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PAIRWISE S-NORMAL SPACES

by

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ABSTRACT. Extending the concept of s - normal spaces to bitopological spaces, the concept of pairwise s - normal spaces is introduced. A space X is said to be s - normal if any two disjoint semi - closed subsets of X can be separated by disjoint semi - open sets. A space (X, τ_1, τ_2) is said to be pairwise s - normal if for any τ_i- semi-closed set A and a τ_j - semi-closed set B disjoint from A, there exists a τ_i - semi - open set U and a τ_j - semi - open set V such that $A \subseteq V$, $B \subseteq U$ and $U \cap V = \phi$ where $i \neq j$; $i, j = 1, 2$. Several characterizations and other results concerning pairwise s - normal spaces have been obtained.

In [1], the authors introduced the concept of s-normal spaces. A space X is said to be s-normal if for any two disjoint semi-closed subsets A and B of X there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. The purpose of the present paper is to extend this concept to bitopological spaces. It is shown that a pairwise normal space of Kelly [7] need not be pairwise s-normal and a pairwise s-normal space need not be pairwise normal.

A set $A \subseteq X$ is said to be *semi-open* [9] if there exists an open set $U \subseteq X$ such that $U \subseteq A \subseteq \mathrm{cl}\, U$, $\mathrm{cl}\, U$ denoting the closure of U. A complement of a semi-open set is said to be *semi-closed* [3]. The semi-closure of A, denoted by scl A, is the intersection of all semi-closed sets containing A. The semi-interior of A, denoted by sint A, is the union of all semi-open sets contained in A. In section 1 several properties of pairwise s-normal spaces are studied and examples are given in section 2.

§1. Pairwise s-normal Spaces.

1.1 DEFINITION [7]. A space (X, τ_1, τ_2) is said to be *pairwise normal* if for each τ_i-closed set A and a τ_j-closed set B disjoint from A, there exists a τ_j-open set U and a τ_i-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$, where i, j = 1, 2; $i \neq j$.

1.2 DEFINITION. A space (X, τ_1, τ_2) is said to be *pairwise s-normal* if for any τ_i-semi-closed set A and a τ_j-semi-closed set B disjoint from A, there exists a τ_i-semi-open set V and a τ_j-semi-open set U such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$, where i, j = 1, 2; $i \neq j$.

1.3 DEFINITION. A space X is said to be *pairwise semi-normal* if for every τ_i-closed set A and a τ_j-closed set B disjoint from A, there exists a τ_i-semi-open set U and a τ_j-semi-open set V such that $A \subseteq V$, $B \subseteq U$ and $U \cap V = \emptyset$, where i, j = 1, 2; $i \neq j$.

In fact the concept of pairwise semi-normal spaces defined above is due to Maheshwary and Prasad [11] who called them pairwise s-normal spaces.

Obviously, every pairwise s-normal space is pairwise semi-normal. But the converse need not be true as can be seen from Examples 2.1.

1.4 REMARK. The examples 2.1 and 2.2 show that a pairwise normal space need not be pairwise s-normal and a pairwise s-normal may fail to be pairwise normal.

1.5 THEOREM. *A space* (X, τ_1, τ_2) *is pairwise* s-*normal if and only if for every* τ_i-*semi-closed set* A *and a* τ_j-*semi-open set*

B *containing* A, *there exists a* τ_j*- semi-open set* U *such that* $A \subseteq U \subseteq \tau_i\text{-scl } U \subseteq B$, $i \neq j$; $i, j = 1, 2$.

Proof. Let (X, τ_1, τ_2) be pairwise s-normal. Let F be a τ_i-semi-closed set and U a τ_j- semi-open set containing F, $i \neq j$; $i, j = 1, 2$. Hence there exists a τ_i - semi-open set G and a τ_j - semi-open set K such that $F \subseteq K$, $(X - U) \subseteq G$ and $G \cap K = \varnothing$. That is, $F \subseteq K \subseteq X - G \subseteq U$ which implies that $F \subseteq K \subseteq \tau_i\text{-scl } K \subseteq X - G \subseteq U$, since G is τ_i-semi- open. Thus there exists a τ_j - semi-open set K containing F such that $F \subseteq K \subseteq \tau_j\text{-scl} K \subseteq U$ where $i \neq j$; $i, j = 1, 2$.

Conversely, let F_1 be a τ_i- semi-closed set and let F_2 be a τ_j-semi- closed set disjoint from F_1, $i \neq j$; $i, j = 1, 2$. Then $X - F_2$ is a τ_j-semi- open set containing F_1. Hence in view of the hypothesis, there exists a τ_j - semi- open set V such that $F_1 \subseteq V \subseteq \tau_i\text{-scl } V \subseteq X - F_2$. Now $F_1 \subseteq V$ and $F_2 \subseteq X - \tau_i\text{-scl} V$. Thus X is pairwise s- normal.

1.6 DEFINITION. A real valued function f on a space X is said to be *quasi- lower semi- continuous* (denoted as q.l.s.c.) if the set $\{x : f(x) > a\}$ is a semi- open subset of X and f is said to be *quasi- upper semi- continuous* (denoted as q.u.s.c.) if the set $\{x : f(x) < b\}$ is a semi- open subset of X where a and b are any two real numbers.

1.7 LEMMA. *Let* D *be any dense subset of the set of all positive real numbers. To each* $t \in D$ *there corresponds a* τ_j *- semi-open subset* U_t *of a space* (X, τ_1, τ_2) *such that* $t < s$ *in* D *implies that* $\tau_i\text{-scl } U_t \subseteq U_s$ *and* $\bigcup_{t \in D} U_t = X$. *Then the function* f *defined as* $f(x) = \inf\{t : x \in U_t\}$ is τ_j - q.u.s.c. *and* τ_i - q.l.s.c., $i \neq j$; $i, j = 1, 2$.

Proof. The set $f^{-1}\{t : t < b\}$ is τ_j - semi- open, $j = 1, 2$, since each U_t is τ_j - semi- open and $f^{-1}\{t : t < b\} = \bigcup\{U_t : t \in D, t < b\}$ (Lemma 4.2 [6]). Thus f is τ_j - q.u.s.c. Now $f^{-1}\{t : t > a\} = X - f^{-1}\{t : t \leq a\} = X - \bigcap\{U_t : t > a \text{ and } t \in D\}$ in view of Lemma 4.2 of [6]. It can be proved as in the proof of Lemma 7 of [1] that $\bigcap\{U_t : t \in D, t > a\} = \bigcap\{\tau_i\text{- scl } U_t : t \in D, t > a\}$, $i \neq j$; $i, j = 1, 2$. Thus $f^{-1}\{t :$

$t > a\} = X - \bigcap\{\tau_i\text{-}\mathrm{scl}\ U_t : t \in D,\ t > a\}$ a τ_i - semi- open set. Thus f is τ_i - q.l.s.c.

1.8 THEOREM. *A space* (X, τ_1, τ_2) *is pairwise* s- *normal if and only if for a* τ_i - *semi-closed set* A *and a* τ_j - *semi - closed set* B *disjoint from* A, *there exists a* τ_j - *q. u. s. c. and* τ_i - *q. l. s. c. function* $f : X \to [0, 1]$ *such that* $f(x) = 0$ *for* $a \in A$ *and* $f(x) = 1$ *for* $x \in B$.

Proof. The easy proof of the 'if' part is omitted.

To prove the 'only if' part, let X be pairwise s - normal and let A be a τ_i - semi - closed set and B be a τ_j - semi - closed set disjoint from A where $i \neq j$. Let Q be the set of all positive rational num bers. For each $t \in Q$ let us define a τ_j - semi- open set U_t, as follows: For $t > 1$, let $U_t = X$. Let $U_1 = X - B$, which is a τ_j - semi - open set contained A. Therefore, in view of Theorem 1.5, there exist a τ_j - semi - open set, say U_0, such that $A \subseteq U_0 \subseteq \tau_i\text{-scl}\ U_0 \subseteq U_1 = X - B$. Let $\{t_n : n \in \mathbb{N}\}$ be the sequence of rational numbers in [0, 1] with $t_1 = 0$ and $t_2 = 1$. For each $n \geq 3$, we shall inductively define the set U_{t_n} in the following way. Let t_k be the largest number such that $t_k < t_n$ and t_s be the smallest number such that $t_n < t_s$ where k, $s < n$. Now corresponding to t_k and t_s, the U_{t_k} and U_{t_s} are defined as: U_{t_s} is a τ_j - semi - open set containing τ_i–slc U_{t_k}. In view of Theorem 1.5 there exists a τ_j - semi - open set, say U_{t_n}, such that $\tau_i\text{-slc}\ U_{t_k} \subseteq U_{t_n} \subseteq \tau_i\text{-slc}\ U_{t_n}$. Thus, U_t is defined for each $t \in Q$ such that for $t_1 < t_2$, U_{t_2} is a τ_j - semi - open set containing τ_i - slc U_{t_1}, $i \neq j$; i, j = 1, 2 and $\bigcup_{t \in D} U_t = X$.

Let us define a real valued function f on X as $f(x) = \inf\{t : x \in U_t\}$. In view of Lemma 1.7, f is τ_j - q.u.s.c. and τ_i - q.l.s.c., $i \neq j$; i, j = 1, 2 . It can be easily verified that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

1.9 DEFINITION [10]. A space (X, τ_1, τ_2) is said to be *pairwise-semi-* T_1 if for any two distinct points x and y of X there exists a τ_1 - semi - open set U and a τ_2 - semi - open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

1.10 DEFINITION [2]. A space (X, τ_1, τ_2) is said to be *pairwise completely s-regular* if for any point x and a τ_i- closed set F not containing x, there exists a τ_i - q.u.s.c. and τ_j - q.l.s.c. function f on X such that $f(x) = 0$ and $f(F) = 1$ where $i \neq j$; $i, j = 1, 2$.

1.11 DEFINITION [12]. A space (X, τ_1, τ_2) is said to be *pairwise s-regular* if for each τ_i - closed set F and a point $x \notin F$, there exists a τ_j - semi- open set U and a τ_i - semi- open set V such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ where $i \neq j$; $i, j = 1, 2$.

1.12 REMARK. In view of Theorem 1.8 it can be observed that every pairwise semi-τ_1 pairwise s-normal space is pairwise completely s-regular. In 1.14 we prove that a pairwise s-normal space is pairwise completely s-regular if and only if it is pairwise s-regular. In 2.3 we give examples to show that a pairwise semi-τ_1 space may fail to be either pairwise s-normal or pairwise s-regular. Also, the space in Example 2.11 is pairwise s-regular and pairwise semi-τ_1 but not pairwise s-normal.

1.13 LEMMA [14]. *A space* (X, τ_1, τ_2) *is pairwise* s-*regular if and only if for each point* $x \in X$ *and each* τ_i - *open set* V *containing* x, *there exists a* τ_i - *semi-open set* U *such that* $x \in U \subseteq \tau_j$ - scl $U \subseteq V$ *where* $i \neq j$; $i, j = 1, 2$.

1.14 THEOREM. *A pairwise* s- *normal space is pairwise completely* s - *regular if and only if it is pairwise* s - *regular.*

Proof. The 'only if' part is immediate in view of the fact that every pairwise completely s-regular space is pairwise s-regular.

To prove the 'if' part, let F be a τ_i - closed set and let $x \in X - F$. Then $X - F$ is a τ_i - open set containing x. Hence in view of Lemma 1.13, there exists a τ_i - semi-open set V such that $x \in V \subseteq \tau_j$ - scl $V \subseteq X - F$ where $i \neq j$; $i, j = 1, 2$. Now in view of Theorem 1.8 there exists a τ_i - q.u.s.c. and τ_j - q.l.s.c. function f on X such that $f(F) = 1$ and $f(\tau_j\text{-scl } V) = 0$. Thus $f(x) = 0$ and $f(F) = 1$. Hence X is pairwise completely s-regular.

In [8], Lane proved that if (X, τ_1, τ_2) is a pairwise normal space, g and f are functions on X such that f is τ_1 - l.s.c. and g is τ_2 - u.s.c. and $g(x) \leq f(x)$ for every $x \in X$, then there exists a τ_1-l.s.c. and τ_2 - u.s.c. function h on X such that $g \leq h \leq f$. We obtain a similar result for pairwise s-normal spaces.

1.15 THEOREM. *If a space* (X, τ_1, τ_2) *is pairwise s-normal, then for every pair of functions* f *and* g *defined on* X *such that* f *is* τ_1 - q.l.s.c. *and* g *is* τ_2 - q.u.s.c. *and* $g(x) \leq f(x)$ *for every* $x \in X$, *there exists a* τ_1 - q.l.s.c. *and* τ_2 - q.u.s.c. *function* h *on* X *such that* $g \leq h \leq f$.

Proof. Let P be the power set of X and ρ be the relation defined on P as $A \rho B$ if only if $\tau_1\text{-scl } A \subseteq \tau_2\text{-sint } B$. The relation ρ satisfies the following three conditions: (i) Let $A = \{A_1, A_2, \ldots, A_m\}$, and $B = \{B_1, B_2, \ldots, B_n\}$ be two finite subcollections of P. Suppose that $A \rho B$. That is, $A_i \rho B_j$ for every i and j, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. $A_i \rho B_j$ implies that $\tau_1\text{-scl } A_i \subseteq \tau_2\text{-sint } B_j$. Hence in view of Theorem 7, there exists a τ_2 - semi-open set $C \subseteq X$ such that $\tau_1\text{-scl } A_i \subseteq C \subseteq \tau_1\text{-scl } C \subseteq \tau_2\text{-sint } B_j$. Thus there exists a $C \in P$ such that $A \rho C$ and $C \rho B$. (ii) Let $A, B \in P$. We shall prove that $A \subseteq B$ implies that $A \bar{\rho} B$ where $\bar{\rho}$ is defined as: $A \bar{\rho} B$ if and only if $B \rho D$ implies $A \rho D$ and $C \rho A$ implies $C \rho B$ for any C and D belonging to P. Let $A \subseteq B$. Let C an D be any two members of P. Then $B \rho D$ implies that $\tau_1\text{-scl} \subseteq B \subseteq \tau_2\text{-sint } D$. Therefore, $\tau_1\text{-scl } A \subseteq \tau_2\text{-sint } D$ which means that $A \rho D$. Also $C \rho A$ implies that $\tau_1\text{-scl } C \subseteq \tau_2\text{-sint } A \subseteq \tau_2\text{-sint } B$. Hence $C \rho B$. That is, $A \subseteq B$ implies that $A \bar{\rho} B$. (iii) Let $A \rho B$. Then $\tau_1\text{-scl } A \subseteq \tau_2\text{-sint } B$ which means that $A \subseteq B$. Hence in view of Lemma 1 of [5] ρ satisfies the following properties (a) and (b):

(a) Let U an V be two countable subcollections of P. Let A, $B \in P$ such that $U \overline{\rho} A$, $A \rho V$, $U \rho B$ and $B \overline{\rho} V$. Then there exists a $C \in P$ such that $U \rho C$ and $C \rho V$.

(b) For any finite subcollection U of P, there exist A, $B \in P$ such that (i) $U \overline{\rho} A$ and $A \rho C$ whenever $U \rho C$, (ii) $B \overline{\rho} U$ and $C \rho B$ whenever $C \rho U$ where $c \in P$.

Let σ be the natural order in the set Q of rational numbers. Let F and G be two functions defined from Q into the power set P of X as $F(t) = \{x \in X : f(x) \leq t\}$ and $G(t) = \{x : g(x) < t\}$. Since f is τ_1 - q.l.s.c. and g is τ_2 - q.u.s.c., F(t) is τ_1 - semi-closed and G(t) in τ_2 - semi - open. Since σ is the natural order on Q, we have for F, $G \in P^Q$, $F \rho^{\sigma} G$, $F \overline{\rho}^{\sigma} G$ and $G \overline{\rho}^{\sigma} G$. Therefore in view of Lemma 2 of [5] there exists a function U from Q into X such that $F \rho^{\sigma} U$, $U \rho^{\sigma} U$ and $U \rho^{\sigma} G$. That is, $t_1 < t_2$ in Q, $F(t_1) \rho U(t_2)$, $U(t_1) \rho U(t_2)$ and $U(t_1) \rho G(t_2)$. Since $F(t_1)$ is τ_1 - semi - closed and $G(t_2)$ is τ_2 - semi - open, we have from the above relation $F(t_1) \subset \tau_2$ - sint $U(t_2)$, τ_1 - slc $U(t_1) \subseteq \tau_2$ - sint $U(t_2)$ and τ_1 - scl $U(t_1) \subseteq G(t_2)$. Now for each x in X, let us define a function h from X to Q as $h(x) = \inf\{t \in Q : x \in U(t)\}$. h is a real valued function on X such that $g(x) \leq h(x) \leq f(x)$ for each x in X. Now it remains to be proved that h is τ_1 - q.l.s.c. and τ_2 - q.u.s.c. Let $x \in X$ and let ε be a positive number. We can choose a t' in Q such that $h(x) - \varepsilon < t' < h(x)$. There exists a t in Q such that $t' < t < h(x)$. Since $t < h(x)$, $x \notin U(t)$. Also, since $t' < t$, τ_1 - scl $U(t') \subseteq \tau_2$ - sint $U(t) \subseteq U(t)$. Thus $x \in X - \tau_1$ - scl $U(t')$, a τ_1 - semi - open set. Now consider $t \in Q$ such that $t < t'$. We have $U(t) \subseteq \tau_1$ - scl $U(t) \subseteq \tau_2$ - sint $U(t')$ $\subseteq \tau_1$ - scl $U(t')$. So, if $p \in X - \tau_1$ - scl $U(t')$, then for $t < t'$, $p \notin U(t)$. Therefore, $h(p) \geq t'$. Thus for $p \in X - \tau_1$ - scl $U(t')$, $h(x) - \varepsilon < h(p)$. Hence h is τ_1 - q. l. s. c. Let us now take t' in Q such that $h(x) < t' < h(x) + \varepsilon$. Choose t in Q such that $h(x) < t < t'$. Since $h(x) < t$, $x \in U(t)$. Since $t < t'$, τ_1 - scl $U(t) \subseteq \tau_2$ - sint $U(t')$. Thus $x \in \tau_2$ - sint $U(t')$, a τ_2 - semi - open set. Hence h is τ_2 - q.u.s.c. Thus the proof is complete.

1.16 DEFINITION [13]. A subset A of a space X is said to be an *α - set* if $A \subseteq int(cl(int\ A)))$.

1.17 LEMMA [1]. *Let* $Y \subseteq X$ *be semi - closed and* α. *If* A *is a semi- closed subset of* Y, *then* A *is a semi - closed subset of* X.

1.18 THEOREM. *Every bi- α, bi - semi - closed subset of pair- wise* s - *normal space is pairwise* s - *normal.*

Proof. Using Lemma 17 and the fact that the intersection of a semi- open set with an α - set is semi - open [13], the result can be easily proved.

1.19 DEFINITIONS. A function f: X → Y is said to be *semi- continuos* [9] (respectively, *irresolute* [4]) if $f^{-1}(A)$ is semi - open for every open (respectively, semi - open) subset A of Y. f : X→Y is said to be *semi- closed* [15] if the image of every closed subset of X is semi - closed.

1.20 DEFINITION. A function f: X → Y is said to be *pre- semi- closed* if the image of every semi - closed subset of X is semi- closed.

Clearly, every pre - semi - closed function is semi - closed. But the converse is not necessarily true as can be seen from the following example.

1.21 EXAMPLE. Let X = {a, b, c}, τ_1 = {ø, X, {a}} and τ_2 = {ø, X, {a}, {a, b }}. Let f: (X,τ_1) → (X,τ_2) be the identity function. Then f is semi - closed but not pre - semi - closed.

1.22 DEFINITIONS. A function f: (X, τ_1, τ_2) → (Y, U_1, U_2) is said to be *pairwise semi - continuos* (respectively, *irresolute, semi - closed, pre - semi - closed*) if f: (X, τ_1) → (Y, U_1) and f: (X, τ_2) → (Y, U_2) are semi - continuos (respectively, irresolute, semi - closed, pre - semi - closed).

1.23 THEOREM. *A pairwise irresolute pairwise pre-semi-closed image of a pairwise s-normal space is pairwise s-normal.*

Proof. Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, U_1, U_2)$ be pairwise irresolute pairwise pre-semi-closed and (X, τ_1, τ_2) be pairwise s-normal. Let A be a U_i- semi-closed and B be a U_j- semi-closed subsets of Y such that $A \cap B = \emptyset$, $i \neq j$; $i, j = 1, 2$.
Then there exists a τ_j- semi-open set P and τ_i- semi-open set K containing the τ_i- semi- open set $f^{-1}(A)$ and τ_j- semi- closed set $f^{-1}(B)$ respectively and $P \cap K = \emptyset$. Let $P_1 = Y - f(X - P)$ and $K_1 = Y - f(X - K)$. Since f is pairwise pre-semi-closed, P_1 is U_j- semi-open and K_1 is U_i- semi-open where $i \neq j$, i, j, = 1, 2. Also $A \subseteq P_1$ and $B \subseteq K_1$ where $P_1 \cap K_1 = \emptyset$. Thus (Y, U_1, U_2) is pairwise s-normal.

1.24 THEOREM. *Every pairwise semi-continuos, pairwise pre-semi-closed image of a pairwise s-normal space is pairwise semi-normal.*

Proof. Is similar to the prof of Theorem 1.23.

§2 Examples.

2.1 Examples of pairwise normal but not pairwise s-normal spaces.

2.1.1. Let $X = \{ \{0\} \cup \mathbb{N} \cup \{j + (1/n) : j, n \in \mathbb{N} - \{1\} \}$ where $\mathbb{N}$ is the set of positive integers. Let τ_1 be generated by the following open set base: (1) the relative open sets from the set of real numbers in X - {0, 1}; (2) all subsets of the form $\{0\} \cup \{j + (1/2n), j \geq k; k, n \in \mathbb{N}$ where $k \geq 2\}$; and (3), all subsets of the form $\{1\} \cup \{j + (1/(2n+1)), j \geq p; p, n \in \mathbb{N}$ and $p \geq 2\}$. Let τ_2 be the topology defined by the following open set base: (1) the relative open sets from the set of real numbers in X - {0, 1}. (2) all subsets of the form $\{0\} \cup \{j + (1/(2n+2)); j \geq k; k, n \in \mathbb{N}$ and $k \geq 2\}$; and (3), all subsets of the form $\{1\} \cup \{j + (1/(2n+3)), j \geq p, p, n \in \mathbb{N}$

and $p \geq 2\}$. Then (X, τ_1, τ_2) is a pairwise normal but not pairwise s-normal. X is a pairwise normal space because, for any two disjoint sets A and B such that A is τ_1-closed and B is τ_2-closed but neither of which contains either $\{0\}$ or $\{1\}$, we can easily find τ_1-open sets and τ_2-open sets satisfying the required condition. Also, it should be noted that both $\{0\}$ and $\{1\}$ are τ_1-closed as well as τ_2-closed. Any set of the form $\{1\} \cup \{j + (1/(2n+3)), j \geq 2$ and for some $n \in \mathbb{N}\}$ is τ_1-closed as well as τ_2-semi-open and $\{1\} \cup \{j + (1/(2n+3)), j \geq 2, n \in \mathbb{N}\} = U$ is τ_2-open. Also $\{0\} \cup \{j + (1/2n); j \geq 2$ and for some $n \in \mathbb{N}\}$ is τ_2-closed as well as τ_1-semi-open and $V = \{0\} \cup \{j + (1/2n), j \geq 2, n \in \mathbb{N}\}$ is τ_1-open and $U \cap V = \emptyset$. Thus (X, τ_1, τ_2) is pairwise normal. X is not pairwise s-normal because, consider the τ_1-semi-closed set $U = \{1\} \cup \{2n: n \in \mathbb{N}\} \cup \{2n + (1/2n) : n \in \mathbb{N}\}$ and τ_2-semi-closed set $V = \{0\} \cup \{2n + 1 : n \in \mathbb{N}\} \cup \{2n + 1 + (1/(2n+1)) : n \in \mathbb{N}\}$. Any τ_1-semi-open set containing $\{0\}$ has to contain a set of the form $\{j + (1/2n) : j \geq p, p \geq 2$ and for some $n \in \mathbb{N}\}$ which has to intersect U. Therefore there is not τ_1-semi-open set G containing V and a τ_2-semi-open set H containing U such that $G \cap H = \emptyset$. Thus (X, τ_1, τ_2) is not pairwise s-normal.

2.1.2. Let $X = [-1, 1]$ and $\tau_1 = \{\emptyset, X, [-1, b), b > 0\}$ and $\tau_2 = \{\emptyset, X, [-1, 1/2^n), n = 1, 2, \ldots\}$. In both topologies, a non-empty semi-open set is the super set of a non-empty open set. This space (X, τ_1, τ_2) is vacuously pairwise normal since every τ_1-closed set intersect every τ_2-closed set. X is not pairwise s-normal because for some $b > 0$, the set of rationals in $[b, 1]$ is τ_1-semi-closed and the set of irrationals in $[b, 1]$ is τ_2-semi-closed. But there is not τ_1-semi-open set U and τ_2-semi-open set V such that $U \cap V = \emptyset$ and U containing the set of irrationals in $[b, 1]$ and V containing the set of rationals in $[b, 1]$.

2.1.3. Let $X = (0, 1)$ and $\tau_1 = \{U_n = (0, 1 - \frac{1}{n}); n = 2, 3, \ldots\} \cup \{X, \emptyset\}$ and $\tau_2 = \{U_n = (0, \frac{1}{n}); n = 2, 3, \ldots\} \cup \{X, \emptyset\}$. (X, τ_1, τ_2) is vacuously pairwise normal since every non-empty τ_1-closed set as well as every non-empty τ_2 - closed set contains points very close to 1. Since every τ_1 - semi- open set intersect every τ_2 - semi - open set and since there are τ_1 - semi- closed sets disjoint from τ_2 - semi-closed sets, X is not a pairwise s - normal space.

2.2 Examples of pairwise s-normal but not pairwise normal spaces.

2.2.1. Let X = [-1, 1] and τ_1 be generated by the family {[-1, b), b > 0; (a, 1], a < 0}. Hence, sets of the form (a, b), a < 0, b > 0 will also be open. Let τ_2 = {[-1, 0), (0, 1], {1}, {-1}, {-1, 1}, [-1, 0) $\cup$ (0, 1]}. This space (X, τ_1, τ_2) is not pairwise normal because [b, 1], b > 0, is τ_1 - closed and {0} is τ_2 - closed. Every τ_1 - open set containing {0} contains an interval with 0 as interior point and hence intersect the smallest τ_2 - open set (0, 1] containing [b, 1]. It is easy to verify that X is pairwise normal.

2.2.2. Let X = (0, 1) and τ_1 be the topology generated by sets of the form $S_a = \{x \in X \mid x > a, a \in X\}$ and $\tau_2 = \{U_n \mid U_n = (0, 1/2^n); n = 1, 2, ...\} \cup \{X, \emptyset\}$. (X, τ_1, τ_2) is not pairwise normal because for some $n \in \mathbb{N}$, $[1/2^n, 1)$ is τ_2 - closed and (0, a] where $1/2^{n+1} < a < 1/2^n$ is τ_1 - closed and $(0, a] \cap [1/2^n, 1) = \emptyset$. Then every τ_2 - open set containing (0, a] intersect any τ_1 - open set containing $[1/2^n, 1)$. Since in both topologies, a super set of non - empty open set is semi-open, it is easy to verify that (X, τ_1, τ_2) is pairwise s- normal.

2.3 Examples of a pairwise semi-T_1 space which is neither pairwise s-regular nor pairwise s-normal.

2.3.1. Let X = {-1, 1} and τ_1 be generated by the family {[-1, b), b > 0; (a, 1], a < 0}. Then the sets of the form (a, b) are also open. Let τ_2 be defined as follows: For each $x \in [-1, 0]$, a basic open set is of the form $[-1, 1/2^n)$; n = 1, 2, ... and for $x \in (0, 1/2)$, a basic open set is of the form $(0, 1/2^n)$; n = 1, 2, ... and for $x \in [1/2, 1]$ X is the neighborhood. Then (X, τ_1, τ_2) is pairwise semi-T_1 but neither pairwise s-regular nor pairwise s-normal. X is not pairwise s-regular because [-1, a], a < 0 is τ_1-closed and $0 \notin [-1, a]$. Every τ_1-semi-open set containing {0} will intersect any τ_2-semi-open set containing [-1, a]. X is not pairwise semi-normal and hence not pairwise s-normal since [-1, a], a < 0 is τ_1-closed and $[1/2^n, 1]$ for some $n \in X$ is τ_2-closed. Every τ_1-semi-open set containing $[1/2^n, 1]$ has to contain an interval with 0 as interior point and hence intersects every τ_2-semi-open set containing [-1, a].

2.3.2. Let X = (0, 1) and let $\tau_1 = \{\emptyset, X_1 \{(0, 1/2^n), n = 1, 2, ...\}\}$ and let τ_2 be the co-finite topology. Then (X, τ_1, τ_2) is pairwise semi-T_1. But it is not pairwise s-regular because $[1/2^n, 1)$ for some $n \in \mathbb{N}$ is τ_1-closed and let $a < 1/2^n$. Every τ_2-semi-open set containing $[1/2^n, 1)$ intersects every τ_1-semi-open set containing a. X is not pairwise s-normal because consider the τ_1-semi-closed set $\{b\} \cup [a, 1)$ where b < a. Let c and d be two distinct points of X such that b < c < d < a. Then {c, d} is τ_2-semi-closed. Every τ_2-semi-open set containing $\{b\} \cup [a, 1]$ intersects every τ_1-semi-open set containing {c, d}.

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