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A BANACH FIXED POINT THEOREM FOR TOPOLOGICAL SPACES

by

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§1. Introduction

The purpose of this paper is to prove a fixed point theorem for Hausdorff first countable topological spaces and to show, that it is really a generalization of the following well-known Banach Fixed Point Theorem: Let $f:(X, d) \to (X, d)$ be a contraction mapping from a complete metric space into itself. Let $x \in X$. Then the sequence $\{f^{(n)}(x)\}$ converges to a point $z_0 \in X$ and $f(z_0) = z_0$. Furthermore, z_0 is the unique fixed point for f. (The definitions of notions not defined here can be found in [1]).

§2. Result

Before presenting our theorem, we need to establish some terminology and to prove one lemma.

DEFINITION 1. Let X be an arbitrary topological space. Let $f: X \to X$ be a function.Then f is said to be topologically contractive (in what follows t-contractive) iff

(*) *for every open cover $\mathcal{T}$ of X and for every couple of points*

$a, b \in X$ there exists $n \in \mathbb{N}$ such that $\forall k \geq n$, $\exists\, U \in \Upsilon$ such that $f^k(a) \in U$ and $f^k(b) \in U$ holds.

LEMMA 1. *Let X be a Hausdorff first countable space. Let* $\{x_n\}$ *be a sequence of points of X. If*

(1) *for every open cover* Υ *of the set* X *there exists* $U \in \Upsilon$ *such that the set* $U \cap \{x_n\}$ *has at least two points holds, then the sequence* $\{x_n\}$ *has a convergent subsequence.*

Proof. Suppose, contrary to what we wish to prove, there exists no convergent subsequence of $\{x_n\}$. Then $\forall x \in X$ exists an open neighborhood O_x of x such, that the set $O_x \cap \{x_n\}$ is finite or empty. Otherwise, by the Hausdorff and first countability properties of the space one could construct a subsequence converging to x. So the set U_x defined as $U_x = O_x - ((O_x \cap \{x_n\}) - \{x\})$ is an open neighborhood of x such that $U_x \cap \{x_n\} \subset \{x\}$ holds. Let us define an open cover Υ of X with aid of the U_x's. Put $\Upsilon = \{U_x : x \in X\}$. Then $\forall U \in \Upsilon$ the set $U \cap \{x_n\}$ is empty or has only one point which is in contradiction with (1).

THEOREM 1. *Let* X *be a Hausdorff first countable topological space. Let* $f: X \to X$ *be a continuous, t-contractive function. Let* $x \in X$. *Then*

(A) *The sequence* $\{f^{(n)}(x)\}$ *has a convergent subsequence which converges to a point* $z_0 \in X$ *and* $f(z_0) = z_0$.

(B) *There is one and only one fixed point for* f.

Proof. Let us denote the sequence $\{x, f(x), \ldots, f^n(x), \ldots\}$ by $\{x_n\}$. First we show that (A) holds. We distinguish two cases.

(I) The set $I = \{x_n : n \in \mathbb{N}\}$ is finite. We will prove that in this case $f(x) = x$ holds. Suppose, to the contrary, that $f(x) \neq x$. Then we can define an open cover Υ of X as follows: $\Upsilon = \{X - \{x_n\}\} \cup \{O_i : i = 1, 2, \ldots, k\}$ where k is a number of distinct points of $\{x_n\}$ and O_i are disjoint open sets such that $x_i \in O_i$ holds for $i = 1, 2, \ldots, k$. Since f is t-contractive, the points x, f(x) and the cover Υ have to fulfill the assertion (*) of Definition 1. But it is easy to see from construction of Υ that this is not true. We have a contradiction.

The assertion $f(x) \neq x$ is false.

(II) The set $I = \{x_n : n \in \mathbb{N}\}$ is infinite. In this case $\ell, k \in \mathbb{N}$, $k \neq \ell$ implies $f^{\ell}(x) \neq f^k(x)$ and, of course $f(x) \neq x$. Again, the points x, f(x) and an arbitrary open cover $\mathcal{T}$ of X fulfill (*) of Definition 1. Therefore the assumptions of Lemma 1 are fulfilled for $\{x_n\}$ and it has a convergent subsequence $\{a_n\}$ which converges to a point z_0. Suppose, contrary to what we wish to prove, $f(z_0) \neq z_0$. Then there exist two disjoint open sets U, O such that $z_0 \in U$, $f(z_0) \in O$. The continuity of f and $a_n \to z_0$ imply that there exists k $\in \mathbb{N}$ such that $\forall j \geq k$, $a_j \in U$ and $f(a_j) \in O$. Let us define an open cover $\mathcal{C}$ of X as follows: put $A = \{z_0, a_k, a_{k+1}, \ldots\}$; $B = \{f(z_0), f(a_k), f(a_{k+1}), \ldots\}$; $W = X-(A \cup B)$ and $\mathcal{C} = \{U, O, W\}$. The sets A, B are closed so $\mathcal{C}$ is an open cover of X. Then the points x, f(x) and the cover $\mathcal{C}$ must satisfy the condition (*). Therefore there exists $n \in \mathbb{N}$ such that $\forall m \geq n\ \exists Y \in \mathcal{C}$ such that $f^m(x) \in Y$ and $f^m(f(x)) \in Y$ holds. Let us take a $m \geq n$ such that $f^m(x) = a_i$ for some $a_i \in U$, where $i \geq k$. Except for U, $a_i = f^m(x)$ belongs to no other set from $\mathcal{C}$. To fulfill (*) $f^m(f(x)) \in U$ must be true but $f^m(f(x)) = f(a_i) \in O$ and U and O are disjoint. This is a contradiction. So $f(z_0) = z_0$ holds and (A) is proved.

To prove (B), suppose that there exist two points $y, z \in X$, $y \neq z$ such that $y = f(y)$, $z = f(z)$. Take two disjoint open sets U, V for which $y \in U$ and $z \in V$ holds. Let us consider the open cover $\mathcal{a}$ of X defined by $\mathcal{a} = \{U, V, X - \{y, z\}\}$. Since $\forall n \in \mathbb{N}$, $f^n(y) = y$ and $f^n(z) = z$ the points y, z and the cover $\mathcal{a}$ do not fulfill the condition (*) from Definition 1. This is a contradiction. ■

The next example shows that the theorem fails if the function f is not continuous.

EXAMPLE 1. Let $X = \{1, 1/2, \ldots, 1/n, \ldots\} \cup \{0\}$ with the natural metric inherited from $\mathbb{R}$. Let us define $f: X \to X$ as follows: $f(0) = 1$ and $f(1/k) = 1/(k + 1)$ for $k = 1, 2, \ldots$.
Then X is a complete metric space, f is t-contractive, but f has no fixed point.

In topological spaces, the concept of "completeness" has no meaning. Therefore our theorem works also for noncomplete metric spaces. The next example illustrates this and also shows that the assumption of t-contractivity of f cannot be omitted.

EXAMPLE 2. Let the sets $X = \mathbb{Q}$, $Y = \mathbb{Q} - \{0\}$ be equipped with the natural topology inherited from the real numbers. Let the functions $f: X \to X$, $g: Y \to Y$ be defined as follows:

$\forall x \in X,\ f(x) = x/2$

$\forall y \in Y,\ g(y) = y/2$

The space X and the function f fulfill the hypotheses of Theorem 1. The fixed point is 0. But it is easy to see, that (0 being not a point of Y) the function g is not t-contractive. It has no fixed point.

Next we show that Theorem 1 is really a generalization of the Banach Fixed Point Theorem. We recall that a function $f: X \to X$ defined on a metric space (X, d) is said to be *contractive* provided there exists a real number α such that $0 \leq \alpha < 1$ and for all x and y in X, $d(f(x), f(y)) \leq \alpha\, d(x, y)$. We have to prove the following assertion:

LEMMA 2. *If* X *is a complete metric space and* $f: X \to X$ *is a contractive function then* f *is continuous and t-contractive.*

Proof. Let us show the t-contractiveness of f. To prove this, we use the fact, that (according to Banach Fixed Point Theorem) our f has a fixed point z and that

(**) $\forall x \in X, \forall O$ open, $z \in O, \exists n \in \mathbb{N}, \forall \ell \geq n\ f^{\ell}(x) \in O$.

So if $a, b \in X$ and Υ is an open cover of X, let us take a set $U \in \Upsilon$ such that $z \in U$. Then by (**) $\exists n \in \mathbb{N}, \forall \ell \geq n\ f^{\ell}(a) \in U$ and $f^{\ell}(b) \in U$. ■

Open problem. Let X be a metric space, let $f: X \to X$ be a continuous t-contractive function. Under which hypotheses is f contractive?

REFERENCES

[1] KASRIEL, R. H., *Undergraduate Topology.* W. B. Saunders Company, Philadelphia, London, Toronto, 1971.

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