^2 + \frac{5a_5}{8}h^4\right] \quad (1)$$

To calculate the second derivative of d with respect to ε, we consider:

$$\Delta^u(0) = 2\int_{-\pi}^{0} X(q(\theta), 0) \wedge \frac{\partial}{\partial \varepsilon} DX(q(\theta), 0) \frac{\partial q^u}{\partial \varepsilon}(0, \theta)\, d\theta$$

$$\Delta^s(0) = 2\int_{\pi}^{0} X(q(\theta), 0) \wedge \frac{\partial}{\partial \varepsilon} DX(q(\theta), 0) \frac{\partial q^s}{\partial \varepsilon}(0, \theta)\, d\theta \quad \text{and}$$

$$\frac{\partial q^u}{\partial \varepsilon}(0, \theta) = \exp\int_{-\pi}^{0}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}dt \int_{-\pi}^{0} \exp\int_{-\pi}^{t} -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}ds \cdot \begin{pmatrix} -f_a(h\cos t) \\ 0 \end{pmatrix}dt$$

$$= \begin{pmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\int_{-\pi}^{0}\begin{pmatrix} -\cos t & \sin t \\ -\sin t & -\cos t \end{pmatrix}\begin{pmatrix} -f_a(h\cos t) \\ 0 \end{pmatrix}dt$$

$$\frac{\partial q^s}{\partial \varepsilon}(0, \theta) = \begin{pmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\int_{\pi}^{0}\begin{pmatrix} -\cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}\begin{pmatrix} -f_a(h\cos t) \\ 0 \end{pmatrix}dt$$

Then

$$\Delta^{u}(0) = 2\int_{-\pi}^{0} h\cos\theta\left[P^{u}\cos\theta f_{a}'(h\cos\theta) + Q^{u}\sin\theta f_{a}'(h\cos\theta)\right]d\theta$$

and

$$\Delta^{s}(0) = 2\int_{\pi}^{0} h\cos\theta\left[P^{s}\cos\theta f_{a}'(h\cos\theta) + Q^{s}\sin\theta f_{a}'(h\cos\theta)\right]d\theta$$

where

$$P^{u,s} = a_1 h\left[\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} \pm \frac{\pi}{2}\right] + a_2 h^2\left[\sin\theta\left(\frac{2}{3} + \frac{1}{3}\cos^2\theta\right)\right]$$

$$+ a_3 h^3\left[\frac{3\theta}{8} + \frac{\sin\theta\cos\theta}{2} + \frac{\sin\theta(2\cos^3\theta - \cos\theta)}{8} \pm \frac{3\pi}{8}\right]$$

$$+ a_4 h^4\left[\sin\theta\left(\frac{1}{3} + \frac{2}{3}\cos^2\theta + \frac{(1-\cos^2\theta)^2}{5}\right)\right]$$

$$+ a_5 h^5\left[\pm\frac{5\pi}{16} + \frac{5\theta}{16} + \frac{\sin\theta\cos^5\theta}{6} + \frac{5\sin\theta\cos\theta}{12} + \frac{5\sin\theta(2\cos^3\theta - \cos\theta)}{48}\right]$$

$$Q^{u,s} = -a_1 h\left[\frac{\cos^2\theta - 1}{2}\right] - a_2 h^2\left[\frac{\cos^3\theta + 1}{3}\right] - a_3 h^3\left[\frac{\cos^4\theta - 1}{4}\right]$$

$$- a_4 h^4\left[\frac{\cos^5\theta - 1}{5}\right] - a_5 h^5\left[\frac{\cos^6\theta - 1}{6}\right]$$

If we consider $h > 0$ such that:

$$a_1 + \frac{3a_3}{4}h^2 + \frac{5a_5}{8}h^4 = 0,$$

an easy calculation shows that $\Delta^u(0) - \Delta^s(0) = 0$. Hence we have proved that:

$$\frac{\partial^2 d}{\partial \varepsilon^2}(h^*, 0) = 0 \qquad (2)$$

and

$$\frac{\partial^3 d}{\partial h \partial \varepsilon^2}(h^*, 0) = 0 \qquad (3)$$

for

$$h^* = \left[\frac{-3a_3 \pm 3\sqrt{a_3^2 - \frac{40}{9}a_1 a_5}}{5a_5}\right]^{\frac{1}{2}}$$

By computing the third derivative of d with respect to ε in $(h^*, 0)$ we get:

$$\operatorname{sign}\frac{\partial^3 d}{\partial \varepsilon^3}(h^*, 0) = -\operatorname{sign} a_5. \qquad (4)$$

3.2. Let $f_a(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \in \mathbb{R}[x]$. We consider as in 3.1, the vector field $X^{\varepsilon f_a}(x, y) = (y - \varepsilon f(x), -x)$. By 3.1 (1), we have for a given $h > 0$,

$\frac{\partial d}{\partial \varepsilon}(h, 0) = -\pi h^2\left[a_1 + \frac{3a_3}{4}h^2 + \frac{5}{8}a_5 h^4\right]$. Then $\frac{\partial d}{\partial \varepsilon}(h, 0) = 0$ if and

only if $h^2 = \frac{3}{5}\left[\frac{-a_3 \pm \sqrt{\Delta}}{a_5}\right]$ or $h = 0$, where $\Delta = a_3^2 - \frac{40}{9}a_1 a_5$.

THEOREM A. *Let $\Delta \le 0$, or let $\Delta > 0$ and a_1, a_3, a_5 have the same sign. Then, there exists $\tilde{\varepsilon} > 0$ such that $X^{\varepsilon f_a}$ has no limit cycle, $\forall \varepsilon$, $0 < \varepsilon < \tilde{\varepsilon}$.*

Proof. Let $h_0 = \min\{|\alpha_i| / f_a'(\alpha_i) = 0\}$. If $a_1 \neq 0$, then $h_0 > 0$ and $\operatorname{div} X^{\varepsilon f_a} = -\varepsilon f_a'(x)$ has constant sign on the band $B_0 = \{(x, y) \in \mathbb{R}^2 / \|x\| < h_0\}$. Therefore $X^{\varepsilon f_a}$ has no limit cycle on B_0, $\forall \varepsilon > 0$.

i) If $\Delta < 0$ then $\partial d/\partial\varepsilon\ (h, 0) \neq 0$, $\forall h > 0$.

Let us remark that there are no limit cycles coming from infinity as is shown in [1]. In fact $\Delta < 0$ implies $a_5 \neq 0$ and it is enough to show the existence of $M > 0$ such that for all $y_0 > M$ there is no closed orbits of $X^{\varepsilon f_a}$ through $(0, y_0)$. We assume $a_5 > 0$. Consider the rectangle $R_\beta = \{(x, y) \,/\, |x| \leq \beta, |y| \leq 1\}$. Let $\omega \leq 0 \leq z$ be the roots of f_a such that all other roots are contained in the interval (a, b). Let k be the supremum of $|xf_a(x)|$ on [a, b]. Chose $\beta > 0$ so that $\beta > |\omega|, \beta > z$ and $f_a(\beta) > 2k(z - w)$.

Let $M > 0$ be such that the disc of center 0 and radius M contains R_β. Let $\tilde{\varepsilon} > 0$ be so small that if $\varepsilon \leq \tilde{\varepsilon}$ the segment of the orbit of $X^{\varepsilon f_a}$ between (0, M) and the first return to the positive vertical axis does not intersect R_β. We take also $\tilde{\varepsilon}$ so small that $|\varepsilon f_a(x)| < \frac{1}{2}$, $\forall x \in [\omega, z]$.

Let $y_0 > M$ and $P(y_0)$ be the first return of the positive orbit through $(0,y_0)$ to the positive vertical axis. Consider the function $V_\varepsilon(x, y) = \frac{1}{2\varepsilon}(x^2 + y^2)$. By evaluating the difference $V_\varepsilon(0, P(y_0)) - V_\varepsilon(0,y_0) = \int_0^T \dot{V}_\varepsilon dt$, where T is the time between $(0,y_0)$ and $(0,P(y_0))$,

we may write

$$\int_0^T \dot{V}_\varepsilon dt = \int_0^{T_1} \dot{V}_\varepsilon dt + \int_{T_1}^{T_2} \dot{V}_\varepsilon dt + \int_{T_2}^{T_3} \dot{V}_\varepsilon dt + \int_{T_3}^{T_4} \dot{V}_\varepsilon dt + \int_{T_4}^{T} \dot{V}_\varepsilon dt$$

where $T_1 < T_2 < T_3 < T_4$ are such that $x(y_0, T_1, \varepsilon) = x(y_0, T_2, \varepsilon) = z$, $x(y_0, T_3, \varepsilon) = x(y_0, T_4, \varepsilon) = \omega$ and $(x(y_0, t, \varepsilon), y(y_0, t, \varepsilon))$ is the considered orbit. Now, the above assumptions imply that

$$\left|\int_0^{T_1} \dot{V}_\varepsilon dt\right| \leq 2kz, \quad \left|\int_{T_2}^{T_3} \dot{V}_\varepsilon dt\right| \leq 2k(z - \omega), \quad \left|\int_{T_4}^{T} \dot{V}_\varepsilon dt\right| \leq -2k\omega,$$

$\int_{T_1}^{T_2} \dot{V}_\varepsilon \, dt \leq 2f_a(\beta) < 4k(z - \omega)$, $\int_{T_3}^{T_4} \dot{V}_\varepsilon \, dt < 0$ and therefore $\int_0^T \dot{V}_\varepsilon \, dt < 0$, and $V_\varepsilon(0, P(y_0)) < V_\varepsilon(0, y_0)$. Then, $X^{\varepsilon f_a}$ has no limit cycle on $\mathbb{R}^2$, for $\varepsilon \leq \tilde{\varepsilon}$.

ii) If $\Delta = 0$ and $a_1 \neq 0$, $a_5 \neq 0$, then $\frac{\partial d}{\partial \varepsilon}(h, 0) = 0$ for $h > 0$ if and only if

$h = h_c = \sqrt{-\frac{3a_3}{5a_5}}$. Since $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h_c, 0) = 0$ and $\frac{\partial^3 d}{\partial h^2 \partial \varepsilon}(h_c, 0) = -\frac{9\pi a_3^2}{a_5} \neq 0$,

then in order to prove that $d(h, \varepsilon)$ has a local extremum at $(h_c, 0)$, we have in the Taylor's formula, by 3.1 (1), (2), (3) and (4):

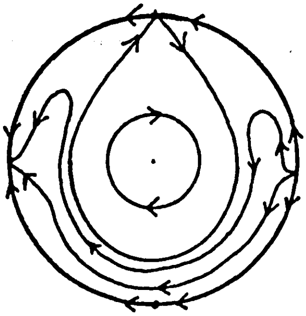
$$d(h, \varepsilon) = \frac{\varepsilon}{2}\left[\frac{\partial^3 d}{\partial h^2 \partial \varepsilon}(h_c, 0)(h - h_c)^2 + \frac{1}{3}\frac{\partial^3 d}{\partial \varepsilon^3}(h_c, 0)\varepsilon^2\right] + \text{h.o.t.}$$

But $\frac{4}{3}\frac{\partial^3 d}{\partial h^2 \partial \varepsilon}(h_c, 0) \cdot \frac{\partial^3 d}{\partial \varepsilon^3}(h_c, 0) \neq 0$ and it has the same sign as

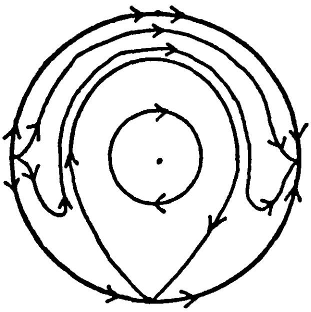
$12\pi \frac{a_3^2}{a_5}$. sgn $a_5 > 0$. Hence $d(h, \varepsilon)$ has a local maximum at $(h_c, 0)$ if $\frac{\partial^3 d}{\partial \varepsilon^3}(h_c, 0) < 0$ i. e. if $a_5 > 0$ and a local minimum if $\frac{\partial^3 d}{\partial \varepsilon^3}(h_c, 0) > 0$ i.e. if $a_5 < 0$. Similarly to i) there exists $\tilde{\varepsilon}_2 > 0$ such that $d(h, \varepsilon) \neq 0$ for all $h > 0$ and $\varepsilon \leq \tilde{\varepsilon}_2$ and there are no limit cycles coming from infinity or growing from the origin.

iii) If $\Delta = 0$, $a_1 = a_3 = a_5 = 0$, the vector field $X^{\varepsilon f_a}$ is a nonlinear center and then it has no limit cycles.

iv) $\Delta > 0$ and a_1, a_3, a_5 have the same sign or $\Delta = 0$, $a_1 a_5 = 0$ ($\Rightarrow a_3 = 0$). If $a_2 = a_4 = 0$ the function div $X^{\varepsilon f_a}$ is of constant sign, $\forall \varepsilon > 0$. If a_2 and a_4 are not both zero let us consider the Poincaré compactification of the vector field $X^{\varepsilon P}$, where $P(x)$ is the even polynomial $a_2 x^2 + a_4 x^4$.



$a_4 > 0$ or $a_4 = 0$ and $a_2 > 0$



$a_4 < 0$ or $a_4 = 0$ and $a_2 < 0$

Define V: $\mathbb{R}^2 \to \mathbb{R}^-$, V(0) = 0, V(z) = intersection of the flow of $X^{\varepsilon P}$ trough z, with the negative vertical axis, if $a_4 > 0$ (or $a_4 = 0$ and $a_2 > 0$) or with the positive vertical axis, if $a_4 < 0$ (or $a_4 = 0$ and $a_2 < 0$).
V is a C^∞ first integral of $X^{\varepsilon P}$ and we have:

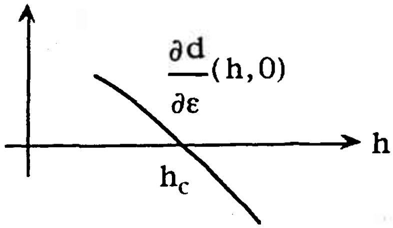
$$DV \cdot X^{\varepsilon P} = \frac{\partial V}{\partial x}(y - \varepsilon P(x)) + \frac{\partial V}{\partial y}(-x) = 0.$$

Then $\partial V/\partial x = x \cdot k(x, y)$ and $\partial V/\partial y = (y - P(x)) \cdot k(x, y)$, with $k(x, y) < 0$ if $(x, y) \neq (0, 0)$ (the origin is the unique maximum of V). Thus $DV \cdot X^{\varepsilon f_a} = -\varepsilon x^2 k(x, y)(a_1 + a_3x^2 + a_5x^4)$ is of constant sign on the plane. Then $X^{\varepsilon f_a}$ no closed orbit for $\varepsilon > 0$.

THEOREM B. *Let* $a_1 \cdot a_5 < 0$. *There exist* $\tilde{\varepsilon} > 0$ *such that* $X^{\varepsilon f_a}$ *has a unique limit cycle on* $\mathbb{R}^2$ *for all* $\varepsilon, 0 < \varepsilon < \tilde{\varepsilon}$.

Proof. Since $a_1 \neq 0$, there exist $h_0 > 0$, as in the proof of Theorem A, such that $X^{\varepsilon f_a}$ has no limit cycle on B_0, $\forall \varepsilon > 0$. We may assume, without loss of generality, that $a_1 < 0$, $a_5 > 0$. (This implies that the origin and the infinite are repellors). The derivative $\frac{\partial d}{\partial \varepsilon}(h, 0)$ has a unique positive root $h_c = \sqrt{\frac{3}{5}\left[\frac{-a_3 + \sqrt{\Delta}}{a_5}\right]}$ with $h_0 < h_c$, and in addition $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h_c, 0) = -2\pi h_c \left[a_1 + \frac{3}{2}a_3 h_c^2 + \frac{15}{8}a_5 h_c^4\right]$

$= -\frac{3\pi}{2} h_c^3 \sqrt{\Delta} \neq 0$ and sign $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h_c, 0) =$ sign a_1. Therefore the graph of $\frac{\partial d}{\partial \varepsilon}$ (h, 0) will be locally at h_c:



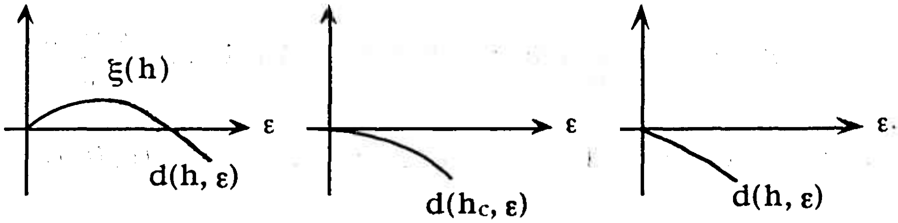
Furthermore, by 3.1 (2) and 3.1 (4) we have $\frac{\partial^2 d}{\partial \varepsilon^2}(h_c, 0) = 0$ and $\frac{\partial^3 d}{\partial \varepsilon^3}(h_c, 0) < 0$, thus $\frac{\partial d}{\partial \varepsilon}(h, 0) = \alpha(h)(h - h_c)$ where

$$\alpha(h) = -\frac{5}{8}\pi a_5 h^2 (h + h_c)\left[h^2 - \frac{3}{5}\left(\frac{-a_3 - \sqrt{\Delta}}{a_5}\right)\right] < 0, \forall h.$$

Consequently $d(h, \varepsilon) = \varepsilon\alpha(h)(h - h_c) - \varepsilon^3\beta(h, \varepsilon)$ with $\beta(h_c, 0) > 0$ for all sufficiently small $\varepsilon > 0$. Then, if $h > h_c$, $d(h, \varepsilon) < 0$ and if $h < h_c$, $\beta(h, \varepsilon) > 0$ for $h_c - \delta < h < h_c$ and $\forall \varepsilon, 0 < \varepsilon < \tilde{\varepsilon}_1$. Formally, by the Division Theorem, one obtains:

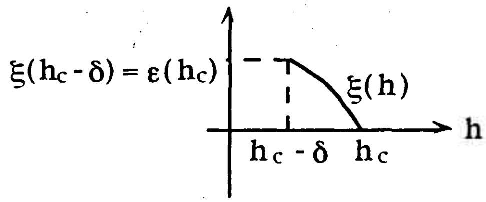
$$d(h, \varepsilon) = \varepsilon\left[\sqrt{\alpha(h)(h - h_c)} + \varepsilon\sqrt{\beta(h, \varepsilon)}\right]\left[\sqrt{\alpha(h)(h - h_c)} - \varepsilon\sqrt{\beta(h, \varepsilon)}\right].$$

Thus $d(h, \varepsilon) = 0$ for $\varepsilon > 0$ if and only if $\sqrt{\alpha(h)(h - h_c)} - \varepsilon\sqrt{\beta(h, \varepsilon)} = 0$. By the Implicit Function Theorem, there exists $\xi(h)$ such that $d(h, \xi(h)) = 0$. We conclude that the graphs of $d(h, \varepsilon)$ in a neighborhood of h_c are as follows:



$h < h_c$ $\qquad$ $h = h_c$ $\qquad$ $h > h_c$

The function $\xi(h)$ is differentiable and has the graph:

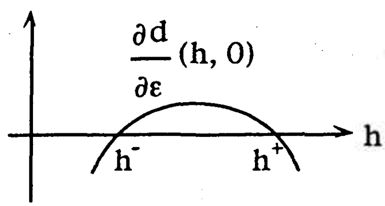


Therefore there exists $\varepsilon(h_c) = \xi(h_c - \delta) > 0$, $\delta > 0$ such that $X^{\varepsilon f_a}$ has a unique limit cycle on the band $B(h_c) = \{(x, y) \in \mathbb{R}^2 / h_c - \delta < \|(x, y)\| < h_c + \delta\}$, $\forall\varepsilon$, $0 < \varepsilon < \varepsilon(h_c)$.
In addition $\forall h < h_c$, $\partial d/\partial\varepsilon(h, 0) > 0$ and $\forall h > h_c$, $\partial d/\partial\varepsilon(h, 0) < 0$ implies that there exist $\tilde{\varepsilon} > 0$ such that $X^{\varepsilon f_a}$ has a unique limit cycle on $\mathbb{R}^2$, $\forall\varepsilon$, $0 < \varepsilon < \tilde{\varepsilon}$. (As before, we can show that there are no limit cycle coming from infinity because $a_5 \neq 0$).

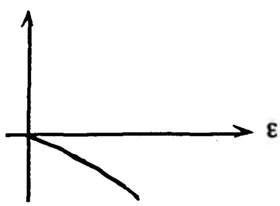
THEOREM C. *Let* $\Delta > 0$ *and let* a_1, a_3, a_5 *be of alternated signs. There exists* $\tilde{\varepsilon} > 0$ *such that* $X^{\varepsilon f_a}$ *has two limit cycles on* $\mathbb{R}^2$, $\forall\varepsilon$, $0 < \varepsilon < \tilde{\varepsilon}$.

Proof. Since $a_1 \neq 0$ there exists $h_0 > 0$, as in the proof of Theorem A such that $X^{\varepsilon f_a}$ has no limit cycle in B_0, $\forall\varepsilon > 0$.
Let us assume, without loss of generality, that $a_1 > 0$ (thus $a_5 > 0$).
The derivative $\dfrac{\partial d}{\partial\varepsilon}(h, 0)$ has two positive roots: $h^* = \sqrt{\dfrac{3}{5}\left[\dfrac{-a_3 \pm \sqrt{\Delta}}{a_5}\right]}$

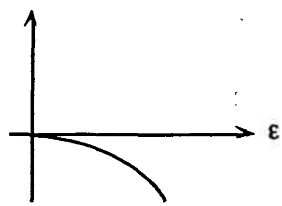
with $h_0 < h^- < h^+$ and $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h^\pm, 0) = \mp \frac{3\pi}{2}\sqrt{\Delta}\,(h^\pm)^3$. Hence $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h^-)$ > 0 and $\frac{\partial^2 d}{\partial h \partial \varepsilon}(h^+) < 0$. This shows that the graph of $\frac{\partial d}{\partial \varepsilon}(h, 0)$ is:



By 3.1 (2) and 3.1 (4), we have $\frac{\partial^2 d}{\partial \varepsilon^2}(h^\pm, 0) = 0$ and $\frac{\partial^3 d}{\partial \varepsilon^3}(h^\pm, 0) < 0$. This implies that $d(h, \varepsilon) = \varepsilon\alpha(h)(h - h^-)(h - h^+) - \varepsilon^3\beta(h, \varepsilon)$ with $\beta(h^\pm, 0) > 0$ for all small $\varepsilon > 0$. Since $\alpha(h) < 0$ for all h, $d(h, \varepsilon) < 0$ if $h < h^-$, $\forall\varepsilon$ small enough. This says that the graphs of $d(h, \varepsilon)$ for $h \leq h^-$ are:



$h^- - \delta < h < h^-$



$h = h^-$

Now, if h^- and h^+ are sufficiently close to each other (that is, if $a_1 \cdot a_5$ is small), we have $\beta(h, \varepsilon) > 0$, $\forall h$, $h^- \leq h \leq h^+$, $\forall \varepsilon$, $0 < \varepsilon < \tilde{\varepsilon}$. Then, formally, by the Division Theorem

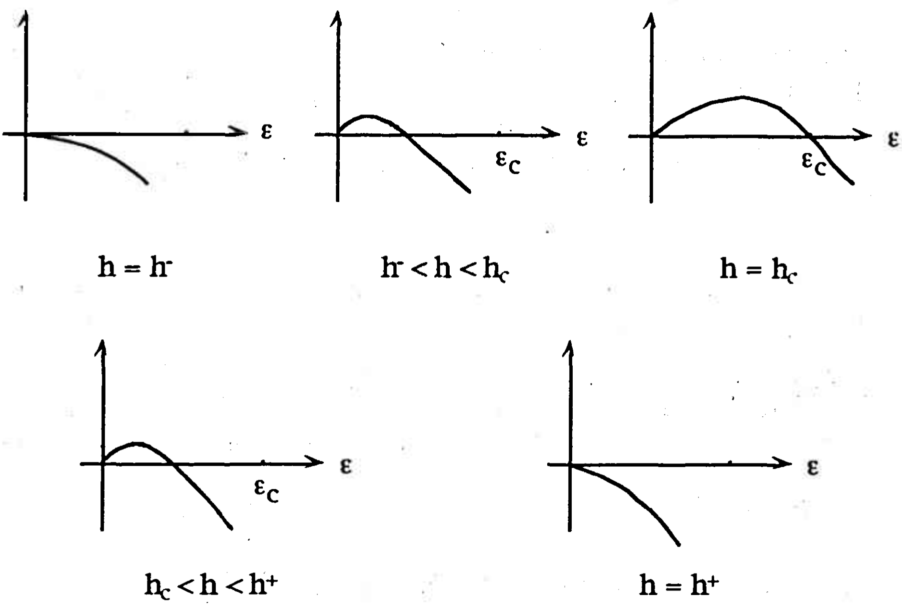
$$d(h,\varepsilon) = \varepsilon\left[\sqrt{\alpha(h)(h-h^-)(h-h^+)} + \varepsilon\sqrt{\beta(h,\varepsilon)}\right] \cdot \left[\sqrt{\alpha(h)(h-h^-)(h-h^+)} - \varepsilon\sqrt{\beta(h,\varepsilon)}\right].$$

So $d(h, \varepsilon) = 0$ for $\varepsilon > 0$ if and only if $\sqrt{\alpha(h)(h-h^-)(h-h^+)} - \varepsilon\sqrt{\beta(h,\varepsilon)} = 0$; the Implicit Function Theorem implies the existence or $\xi(h)$ such that $d(h, \xi(h)) = 0$. Moreover

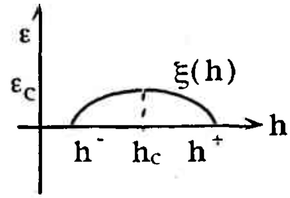
$$\xi'(h) = 0 \Leftrightarrow \frac{d}{dh}\left[\alpha(h)(h-h^-)(h-h^+)\right] = 0 \Leftrightarrow \frac{\partial^2 d}{\partial h \partial \varepsilon}(h, 0) = 0.$$

The last equation has a unique positive root h_c such that $h^- < h_c < h^+$.

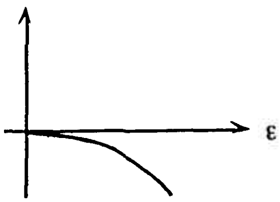
From the above arguments we can conclude that the graphs of $d(h, \varepsilon)$ for $h^- \leq h \leq h^+$ are:



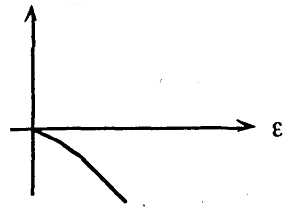
where ε_c is the unique maximum of $\xi(h)$:



Now, if $h \geq h^+$, $d(h, \varepsilon) < 0$ and we have the graphs:



$h = h^+$



$h^+ < h < h^+ + \delta$

This shows that the vector field $X^{\varepsilon f_a}$ has two limit cycles on the band $B = \{(x, y) \in \mathbb{R}^2 / h^- - \delta < \|(x, y)\| < h^+ + \delta\}$, $\forall \varepsilon$, $0 < \varepsilon < \varepsilon_c$. We obtain a similar result if $a_1 < 0$ (then $a_5 < 0$).
As before we conclude that $X^{\varepsilon f_a}$ has two limit cycles on $\mathbb{R}^2$, $\forall \varepsilon$, $0 < \varepsilon < \tilde{\varepsilon}$ with $\tilde{\varepsilon} \leq \varepsilon_c$. ■

All the above results together with those of [1], are summarized in the following:

3.3. Conclusion. Let $a = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$, $f_a(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ and let us consider the Liénard equation of degree 5

$$X^{\varepsilon f_a} : \begin{cases} \dot{x} = y - \varepsilon f_a(x) \\ \dot{y} = -x \end{cases}, \text{ for } \varepsilon > 0 \text{ a small parameter.}$$

Finally let $\Delta(a) = a_3^2 - \frac{40}{9} a_1 a_5$. We have:

(1) If $a_1 \neq 0$, the origin is a hyperbolic focus, a repellor if $a_1 < 0$ and an attractor if $a_1 > 0$.

(2) If $a_1 = 0$ and $a_3 \neq 0$, the origin is a weak focus, a repellor if $a_3 < 0$ and an attractor if $a_3 > 0$.(Idem if $a_1 = a_3 = 0$, $a_5 \neq 0$).

(3) If $a_1 = a_3 = a_5 = 0$, the origin is a center (usually a nonlinear one).

(4) If $a_5 > 0$, the infinite is a repellor and if $a_5 < 0$, the infinite is an attractor. (Idem if $a_5 = 0$ and $a_3 \neq 0$, or $a_5 = a_3 = 0$ and $a_1 \neq 0$) ([3]).

(5) When a_1 changes sign, with a_3 or $a_5 \neq 0$ then we have a Hopf bifurcation at the origin.

(6) When a_5 changes sign, with a_3 or $a_1 \neq 0$ then we have a bifurcation at infinity: a limit cycle appears from infinity.

(7) When a_3 changes sign with $a_1 = 0$ and $a_5 \neq 0$ we have a generalized Hopf bifurcation at the origin. If $a_1 \neq 0$ and $a_5 = 0$ we have a bifurcation at infinity.

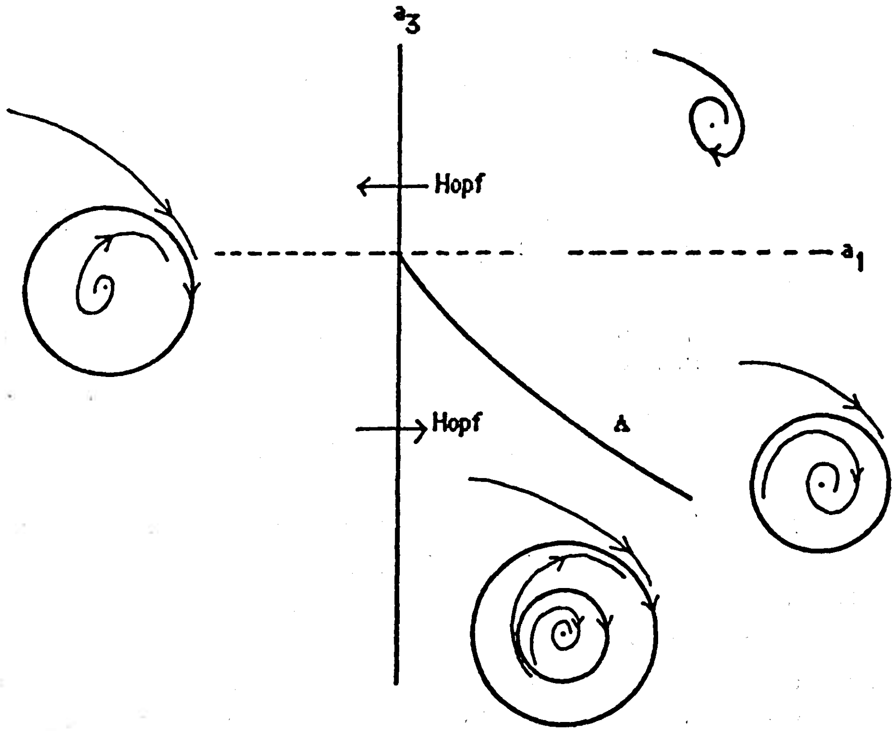
(8) If $\Delta(a) \leq 0$, there exists $\varepsilon(a)$ such that $X^{\varepsilon f_a}$ has no limit cycle for all ε, $0 < \varepsilon < \varepsilon(a)$.

(9) If $\Delta(a) > 0$, and a_1, a_3, a_5 have the same sign, there exists $\varepsilon(a)$ such that $X^{\varepsilon f_a}$ has no limit cycles for all ε, $0 < \varepsilon < \varepsilon(a)$.

(10) If $a_1 \cdot a_5 < 0$, there exists $\varepsilon(a)$ such that $X^{\varepsilon f_a}$ has a unique limit cycle for all ε, $0 < \varepsilon < \varepsilon(a)$.

(11) If $\Delta(a) > 0$, and a_1, a_3, a_5 are of alternated sign, then there exists $\varepsilon(a)$, $\varepsilon_c(a)$, $\varepsilon_c(a) < \varepsilon(a)$, such that
$X^{\varepsilon f_a}$ has two limit cycles for all ε, $0 < \varepsilon < \varepsilon_c(a)$,
$X^{\varepsilon f_a}$ has a unique limit cycle (non hyperbolic) if $\varepsilon = \varepsilon_c(a)$, and
$X^{\varepsilon f_a}$ has no limit cycle for $\varepsilon_c(a) < \varepsilon < \varepsilon(a)$.

(12) There exists a neighborhood $\mathcal{U}$ of the origin of $\mathbb{R}^5$ and a cone C containing the plane a_2, a_4 and being tangent to it at the origin, such that the family of vector fields $X^{\varepsilon f_a}$, with $\varepsilon a \in K = \mathcal{U} - C = \{\varepsilon a : a \in \mathbb{R}^5, \varepsilon < \varepsilon(a)\}$, has the following bifurcation diagram:



$a_5 > 0$

(The case $a_5 < 0$ is analogous).

Here Λ denotes the bifurcation set $\{\varepsilon_c(a) \cdot a \in \mathbb{R}^5 / \Delta(a) > 0$ and a_1, a_3, a_5 of alternated signs$\}$ and corresponds to a coalescence of limit cycles.
The hyperplane $a_1 = 0$ is a set of Hopf bifurcations.
The hyperplane $a_5 = 0$ is a set of bifurcation at infinity.

Remark. Indeed, the existence and non existence of limit cycles for the family of vector fields $X^{\varepsilon f_a}$ follows from the analysis of the function $\partial d/\partial\varepsilon(h, 0)$ and its derivatives.
In fact:

i) If $\partial d/_{\partial\varepsilon}(h, 0) \neq 0$ for all $h > 0$ then $X^{\varepsilon f_a}$ has no limit cycle for $\varepsilon \to 0$.

ii) If $\partial d/_{\partial\varepsilon}(h_c, 0) = 0$ and $\partial^2 d/_{\partial h \partial\varepsilon}(h_c, 0) \neq 0$ for $h_c > 0$, then $X^{\varepsilon f_a}$ has, for $\varepsilon \to 0$, a unique limit cycle on a neighborhood of $x^2 + y^2 = h_c^2$.

iii) If $\partial d/_{\partial\varepsilon}(h_c, 0) = \partial^2 d/_{\partial h \partial\varepsilon}(h_c, 0) = 0$ and $\partial^3 d/_{\partial^2 h \partial\varepsilon}(h_c, 0) \neq 0$ then $X^{\varepsilon f_a}$ has, for $\varepsilon \to 0$, at most two limit cycles on a neighborhood of the circle $x^2 + y^2 = h_c^2$. Other situations are not possible because $\partial d/_{\partial\varepsilon}(h, 0)$ has at most two positive roots.

Finally, remark that the calculus of the derivatives $\partial^2 d/_{\partial\varepsilon^2}(h, 0)$ $(= 0)$ and $\partial^3 d/_{\partial\varepsilon^3}(h, 0)$ $(\neq 0)$ is necessary to assure that the function $d(h, \varepsilon)$ is well-shaped, that is $d(h_c, \varepsilon)$ is not identically zero for h_c a root of $\partial d/_{\partial\varepsilon}(h, 0)$.

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