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ON TWO SYSTEMS OF ORTHOGONAL POLYNOMIALS RELATED TO THE POLLACZEK POLYNOMIALS

by

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ABSTRACT. The spectral properties of two systems of orthogonal polynomials related to the Pollaczek polynomials and of their corresponding Jacobi operators are examined. The continued fractions and orthogonality measures of the polynomials and the spectra and spectral resolutions of the operators are determined. End point and embedded eigenvalues are detected for appropriate values of the parameters. Explicit representations of the polynomials in terms of the Pollaczek polynomials are included.

§1. Introduction. We study in this paper two systems of orthogonal polynomials determined by recurrence relations of the form

$$\begin{aligned} x\,S_{2n}(x) &= S_{2n+1}(x) + a_n^{(0)}\,S_{2n-1}(x) \\ x\,S_{2n+1}(x) &= S_{2n+2}(x) + a_n^{(1)}\,S_{2n}(x), \quad n \geq 0 \end{aligned} \tag{1.1}$$

and initial conditions

$$S_{-1}(x) = 0, \quad S_0(x) = 1 \tag{1.2}$$

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An early example, $a_n^{(0)} = a,\ a_n^{(1)} = b$, a, b, > 0, is in [13], p. 91 (see also [8]). Rank 2 perturbations of this particular system are studied in [12]. The first system, denoted with $\{p_n(x)\}$, is given by

$$a_0^{(1)} = 2\lambda bc;\quad a_n^{(0)} = a,\quad a_n^{(1)} = \frac{2\lambda + n}{n} b,\ n \geq 1. \tag{1.3}$$

The second, $\{p_n^{(1)}(x)\}$, by

$$a_0^{(1)} = a;\quad a_n^{(0)} = \frac{2\lambda + n}{n} b,\quad a_n^{(1)} = a,\ n \geq 1. \tag{1.4}$$

For both systems we assume

$$\lambda > 0;\ a, b, c > 0. \tag{1.5}$$

As a matter of fact, except for Theorem 4.4 below, only the case $\lambda \geq 1/2$ is actually relevant, and could be assumed througout. We observe that $\{p_n^{(1)}(x)\}$ is the system of *first associated (numerator) polynomial* of $\{p_n(x)\}$ (see Section 2 for the appropriate definitions).

Recurrence relations of the form (1.1) have appeared in the study of certain processes in physical chemistry. See [29], [34]. They usually arise in the description of quantum phenomena in terms of the Heisenberg - Jacobi matrix analysis or by diagonalization of Hamiltonians in appropriate L^2 - basis ([5], [7], [17], [26]). As a matter of fact, (1.3), (1.4) are first approximations to the description of some diatomic processes where a, b depend on the components, $\lambda \geq 1/2$ is a coupling parameter and c depends on the initial conditions. The support of the orthogonality measures yields the spectrum of the Jacobi matrices (or the Hamiltonians) involved and, therefore, the energy levels of the processes. Both systems have, under certain assumptions on the initials conditions, endpoint or embedded eigenvalues, which correspond to spectral concentrations suggesting resonances. Both phenomena are rather exotic in orthogonal polynomial systems explicitly given by recurrence relations ([11], [12], [14], [20],), specially of such innocent looking recursions as (1.1). This is the main justification for the present paper.

Recurrence relation (1.1) is also a special case of general symmetric sieved polynomials in the sense of [10]. Indeed, the polynomials $\{p_n(x)\}$ and $\{p_n^{(1)}(x)\}$ are closely related to sieved Pollaczek polynomials. It is worth mentioning, however, that the techniques used in [2], [9], [18] to determine the orthogonality measures are not appropriate in these cases, as neither $\{p_n(x)\}$ nor $\{p_n^{(1)}(x)\}$ seem to originate in a sieving process in the sense of these papers. We will follow, therefore, the more direct approach in [10]. As a matter of fact, it was the research for the present paper which motivated some of the ideas in [10]. We hope this work will illustrate the technical difficulties encountered in the spectral analysis of systems such as (1.3) and (1.4), and how to overcome some of them.

The paper is organized as follows: Section 2 is a brief account of background material. Its main purpose is to fix the language. Section 3 is a review of basic facts about Pollaczek's polynomials. Section 4 deals with the system $\{p_n(x)\}$ given by (1.3), and Section 5, with the system of their first associated (numerator) polynomials. Section 6 contains some final observations.

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§2. Background

Let $\{P_n(x)\}$ be the system of polynomials determined by

$$(A_n x + B_n) P_n(x) = P_{n+1}(x) + C_n P_{n-1}(x), \quad n \geq 0, \tag{2.1}$$

and the initial conditions

$$P_{-1}(x) = 0, \quad P_0(x) = 1 \tag{2.2}$$

If the coefficients A_n, B_n, C_n are real numbers, which we will assume henceforth, the polynomials $\{P_n(x)\}$ will be real polynomials. Clearly $P_n(x)$ has degree $\le n$, and exact degree n if and only if $A_n \neq 0$ for $n \ge 0$.

The system of polynomials $\{P_n^{(i)}(x)\}$, $i = 0, 1, 2, \ldots$, determined by

$$(A_{n+i}\, x + B_{n+i})\, P_n^{(i)}(x) = P_{n+1}^{(i)}(x) + C_{n+i}\, P_{n-1}^{(i)}(x), \quad n \ge 0, \tag{2.3}$$

and the initial conditions

$$P_{-1}^{(i)}(x) = 0, \quad P_0^{(i)}(x) = 1, \tag{2.4}$$

is called the system of i^{th} - associated polynomials of $\{P_n(x)\}$. Clearly $P_n^{(0)}(x) = P_n(x)$ for all $n \ge 0$.

The following well know results will be used in the sequel.

THEOREM 2.1 (Favard). *Let* $\{P_n(x)\}$ *be determined by* (2.1) *and* (2.2). *Then* $\{P_n(x)\}$ *is a system of orthogonal polynomials, i. e., there is a positive measure* μ *supported by the real line* $\mathbb{R}$ *such that*

$$\int_{-\infty}^{+\infty} P_n(x)\, P_m(x)\, d\mu(x) = \lambda_n\, \delta_{mn}, \quad \lambda_n > 0, \quad m, n \ge 0, \tag{2.5}$$

if and only if

$$\frac{C_{n+1}}{A_n\, A_{n+1}} > 0, \quad n \ge 0. \tag{2.6}$$

A measure μ satisfying (2.5) is called an *orthogonality measure or an spectral measure* of the system $\{P_n(x)\}$, and the $P_n(x)$'s are said to be *orthogonal with* respect to μ. If μ is so chosen that $\mu(\mathbb{R}) = 1$, in which case it is called a *normalized orthogonality measure of* $\{P_n(x)\}$, then

$$\lambda_0 = 1; \quad \lambda_n = \frac{A_0}{A_n} C_1 C_2 \ldots C_n, \quad n \ge 1, \tag{2.7}$$

and, if

$$\widehat{P}_n(x) = \frac{P_n(x)}{\sqrt{\lambda_n}}, \quad n \geq 0, \tag{2.8}$$

then

$$\int_{-\infty}^{+\infty} \widehat{P}_n(x)\, \widehat{P}_m(x)\, d\mu(x) = \delta_{mn}, \quad m, n \geq 0, \tag{2.9}$$

$$x\, \widehat{P}_n(x) = b_{n+1} \widehat{P}_{n+1}(x) + a_n \widehat{P}_n(x) + b_n \widehat{P}_{n-1}(x), \; n \geq 0, \tag{2.10}$$

and

$$\widehat{P}_{-1}(x) = 0, \; \widehat{P}_0(x) = 1, \tag{2.11}$$

where

$$a_n = -\frac{B_n}{A_n}, \quad b_{n+1} = \sqrt{\frac{C_{n+1}}{A_n A_{n+1}}}, \quad n \geq 0 \tag{2.12}$$

The system $\{\widehat{P}_n(x)\}$ is called the *system of orthonormal polynomials* of $\{P_n(x)\}$, and μ is called an *orthonormality measure of* $\{\widehat{P}_n(x)\}$. As a matter of fact, any system determined by (2.10) and (2.11) with a_n, b_n real and $b_{n+1} > 0$, $n \geq 0$, is orthonormal for some normalized measure. This follows at once from Favard's theorem and from (2.7).

With the notations above we have:

THEOREM 2.2 *In order for the normalized orthogonality measure* μ *of* $\{P_n(x)\}$ *to be compactly supported it is necessary and sufficient that there is* $M > 0$ *such that*

$$\left|\frac{B_n}{A_n}\right| \leq \frac{M}{3}, \quad \sqrt{\frac{C_{n+1}}{A_n A_{n+1}}} \leq \frac{M}{3}, \quad n \geq 0, \tag{2.13}$$

in which case Supp $\mu \subseteq [-M, M]$. *If* v *is any other orthogonality measure of* $\{P_n(x)\}$ *then* $v = v(\mathbb{R})\,\mu$, *and* μ *is the only measure for which* (2.5) *and* (2.7) *hold.*

Remark 2.1. If $A_n = 1$ for all $n \geq 0$, the leading coefficient of $P_n(x)$ is 1 for all $n \geq 0$, i. e., $P_n(x)$ is a *monic polynomial.* This is the case of the polynomials $\{S_n(x)\}$ determined by (1.1) and (1.2) and in particular of $\{p_n(x)\}$ and $\{p_n^{(1)}(x)\}$.

Remark 2.2. If $\{P_n(x)\}$, given by (2.1) and (2.2), is a system of orthogonal polynomials, then the same is true of $\{P_n^{(i)}(x)\}$, $i = 0, 1, 2, \ldots$, and their normalized orthogonality measures μ_i have compact support in $[-M, M]$ if (2.13) holds. In general $\mu_i \neq \mu_j$ if $i \neq j$. We observe that if $|B_n/A_n| \leq M_0$,

$$\sqrt{\frac{C_{2n+1}}{A_{2n}\,A_{2n+1}}} \leq M_1 \quad \text{and} \quad \sqrt{\frac{C_{2n+2}}{A_{2n+1}\,A_{2n+2}}} \leq M_2, \quad n \geq 0, \text{ then}$$

$$\operatorname{Supp} \mu_i \subseteq [-M, M], \; M = M_0 + M_1 + M_2, \; i = 0, 1, 2, \ldots \qquad (2.14)$$

Let $P(x)$, $x \in \mathbb{C}$, denote the limit of the continued fraction

$$\frac{1}{\lfloor A_0 x + B_0} - \frac{C_1}{\lfloor A_1 x + B_1} - \frac{C_2}{\lfloor A_2 x + B_2} - \ldots \qquad (2.15)$$

wherever it exists. Then (see [13], Chap. III)

$$P(x) = \lim_{n \to \infty} \frac{P_{n-1}^{(1)}(x)}{P_n(x)}. \qquad (2.16)$$

Assume $\{P_n(x)\}$ is a system of orthogonal polynomials given by (2.1) and (2.2). Also assume that (2.13) holds. Then:

THEOREM 2.3. (Markov). *The limit* (2.16) *exists for all* $z \in \mathbb{C} - [-M, M]$, *and if* μ *is the normalized orthogonality measure of* $\{P_n(x)\}$, *then*

$$P(z) = \lim_{n \to \infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)} = \frac{1}{A_0} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{z - t}, \quad z \notin [-M, M]. \qquad (2.17)$$

Furthermore, the convergence is uniform on compact subsets of $\mathbb{C}$ - [-M, M].

The Stieltjes - Perron inversion formula ([3], [6], [8], [13], [28]) allows to recover μ from P(z) via

$$\int_{-\infty}^{+\infty} f(x)\,d\mu(x) = \lim_{\varepsilon \to 0^+} \frac{A_0}{2\pi i} \int_{-\infty}^{+\infty} \left\{P(x-i\varepsilon) - P(x+i\varepsilon)\right\} f(x)\,dx, \quad (2.18)$$

which holds for any bounded measurable function f on $\mathbb{R}$. Let

$$\sigma(x) := \int_{-\infty}^{x} d\mu(t) = \mu((-\infty, x]) \quad (2.19)$$

be the distribution function of μ. It follows from (2.18) that

$$\sigma(x) - \sigma(x_0 - 0) = \lim_{\varepsilon \to 0^+} \frac{A_0}{2\pi i} \int_{x_0}^{x} \left(P(t - i\varepsilon) - P(t + i\varepsilon)\right) dt, \quad (2.20)$$

where $\sigma(x_0 - 0) = \lim_{t \to x_0^-} \sigma(t)$, and in particular, that

$$\sigma(x) = \lim_{\varepsilon \to 0^+} \frac{A_0}{2\pi i} \int_{-\infty}^{x} \left(P(t - i\varepsilon) - P(t + i\varepsilon)\right) dt. \quad (2.21)$$

Remark 2.3. In practice, P(z) is determined, regardless of Theorem 2.3, by calculating $\lim_{n \to \infty} P^{(1)}_{n-1}(z)/P_n(z)$ from the asymptotics of the polynomials or by any other means. Then (2.18) is used to determine μ. The function P(z) itself will be called hereafter the *continued fraction of* $\{P_n(x)\}$.

Clearly Suppμ is the set of points of increase of σ, i.e., the set of points x in $\mathbb{R}$ such that $\sigma(x_1) < \sigma(x_2)$ for all x_1, x_2 with $x_1 < x < x_2$. If $\sigma(x) \neq \sigma(x - 0)$ then $x \in$ Suppμ, and the set of such points, which is necessarily countable, is called the *point support* of μ

and is denoted with P_μ. Points in P_μ are also called **mass points of** μ, and $x \in P_\mu$ if and only if

$$\mu(\{x\}) = \sigma(x) - \sigma(x - 0) \neq 0. \qquad (2.22)$$

The following characterization of points in P_μ is useful:

THEOREM 2.4. *Assume* μ *is the normalized orthogonality measure of* $\{P_n(x)\}$ *and that* $Supp\,\mu \subseteq [-M, M]$, $M > 0$. *Let* $\{\hat{P}_n(x)\}$ *be the orthonormal system of* $\{P_n(x)\}$. *Then* $x \in P_\mu$ *if and only if*

$$\sum_{n=0}^{\infty} \hat{P}_n^2(x) < +\infty,$$

and in such case

$$\mu(\{x\}) = \frac{1}{\sum_{n=0}^{\infty} \hat{P}_n^2(x)}. \qquad (2.23)$$

Let D_μ be the set of isolated points of $Supp\mu$. Then $D_\mu \subseteq P_\mu$. The set D_μ is called the discrete support of μ and a point in D_μ is called an *isolated mass point of* μ. Under the assumptions of Theorem 2.4, we have:

THEOREM 2.5. *A point* x *is in* D_μ *if and only if* x *is an isolated pole of* P(z). *In such case,*

$$\mu(\{x\}) = A_0 \, Res(P, x) \qquad (2.24)$$

The set of points x in $Supp\mu$ such that $\sigma(x) = \sigma(x - 0)$ is called the *continuos support of* μ and is denoted with C_μ. Clearly $Supp\mu = C_\mu \cup P_\mu$ and $C_\mu \cap P_\mu = \varnothing$. We observe that C_μ and P_μ may not be closed subsets of $Supp\mu$.

Points in P_μ which are interior to $Supp\mu$ are called *embedded mass points*. A point x in $Supp\mu$ is embedded if and only if x is interior to the clousure of C_μ. Points in P_μ which are in the clousure of C_μ but are not embedded are called *end - point masses.*

The orthogonality measure μ has the Lebesgue decomposition

$$\mu = \mu_c + \mu_p + \mu_s \tag{2.25}$$

where μ_c is *absolutely continuous* and carried by a subset of C_μ, μ_s is *singular continuous* and carried by a subset of C_μ of Lebesgue measure 0, and μ_p is a jump measure carried by P_μ. The measure μ_c can be uniquely written in the form

$$d\mu_c = \varphi(x)\, dx \tag{2.26}$$

where $\varphi(x)$ is integrable with respect to Lebesgue measure and is called the *weight function* of μ (or of $\{P_n(x)\}$).

Assume the normalized orthogonality measure μ of $\{P_n(x)\}$ is compactly supported. We have

THEOREM 2.6. *For each* $\varepsilon > 0$ *let*

$$P_{(\varepsilon)}(x) = \frac{1}{2\pi i}\Big(P(x - i\varepsilon) - P(x + i\varepsilon)\Big). \tag{2.27}$$

Let $\mathbb{I}$ *be an open interval of* $\mathbb{R}$ *such that for each* a, b *in* $\mathbb{I}$ *with* $a < b$ *there are* $\varepsilon(a, b) > 0$ *and* $C(a, b) > 0$ *such that* $|P_\varepsilon(x)| \leq C(a, b)$ *for almost of* x *in* [a, b] *and all* $0 < \varepsilon < \varepsilon(a, b)$. *Further assume that* $\lim_{\varepsilon \to 0} P_{(\varepsilon)}(x)$ *exist a.e. on* $\mathbb{I}$. *Then*

$$\varphi(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} P_{(\varepsilon)}(x), \text{ a.e. on } \mathbb{I}. \tag{2.28}$$

Moreover $\mathbb{I} \cap P_\mu = \emptyset$, *and if* $\varphi(x) \neq 0$ *a.e. on* $\mathbb{I}$ *then* $\mathbb{I} \subseteq C_\mu$.

Theorem 2.6. is an easy consequence of Lebesgue's dominated convergence and Levi's theorems (see [27], p.36 and 32).

A tridiagonal symmetric real matrix

$$\mathbb{J} = \begin{bmatrix} a_0 & b_1 & 0 & 0 & 0\ldots & 0 \\ b_1 & a_1 & b_2 & 0 & 0\ldots & 0 \\ 0 & b_2 & a_3 & b_3 & 0\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \end{bmatrix} \tag{2.29}$$

such that $b_n > 0$ for all $n \geq 1$ is called a *Jacobi matrix*. The matrix $\mathbb{J}$ is *bounded* if there is $M > 0$ such that

$$|a_n| \leq M/3, \; b_{n+1} \leq M/3, \; n \geq 0. \tag{2.30}$$

Let $\ell_2(\mathbb{C})$ be the Hilbert space of complex sequences (x_n), $n \geq 0$, such that

$$\sum_{n=0}^{\infty} |x_n|^2 < +\infty,$$

with the inner product

$$((x_n) \, ; \, (y_n)) = \sum_{n=0}^{\infty} x_n \overline{y_n}. \tag{2.31}$$

A bounded Jacobi matrix $\mathbb{J}$ defines on ℓ_2 a bounded linear operator $\widehat{\mathbb{J}}$ by

$$\widehat{\mathbb{J}} e_n = b_{n+1} e_{n+1} + a_n e_n + b_n e_{n-1}, \; n \geq 0, \tag{2.32}$$

and continuous linear extension. Here $e_n = (\delta_{0n}, \delta_{1n}, \ldots)$, $n \geq 0$, is the canonical basis of ℓ_2, and $e_{-1} = (0, 0, \ldots)$; $\mathbb{J}$ is the matrix of $\widehat{\mathbb{J}}$ relative to $\{e_n\}$. The operator $\widehat{\mathbb{J}}$ is called the *Jacobi operator determined by* $\mathbb{J}$. If $\mathbb{J}$ satisfies (2.30) then $\| \widehat{\mathbb{J}} \| \leq M$.

Let $\{P_n(x)\}$ be the system of polynomials *determined by* $\mathbb{J}$ through

$$xP_n(x) = b_{n+1} P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \; n \geq 1 \tag{2.33}$$

and

$$P_0(x) = 1, \;\; P_1(x) = \frac{1}{b_1}(x - a_0) \tag{2.34}$$

Then $\{P_n(x)\}$ is an orthonormal system of polynomials with respect to a positive measure μ such that $\mu(\mathbb{R}) = 1$, called the *polynomials of* $\mathbb{J}$. The measure μ is unique and compactly supported if (2.30) holds, in which case $\operatorname{Supp}\mu \subseteq [-M, M]$. It can be shown that in such case $\operatorname{Supp}\mu$ coincides with the *spectrum* $\sigma_p(\hat{\mathbb{J}})$ of $\hat{\mathbb{J}}$, C_μ is identical to the *continuous spectrum* $\sigma_c(\hat{\mathbb{J}})$, and $\sigma_p(\hat{\mathbb{J}})$, the *point spectrum*, is P_μ. The embedded mass points of μ are the *embedded eigenvalues* of $\hat{\mathbb{J}}$, i.e., the eigenvalues of $\hat{\mathbb{J}}$ which are interior to the spectrum. The points in D_μ are the isolated eigenvalues of $\hat{\mathbb{J}}$. If $(E_t)_{t\in\mathbb{R}}$ is a *right continuous spectral resolution* of $\hat{\mathbb{J}}$ and σ is the distribution of μ (i.e.,$\sigma(x) = \mu((-\infty, x])$) then $\sigma(t) = (E_t\, e_0; e_0)$. Conversely, (E_t) can be obtained from μ by means of

$$E_\lambda e_n = \sum_{j=0}^{\infty} \left\{ \int_{-\infty}^{\lambda} P_n(t)\, P_j(t)\, d\mu(t) \right\} e_j, \; n = 0, 1, 2, \dots \tag{2.35}$$

and continuous linear extension. The system of orthogonal polynomials determined by the matrix may be of help in the search for the spectrum of the matrix.

Standard references for the results in this section, including the connection with functional analysis, are [1], [13], [28], [30], [31]. See also [8], [12], where most of the results are proved, and where the connections between spectral resolutions of Jacobi operators and orthogonality measures of their polynomials, and between spectra and supports, are examined. For continued fractions and their relation to orthogonal polynomials, [3], [13], [28], [33] are excellent sources.

§3. The Pollaczek polynomials

The following notations will be used in the sequel.

With $(x^2 - 1)^{1/2}$ we denote the branch of the square root of $x^2 - 1$ in $\mathbb{C}$ which behaves as x when $x \to \infty$. Hence $(x^2 - 1)^{1/2} = \sqrt{x^2-1}$ for $x > 1$ (here, $\sqrt{\ }$ denotes the usual non-negative square root of a non-negative real number) and $(x^2 - 1)^{1/2} = -\sqrt{x^2-1}$ if $x < -1$. Also $(x^2 - 1)^{1/2} = i\sqrt{x^2-1}$ if $-1 \le x \le 1$. It can be shown that $(x^2 - 1)^{1/2}$ is analytic in $\mathbb{C} - [-1, 1]$ (see [9] for details).

Let

$$\alpha(x) = x + (x^2 - 1)^{1/2}, \beta(x) = x - (x^2 - 1)^{1/2}, x \in \mathbb{C}. \tag{3.1}$$

Then α, β are analytic in $\mathbb{C} - [-1, 1]$, $\alpha(x) + \beta(x) = 2x$, $\alpha(x) - \beta(x) = 2(x^2 - 1)^{1/2}$ and $\alpha(x)\beta(x) = 1$. It follows (see [9]) that $|\alpha(x)| \ge |\beta(x)|$, with $|\alpha(x)| = |\beta(x)|$ if and only if $x \in [-1, 1]$, in which case $\alpha(x) = \overline{\beta(x)}$ and $|\alpha(x)| = 1$.

If a is a complex number and $n \ge 0$ is an integer, the Pochhammer shifted factorial $(a)_n$ is

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)\ldots(a+n-1), & n \ge 1 \end{cases} \tag{3.2}$$

If a is not an integer ≤ 0 then

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \ge 0, \tag{3.3}$$

where $\Gamma(x)$ denotes the Gamma function (See [15], [21], [25]).

Let a, b, c be complex numbers, c not an integer ≤ 0. The hypergeometric series ${}_1\mathbb{F}_1$ and ${}_2\mathbb{F}_1$ are defined by

$${}_1\mathbb{F}_1\left(\begin{matrix} a \\ c \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n} x^n, \tag{3.4}$$

and

$${}_2\mathbb{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} x^n, \ |x| < 1 \tag{3.5}$$

Properties of ${}_1\mathbb{F}_1$, ${}_2\mathbb{F}_1$ can be found in [15], [21], [25]. In particular,

$$ {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt, \quad |x|<1, \tag{3.6} $$

provided that $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. This is known as *Euler's formula.*

Now we define

$$ \left[\int_0^1\right] t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle|\, x\right), \quad |x|<1, \tag{3.7} $$

whenever c and c - b are not integers ≤ 0 and $b > 0$. The integral in (3.7) is called a *Hadamard integral.* Details about the Hadamard integrals can be found in [9], [24]. If a, b, c are analytic functions of x, the Hadamard integral in (3.7) is analytic in x, $|x| < 1$, except where c or c - b are integers ≤ 0.
If (a_n) and (b_n) are sequences, the notation $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$, and we say that (a_n) and (b_n) behave asymptotically equal as $n \to \infty$. The asymptotic formulae

$$ \frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{a-b}, \quad \frac{(a)_n}{(b)_n} \sim \frac{\Gamma(b)}{\Gamma(a)}\, n^{a-b}, \tag{3.8} $$

simple consequences of *Stirling's formula* (see [21], [25]), will be needed.

The associated Pollaczek polynomials $R_n^{(j)}(x)$, $n \geq 0$, $j = 0, 1, 2, \ldots$, are defined through

$$ 2\left[(n+\lambda+a+j)x+b\right]R_n^{(j)}(x) = (n+1+j)R_{n+1}^{(j)} + (n+2\lambda-1+j)R_{n-1}^{(j)}(x), \tag{3.9} $$

$n \geq 0$, and initial conditions

$$ R_{-1}^{(j)}(x) = 0, \quad R_0^{(j)}(x) = 1 \tag{3.10} $$

We will write $R_n^{(0)}(x) = R_n(x)$, and also

$$ R_n(x) = P_n(x;\lambda, a, b), \quad n \geq 0 \tag{3.11} $$

The polynomials $R_n(x)$ are simply called the Pollaczek polynomials; the $R_n^{(j)}(x)$'s are the j^{th} - associates of the $R_n(x)$'s. It is assumed that λ, a, b are real numbers. The positivity condition

$$\lambda > 0 \text{ and } \lambda + a > 0 \tag{3.12}$$

ensures that $\{R_n^{(j)}(x)\}$, j = 0, 1, 2, ..., is a system of orthogonal polynomials.

Let

$$A = -\lambda + \frac{ax+b}{(x^2-1)^{1/2}}, \quad B = -\lambda - \frac{ax+b}{(x^2-1)^{1/2}} \tag{3.13}$$

Darboux's method ([22], Chap. VIII) and some simple analytic continuation arguments allow to prove (see [9] for details) that

THEOREM 3.1. *If* $\{R_n(x)\}$ *is given by* (3.9) *and* λ, a, b *are real numbers,* $\lambda > 0$, $\lambda + a > 0$, *then* $\{R_n(x)\}$ *is a system of orthogonal polynomials, and*

$$R(x) = \lim_{n\to\infty} \frac{R_{n-1}^{(1)}(x)}{R_n(x)} = \beta\int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du$$

$$= -\frac{\beta}{B}\,{}_2F_1\left(\begin{matrix} A+1 & 1 \\ -B+1 & \end{matrix}\,\middle|\,\beta^2\right) \tag{3.14}$$

Furthermore, for $i \geq 1$,

$$R^{(i)}(x) = \lim_{n\to\infty} \frac{R_{n-1}^{(i+1)}(x)}{R_n^{(i)}} = \frac{i+1}{i}\,\beta\,\frac{\displaystyle\int_0^1 u^i(1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du}{\displaystyle\int_0^1 u^{i-1}(1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du}$$

$$= \frac{i+1}{i-B}\,\beta\,\frac{{}_2\mathbb{F}_1\left(\begin{matrix} A+1, & i+1 \\ -B+i+1 & \end{matrix}\middle|\,\beta^2\right)}{{}_2\mathbb{F}_1\left(\begin{matrix} A+1, & i+1 \\ -B+i & \end{matrix}\middle|\,\beta^2\right)}. \tag{3.15}$$

Relations (3.14) *and* (3.15) *hold for* $x \in \mathbb{C} - [-M, M]$, $M = (1/\lambda)(|b| + \lambda\sqrt{2})$.

Remark 3.1. As matter of fact, (3.14) holds for $x \notin \text{Supp } m$, where m is the normalized spectral measure of $\{R_n(x)\}$. Since

$$\int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du$$

is analytic for $x \notin [-1, 1]$ except possibly for simple poles at the points x, if any, where $B(x) = n$, $n = 0, 1, 2, \ldots$, these poles must belong to Supp m, and have to be located on $[-M, 1] \cup (1, M]$.

Remark 3.2. Relations (3.14) and (3.15) still hold for $-1/2 < \lambda < 0$, provided that $0 < \lambda + a + 1 < 1$, but we do not need this (see [9]).

Remark 3.3. Let $\{R_n(x)\}$ be a system of Pollaczek's polynomials and assume that R(x) is given by (3.14). From simple properties of α, β and of A, B, it follows (see [9]) that

$$\frac{1}{2\pi i}\lim_{\varepsilon \to 0+}\Big(R(x-i\varepsilon) - R(x+i\varepsilon)\Big) = -\frac{1}{\pi}\,\text{Im}R(x), \tag{3.16}$$

and also that

$$\text{Im}R(x) = \begin{cases} -\dfrac{|\Gamma(-B)|^2}{\Gamma(2\lambda)}\sqrt{1-x^2}\,\left|(1-\beta^2)^{-A-1}\right|^2, & x \in (-1, 1) \\ 0, \quad x \notin [-1, 1], \quad B(x) \neq 0, 1, 2, \ldots & \end{cases}$$

Hence, the absolutely continuous part m_c of the spectral measure m of $\{R_n(x)\}$ is

$$dm_c = \frac{|\Gamma(-B)|^2}{\pi\Gamma(2\lambda)} \sqrt{1-x^2} \left|(1-\beta^2)^{-A-1}\right|^2 \chi_{(-1,1)}(x)\, dx \tag{3.17}$$

where $\chi_{(-1,1)}(x)$ is the characteristic function of (-1, 1). Also (see [11] for details), for the spectral measure $m^{(1)}$ of $\{R_n^{(1)}(x)\}$, we have

$$dm_c^{(1)} = \frac{2|\Gamma(-B+1)|^2}{\pi\Gamma(2\lambda+1)} \sqrt{1-x^2} \frac{\left|(1-\beta^2)^{-A-1}\right|^2 \chi_{(-1,1)}(x)}{\left|{}_2F_1\left(\begin{matrix} A+1, & 1 \\ -B+1 & \end{matrix}\middle|\beta^2\right)\right|^2} dx \tag{3.18}$$

Observe that if $\lambda \geq 1/2$, the series in the denominator of (3.18) converges absolutely.

We will also need the following result:

THEOREM 3.2. *Let $\{P_n(x)\}$ be the system of orthogonal polynomials defined by the recurrence relation*

$$2\left[nx - \lambda\sqrt{\frac{b}{a}}\right] P_n(x) = (n+1)P_{n+1}(x) + (n+2\lambda-1)P_{n-1}(x), \quad n \geq 2,$$

$$\left[x - \lambda\sqrt{\frac{b}{a}}\right] P_1(x) = P_2(x) + \lambda c\, P_0(x) \tag{3.19}$$

and the initial conditions

$$P_0(x) = 1, \quad P_1(x) = 2x + \sqrt{\frac{b}{a}} + (1-2\lambda c)\sqrt{\frac{b}{a}}, \tag{3.20}$$

where

$$\lambda > 0; \quad a, b, c > 0. \tag{3.21}$$

Then, the continued fraction limit $P(x)$ *of* $\{P_n(x)\}$ *is*

$$P(x) = \frac{R(x)}{c + \left(2x + \frac{a+b}{\sqrt{ab}}\right) R(x)}, \qquad x \in \mathbb{C} - [-M, M], \tag{3.22}$$

where

$$M = 2\lambda \sqrt{\frac{b}{a}} + 2\,\text{máx}\left\{\sqrt{\frac{\lambda c}{2}}, \frac{\sqrt{2\lambda+1}}{2}\right\},$$

$$R(x) = \beta \int_0^{1} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1} du, \tag{3.23}$$

and

$$A = 2\lambda \frac{\sqrt{\frac{b}{a}} + \alpha}{\beta - \alpha}, \qquad B = -2\lambda \frac{\sqrt{\frac{b}{a}} + \beta}{\alpha - \beta} \tag{3.24}$$

Proof. That $\{P_n(x)\}$, as above, is an orthogonal system with respect to a positive measure ε such that Supp ε ⊆ [-M, M], is a consequence of Favard's theorem, Theorem 2.2, and (2.14). Now, from (3.19) and (3.20) we obtain ([11], p. 157)

$$P_0(x) = 1; \; P_n(x) = cR_n(x) + \left(2x + \frac{a+b}{\sqrt{ab}}\right) R_{n-1}^{(1)}(x), \; n \geq 1 \tag{3.25}$$

where

$$R_n(x) = P_n\left(x; \lambda, -\lambda, -\lambda\sqrt{\frac{b}{a}}\right), \; n \geq 0 \tag{3.26}$$

is a Pollaczek polynomial. As for $\{P_n^{(1)}(x)\}$, this system satisfies the recurrence relation of $\{R_n^{(1)}(x)\}$ for $n \geq 0$. Hence, $P_n^{(1)}(x) = R_n^{(1)}(x)$, $n \geq 0$. We observe that $R_n^{(1)}(x)$ is an orthogonal system but $\{R_n(x)\}$ is not (as $R_1(x) = -\lambda\sqrt{b/a}$). Hence, Markov's theorem does not hold for $\{R_n(x)\}$. However, it is easy to prove that we still have (compare with (3.14))

$$\lim_{n\to\infty} \frac{R_{n-1}^{(1)}(x)}{R_n(x)} = \beta \int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du,\ x\in\mathbb{C}-[-M,M], \quad (3.27)$$

with A, B as above, and (3,21) follows at once from (3.25), from $R_n^{(1)}(x) = P_n^{(1)}(x)$, $n \geq 0$, and from (3.27). ■

The associated Laguerre polynomials $\{L_n^{(\alpha)}(x;\, i)\}$, $i = 0, 1, \ldots$, $L_n^{(\alpha)}(x;0) = L_n^{(\alpha)}(x)$, $n \geq 0$, are defined by the recurrence relation

$$(\alpha+2n+2i+1-x)L_n^{(\alpha)}(x;i) = (n+i+1)L_{n+1}^{\alpha}(x;i) + (n+\alpha+i)L_{n-1}^{(\alpha)}(x;i) \quad (3.28)$$

and the initial conditions

$$L_{-1}^{(\alpha)}(x;i) = 0,\ L_0^{\alpha}(x;i) = 1. \quad (3.29)$$

The system $\{L_n^{(\alpha)}(x)\}$ is called the *Laguerre polynomials,* and $\{L_n^{(\alpha)}(x;\, i)\}$, their i^{th} - associates. A deep study of the Laguerre polynomials is in [31]; their i^{th} - associates are carefully examined in [4]. We observe that if $\{P_n^{(i)}(x;\, \lambda,\, a,\, b)\}$ is the system of the associated Pollaczek polynomials then

$$P_n^{(i)}(-1;\lambda,a,b) = (-1)^n L_n^{(2\lambda-1)}\big(2(a-b);\, i\big),\ n \geq 0 \quad (3.30)$$

and

$$P_n^{(i)}(1;\lambda,a,b) = L_n^{(2\lambda-1)}\big(-2(a-b);\, i\big),\ n \geq 0 \quad (3.31)$$

The Pollaczek polynomials were introduced by F. Pollaczek in [23], [24]; see also [31], Appendix. A study of the singular cases of Pollaczek's polynomials is in [9].

§4. The system $\{p_n(x)\}$

We denote with $\{p_n(x)\}$ the system $\{S_n(x)\}$ determined by (1.1) and (1.2) with $a_n^{(0)}$, $a_n^{(1)}$ given by (1.3).
Results in [10] and Theorem 3.2. imply, with

$$\omega = \frac{x^2 - a - b}{2\sqrt{ab}}, \tag{4.1}$$

that

$$p_0(x) = 1; \quad p_{2n}(x) = n\left(\sqrt{ab}\right)^n\left(cR_n(\omega) + \frac{x^2}{\sqrt{ab}}R^{(1)}_{n-1}(\omega)\right), \quad n \geq 1, \tag{4.2}$$

so that $p_{2n}(x) = P_n(\omega)$, where $\{P_n(x)\}$ is the orthogonal system in Theorem 3.2.

The system $\{p^{(1)}_n(x)\}$ of the first associates of $\{p_n(x)\}$ is determined by

$$\begin{aligned} xp^{(1)}_{2n}(x) &= p^{(1)}_{2n+1}(x) + a^{(1)}_n\, p^{(1)}_{2n-1}(x) \\ xp^{(1)}_{2n+1}(x) &= p^{(1)}_{2n+2}(x) + a^{(0)}_{n+1}\, p^{(1)}_{2n}(x) \end{aligned} \tag{4.3}$$

and the initial conditions $p^{(1)}_{-1}(x) = 0$, $p^{(1)}_0(x) = 1$, with $a^{(0)}_n$, $a^{(1)}_{n+1}$ given by (1.3). Again, results in [10] and Theorem 3.2 imply that

$$p^{(1)}_{2n+1}(x) = (n+1)\left(\sqrt{ab}\right)^n xR^{(1)}_n(\omega), \quad n \geq 0 \tag{4.4}$$

It follows from (4.2) and (4.4) that the continued fraction limit p(x) of $\{p_n(x)\}$ is

$$p(x) = \frac{\frac{x}{\sqrt{ab}}R(\omega)}{c + \frac{x^2}{\sqrt{ab}}R(\omega)} \tag{4.5}$$

where

$$\begin{aligned} R(\omega) &= \beta(\omega)\int_0^1 (1-\beta^2(\omega)u)^{-A(\omega)-1}(1-u)^{-B(\omega)-1}\,du \\ &= -\frac{\beta(\omega)}{B(\omega)}\, {}_2F_1\left(\begin{matrix} A(\omega)+1 & 1 \\ -B(\omega)+1 & \end{matrix}\middle|\, \beta^2(\omega)\right), \end{aligned} \tag{4.6}$$

with

$$A(\omega) = 2\lambda \frac{\sqrt{\frac{b}{a}} + \alpha(\omega)}{\beta(\omega) - \alpha(\omega)}, \quad B(\omega) = 2\lambda \frac{\sqrt{\frac{b}{a}} + \beta(\omega)}{\alpha(\omega) - \beta(\omega)} \tag{4.7}$$

This is an analytic function of x for $\omega(x) \notin [-1, 1]$, i.e., for $x \notin \bar{L}$, the clousure of

$$L = \left(-\sqrt{a} - \sqrt{b}, -\left|\sqrt{a} - \sqrt{b}\right|\right) \cup \left(\left|\sqrt{a} - \sqrt{b}\right|, \sqrt{a} + \sqrt{b}\right), \tag{4.8}$$

except perhaps for simple poles, which are necessarily located on $[-M, M] - \bar{L}$, where

$$M = \operatorname{Max}\left\{\sqrt{2\lambda bc} + \sqrt{a}, \ \sqrt{(1+2\lambda)\, b} + \sqrt{a}\right\}. \tag{4.9}$$

We observe that if μ denotes the normalized orthogonality measure of $\{p_n(x)\}$ then $\operatorname{Supp}\mu \subseteq [-M, M]$. Also

$$\int_{-\infty}^{+\infty} p_m(x)\, p_n(x)\, d\mu(x) = \lambda_n \delta_{mn}, \ m, n \geq 0, \tag{4.10}$$

where

$$\lambda_0 = 1; \ \lambda_{2n-1} = \frac{(2\lambda)_n}{(n-1)!} b(ab)^{n-1}c, \quad \lambda_{2n} = \frac{(2\lambda)_n}{(n-1)!}(ab)^n c, \quad n \geq 1 \tag{4.11}$$

A calculation which takes into account (3.16) and (3.17) also shows that the absolutely continuous part μ_c of μ is $d\mu_c(x) = \varphi(x)\, dx$, where

$$\varphi(x) = -\frac{c\,|x|}{\pi\sqrt{ab}} \frac{\chi_{(-1,1)}(\omega)}{\left|c + \frac{x^2}{\sqrt{ab}} R(\omega)\right|^2} \operatorname{Im} R(\omega) \tag{4.12}$$

i.e.,

$$\varphi(x) = \frac{c\,|x|\left|\Gamma(-B)\right|^2}{\pi\sqrt{ab}\,\Gamma(2\lambda)} \cdot \frac{\sqrt{1-\omega^2}\left|(1-\beta^2)^{-A-1}\right|}{\left|c + \frac{x^2}{\sqrt{ab}} \int_0^{1} (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du\right|^2} \tag{4.13}$$

for $x \in L$ and $\varphi(x) = 0$ for $x \in \mathbb{R} - \bar{L}$. Since $\operatorname{Im} R(\omega) < 0$ when $x \in L$, (Remark 3.3), the denominator in (4.12), (4.13) does not vanish on L, and φ is continuous and positive on this set. Hence, $L \subseteq C_\mu$.

To determine whether $\bar{L} \subseteq C_\mu$ amounts to decide whether

$$z_0 = \left|\sqrt{a} - \sqrt{b}\right|, \quad z_1 = \sqrt{a} + \sqrt{b} \tag{4.14}$$

belong to P_μ, i.e., are mass points of μ. This is not an easy task, and little can be said. For this purpose we observe that from (3.30), (3.31) and (4.2) it follows that

$$p_{2n}(z_0) = (-1)^n n(\sqrt{ab})^n \left(c L_n^{(2\lambda-1)}(z_0^*) - \frac{z_0^2}{\sqrt{ab}} L_{n-1}^{(2\lambda-1)}(z_0^*; 1) \right), \ n \geq 1 \tag{4.15}$$

and

$$p_{2n}(z_1) = n(\sqrt{ab})^n \left(c L_n^{(2\lambda-1)}(z_1^*) + \frac{z_1^2}{\sqrt{ab}} L_{n-1}^{(2\lambda-1)}(z_1^*; 1) \right), \quad n \geq 1, \tag{4.16}$$

where $\{L_n^{(\gamma)}(x; j)\}$ is the system of the associated Laguerre polynomials (as in Section 3) and

$$z_0^* = 2\lambda\left(1 - \sqrt{\frac{b}{a}}\right), \quad z_1^* = 2\lambda\left(1 + \sqrt{\frac{b}{a}}\right) \tag{4.17}$$

We observe that z_0 (resp. z_1) is a mass point of μ if and only if the same is true of $-z_0$ (resp. $-z_1$). Note that $\pm z_1$ would be end-point masses of μ, and the same would be true of $\pm z_0$ if $a \neq b$.

Let

$$\ell_n^{(\gamma)}(x) = \sqrt{\frac{n!}{\Gamma(\gamma + n + 1)}}\, L_n^{(\gamma)}(x), \quad n \geq 0 \tag{4.18}$$

be the orthonormal Laguerre polynomials.

THEOREM 4.1. *If* $a = b$, $z_0 = 0$ *is not a mass of* μ.

Proof. From (4.17), $z_0^* = 0$; and from (4.18), (4.16),

$$\frac{p_{2n}^2(0)}{\lambda_{2n}} = c\Gamma(2\lambda)\, n \frac{\left(L_n^{(2\lambda-1)}(0)\right)^2}{\frac{(2\lambda)_n \Gamma(2\lambda)}{n!}} = c\,\Gamma(2\lambda)\, n \left(\ell_n^{(2\lambda-1)}(0)\right)^2. \tag{4.19}$$

Since $z = 0$ is not a mass point of $\{\ell_n^{(2\lambda-1)}(x)\}$ (see [4], [31]), $\sum_{n=0}^{\infty} p_{2n}^2(0)/\lambda_{2n}$ diverges, and so does $\sum_{n=0}^{\infty} p_n^2(0)/\lambda_n$. The assertion then follows from Theorem 2.4. ■

COROLLARY 4.1. *If* $a = b$ *then* $(-2\sqrt{a}, 2\sqrt{a}) \subseteq C_\mu$. *Hence,* μ *has no embedded mass points.*

LEMMA 4.1. *Let* $\lambda > 0$, $b \geq a$. *Then, there is at most one value of* c *for which* z_0 *is a mass point of* $\{p_n(x)\}$.

Proof. Assume z_0 is a mass point of μ. Since $z_0^* \leq 0$, $L_n^{(2\lambda-1)}(z_0^*) \neq 0$ for $n \geq 0$. From (4.11), (4.15) and (4.18) we get

$$\frac{p_{2n}^2(z_0)}{\lambda_{2n}} = \frac{\Gamma(2\lambda)}{c}\left(c - \frac{z_0^2}{\sqrt{ab}}\,\frac{L_{n-1}^{(2\lambda-1)}(z_0^*;1)}{L_n^{(2\lambda-1)}(z_0^*)}\right)^2 n\left(\ell_n^{(2\lambda-1)}(z_0^*)\right)^2.$$

Since $\sum_{n=0}^{\infty} p_{2n}^2(z_0)/\lambda_{2n}$ is convergent, and $n(\ell_n^{(2\lambda-1)}(z_0^*))^2 \sim Cn^{1/2}e^{4\sqrt{-nz_0^*}}$ (where C, which depends on λ, a, b, is independent of n) follows from Perron's formula for the Laguerre polynomials ([31], p. 199), we must have

$$c = \lim_{n\to\infty} \frac{z_0^*}{\sqrt{ab}}\,\frac{L_{n-1}^{(2\lambda-1)}(z_0^*;1)}{L_n^{(2\lambda-1)}(z_0^*)}, \tag{4.20}$$

and the assertion follows. ■

A value of c for which z_0 (resp. z_1) carries a mass of μ, if it exists, will be called an *internal critical initial condition* for λ, a, b (resp. an *external critical initial condition*). The internal critical condition given by (4.20) will be denoted with $c_0(\lambda, a, b)$. If z_0, z_1 do not carry masses for a certain value c, c will be said to be *non*-critical.

THEOREM 4.2. *For* c *to be non-critical it is necessary and sufficient that* $C_\mu = \bar{L}$.

Proof. If $\bar{L} = C_\mu$ then z_0, z_1 are not mass points of μ. Hence, c is non-critical. Conversely, if c is non-critical, $\bar{L} \subseteq C_\mu$, and since p(x) is analytic in $[-M, M] - \bar{L}$, except possibly for isolated singularities, then $P_\mu = D_\mu = \mathrm{Supp}\,\mu \cap ([-M, M] - \bar{L})$, i.e., $C_\mu \cap ([-M, M] - \bar{L}) = \emptyset$. Therefore, $C_\mu \subseteq \bar{L}$, ■

Now we prove a result on the presence of masses of μ at z_0. We will use a result of Askey and Wimp in [4].
We recall that the Tricomi ψ-function $\psi(a, b, x)$ is defined by

$$\psi(a, b, x) = \frac{1}{\Gamma(a)} \int_{0}^{+\infty} e^{-xt} t^{a-1} (1+t)^{b-a-1} dt, \tag{4.21}$$

provided that $\mathrm{Re}(x) > 0$ and a is not an integer ≤ 0. The integral is in the Hadamard sense (See [9]).
For $\mathrm{Re}(a) > 0$, the integral in (4.21) is proper. When z_0^* in (4.17) is < 0, the result of Askey and Wimp is (see (3.5) of [4])

$$\frac{L_{n-1}^{(2\lambda-1)}(z_0^*; 1)}{L_n^{(2\lambda-1)}(z_0^*)} = \frac{\psi(2\lambda, 2\lambda; -z_0^*)}{\psi(2\lambda-1, 2\lambda; -z_0^*)} + O\left(e^{-4\sqrt{-nz_0^*}}\right), \tag{4.22}$$

i.e., (see [18]),

$$\frac{L_{n-1}^{(2\lambda-1)}(z_0^*; 1)}{L_n^{(2\lambda-1)}(z_0^*)} = e^{-z_0^*}(-z_0^*)^{2\lambda-1}\Gamma(-2\lambda+1, -z_0^*) + O\left(e^{-4\sqrt{-nz_0^*}}\right), \tag{4.23}$$

where

$$\Gamma(\alpha, \zeta) = \int_{\zeta}^{+\infty} e^{-t} t^{\alpha-1} dt, \quad \zeta > 0, \tag{4.24}$$

is the incomplete Gamma function (see [22], Chap. II). Let

$$\emptyset(\lambda, a, b) = e^{-z_0^*}(-z_0^*)^{2\lambda-1}\,\Gamma(-2\lambda+1, -z_0^*), \quad z_0^* = 2\lambda\left(1 - \sqrt{\frac{b}{a}}\right). \tag{4.25}$$

Then

THEOREM 4.3. *For* $b > a$ *and* $\lambda > 1/2$, z_0 *is an end-point mass of* $\{p_n(x)\}$, *provided that*

$$c = \frac{z_0^2}{\sqrt{ab}} \varnothing(\lambda, a, b). \tag{4.26}$$

Proof. Since $\lambda > 1/2$, it follows that $\varnothing(\lambda, a, b) > 0$. On the other hand, since $z_0^* < 0$ is not a root of $L_n^{(2\lambda-1)}(x)$ for $n \geq 0$, (4.11), (4.15) and (4.20) yield

$$\frac{p_{2n}^2(z_0)}{\lambda_{2n}} = \frac{\Gamma(2\lambda)}{c} n \left(c - \frac{z_0^2}{\sqrt{ab}} \varnothing(\lambda, a, b) + O\left(e^{-4\sqrt{-nz_0^*}}\right)\right)^2 \left(L_n^{(2\lambda-1)}(z_0^*)\right)^2$$

Hence, if (4.26) holds then

$$\frac{p_{2n}^2(z_0)}{\lambda_{2n}} = \frac{\Gamma(2\lambda)}{c} n \left(O\left(e^{-4\sqrt{-nz_0^*}}\right)\right)^2 \left(L_n^{(2\lambda-1)}(z_0^*)\right)^2$$

Now, Perron's Formula for the Laguerre's polynomials ([31], p. 199) shows that

$$n \left(L_n^{(2\lambda-1)}(z_0^*)\right)^2 \sim C\, n^{1/2} e^{4\sqrt{-nz_0^*}}, \quad n \to +\infty$$

where C is a constant (which depends on z_0^* but not on n). Thus, for some constant $N > 0$,

$$\frac{p_{2n}^2(z_0)}{\lambda_{2n}} \leq N\, n^{1/2}\, e^{-4\sqrt{-nz_0^*}}$$

Hence

$$\sum_{n=0}^{\infty} \frac{p_{2n}^2(z_0)}{\lambda_{2n}} < +\infty.$$

To prove that z_0 is a mass point we still have to show that

$$\sum_{n=0}^{\infty} \frac{p_{2n+1}^2(z_0)}{\lambda_{2n+1}}$$

converges. But this is a consequence of

$$\frac{p_{2n+1}^2(z_0)}{\lambda_{2n+1}} \leq \frac{2}{z_0^2} \left(a \frac{p_{2n+2}^2(z_0)}{\lambda_{2n+2}} + \frac{n+2\lambda}{n} b \frac{p_{2n}^2(z_0)}{\lambda_{2n}}\right), \quad z_0 \neq 0,$$

as follows from the recurrence relation of $\{p_n(x)\}$ and from $(a + b)^2 \leq 2(a^2 + b^2)$. ■

Remark 4.1. The Jacobi matrix $\mathbb{J}$ of $\{p_n(x)\}$ is (2.29) with $a_n = 0$ for $n \geq 0$ and

$$b_1 = \sqrt{2\lambda bc}; b_{2n} = \sqrt{a}, b_{2n+1} = \sqrt{\frac{n+2\lambda}{n}b}, \; n \geq 1 \tag{4.27}$$

Theorem 4.3 then implies that a rank two perturbation of $\hat{\mathbb{J}}$ can add or remove end point eigenvalues.

Now we state some results on the absence of masses either at z_0 or z_1. They also follow from asymptotic formulae in [4].

THEOREM 4.4. *Assume* λ, a, b *are rational or, more generally, algebraic numbers. Also assume that* $0 < \lambda < 1/2$ *and* $b < a$. *Then,* z_0 *is free of masses of* μ.

Proof. Observe that $z_0^* > 0$. Under the assumptions, $\pi/\sqrt{z_0^*}$ is irrational, so that

$$\lim_{n \to \infty} \sup \cos\left(2\sqrt{nz_0^*} - \pi\left(\lambda - \frac{1}{4}\right)\right) = 1$$

A proof of this can be found in [32]. Hence, for some subsequence (n_k) of (n),

$$\lim_{k \to \infty} \cos\left(\sqrt{n_k z_0^*} - \pi\left(\lambda - \frac{1}{4}\right)\right) = 1.$$

Now, from (2.16) of [4], and taking into account that

$$\frac{(2\lambda)_n}{n!} \sim \frac{n^{2\lambda-1}}{\Gamma(2\lambda)}, \quad n \to \infty,$$

we get

$$\left(\ell_{n_k}^{(2\lambda-1)}(z_0^*)\right)^2 \sim \Gamma(2\lambda)\left(z_0^*\right)^{\frac{1-4\lambda}{2}} \exp\left(-z_0^*\right) \frac{1}{\sqrt{n_k}}, \quad k \to \infty,$$

Hence,

$$n_k\left(\ell_{n_k}^{(2\lambda-1)}(z_0)\right)^2 \not\longrightarrow 0,$$

and arguing as in the proof of Lemma 4.1, and using again (2.16) of [4], we obtain

$$c = \lim_{k\to\infty} \frac{z_0^2}{\sqrt{ab}} \frac{L_{n_k-1}^{(2\lambda-1)}(z_0^*;1)}{L_{n_k}^{(2\lambda-1)}(z_0^*)}$$

$$= \lim_{k\to\infty} \frac{z_0^2}{\sqrt{ab}} \frac{1}{2\lambda-1}\left(1-\frac{1}{n_k}\right)^{\lambda-3/4} \exp(-z_0^*)\ {}_1\mathbb{F}_1\left(\begin{matrix}1-2\lambda\\2-2\lambda\end{matrix}\middle| z_0^*\right) \tag{4.28}$$

$$\cdot \frac{\cos\left(2\sqrt{(n_k-1)z_0^*}-\pi\left(\lambda-\frac{1}{4}\right)\right)}{\cos\left(2\sqrt{n_k z_0^*}-\pi\left(\lambda-\frac{1}{4}\right)\right)}$$

Hence

$$\ell = \lim_{k\to\infty} \frac{\cos\left(2\sqrt{(n_k-1)z_0^*}-\pi\left(\lambda-\frac{1}{4}\right)\right)}{\cos\left(2\sqrt{n_k z_0^*}-\pi\left(\lambda-\frac{1}{4}\right)\right)} \tag{4.29}$$

exists. Since $\sqrt{n_k z_0^*} - \sqrt{(n_k-1)z_0^*} \to 0$ as $k \to \infty$, it follows that $\ell > 0$. Given that

$${}_1\mathbb{F}_1\left(\begin{matrix}1-2\lambda\\2-2\lambda\end{matrix}\middle| z_0^*\right) > 0 \text{ if } \lambda < 1/2,$$

it is impossible for c to be > 0. ■

THEOREM 4.5. *Assume* a, b, *are rational or, more generally, algebraic numbers. Then, for each* $n \geq 1$, *there is* $n/2 < \alpha_n < (n+1)/2$, *unique, such that if* $n/2 < \lambda \leq \alpha_n$ *is rational (or algebraic) then* z_1 *is free of masses of* μ.

Proof. The same argument as in the proof of Theorem 4.4 shows that if c exists such that z_1 bears a mass of μ then, for some subsequence (n_k) of $\mathbb{N}$,

$$c = \frac{z_1^2}{\sqrt{ab}} \cdot \frac{1}{1-2\lambda} e^{-z_1^*} {}_1\mathbb{F}_1\left(\begin{matrix}1-2\lambda\\2-2\lambda\end{matrix}\middle| z_1^*\right) \lim_{k\to\infty} \frac{\cos\left(2\sqrt{(n_k-1)z_0^*} - \pi\left(\lambda - \frac{1}{4}\right)\right)}{\cos\left(2\sqrt{n_k z_0^*} - \pi\left(\lambda - \frac{1}{4}\right)\right)}$$

Now, the sign of the right hand side is determined by that of

$$f(\lambda) = \frac{1}{1-2\lambda}\, {}_1\mathbb{F}_1\left(\begin{matrix}1-2\lambda\\2-2\lambda\end{matrix}\middle| z_1^*\right) = \sum_{n=0}^{\infty} \frac{1}{n!\,(1-2\lambda+n)} (z_1^*)^n$$

But $f'(\lambda) > 0$ for $\lambda > 0$, $\lambda \neq n/2$, $n = 1, 2, \ldots$, so that for some α_n, unique, in $(n/2, (n+1)/2)$, $n \geq 1$, $f(\alpha_n) = 0$. Hence, if $n/2 < \lambda \leq \alpha_n$ then $f(\lambda) \leq 0$, and it is impossible that $c > 0$. ■

Remark 4.2. We have not succeeded in establishing results on the existence of masses at z_1. The condition on the rationality or the algebraic character of λ, a, b in Theorems 4.4 and 4.5 is not too restrictive from the physical point of view, where λ, a, b depend rationally or algebraically on certain quantum numbers.

Now we examine the discrete support D_μ. We observe that $P_\mu \subseteq D_\mu \cup \{\pm z_0, \pm z_1\}$, with $P_\mu = D_\mu$ if and only if c is non-critical. All along the rest of this section and in Section 5, the meaning of $A = A(\omega)$, $B = B(\omega)$ will be that of (4.7), with ω as in (4.1).

THEOREM 4.6. *Let* D *be the set of points where the denominator*

$$G(x) = c + \frac{x^2}{\sqrt{ab}} R(\omega)$$

of p(x) *in (4.5) vanishes. Then* $D \subseteq D_\mu \subseteq D \cup \{0\}$. *Furthermore,* $0 \in D_\mu$ *if and only if* $b < a$, *and in such case,* $D_\mu = D \cup \{0\}$.

Proof. Since $F(x) = (x/\sqrt{ab})\, R(\omega)$, the numerator of $p(x)$ in (4.5), and $G(x)$ can not vanish simultaneously, any point in D has to be a pole of $p(x)$. On the other hand, $F(x)$ and $G(x)$ are analytic in $[-M, M] - \bar{L}$, except for simple poles. Since $F(x)$ and $G(x)$ have the same poles, except for $x = 0$, and $p(x) = 1/x$ at any non zero pole x, theses are not singularities of $p(x)$. Thus, the poles of $p(x)$, if any, are zeros of $G(x)$. Hence, $D \subseteq D_\mu \subseteq D \cup \{0\}$. Now we prove that $0 \in D_\mu$ if and only if $b < a$ (we recall from Theorem 4.1. that $0 \notin P_\mu$ if $a = b$). If $a < b$ and $x = 0$ then

$$\omega = -\frac{a+b}{2\sqrt{ab}}, \quad \alpha(\omega) = -\sqrt{\frac{b}{a}}, \quad \beta(\omega) = -\sqrt{\frac{a}{b}}$$

and $A = 0$, $B = -2\lambda$. Thus, at $x = 0$,

$$\int_0^1 (1 - \beta^2 u)^{-A-1}(1 - u)^{-B-1}du = \int_0^1 \left(1 - \frac{a}{b}u\right)^{-1}(1 - u)^{2\lambda-1}du,$$

and, since $\lambda > 0$, the right hand side integral is convergent. Therefore $F(0) = 0$ and $G(0) = c$, so that $p(0) = 0$; i.e., $p(x)$ is analytic at $x = 0$, and vanishes there. Hence, 0 is not a mass point if $a > b$.

Now let $b < a$. Then $B = 0$, and $x = 0$ is a simple pole of

$$F(x) = -\frac{x\beta}{\sqrt{ab}}\frac{1}{B}\,{}_2F_1\left(\begin{matrix} A+1 & 1 \\ -B+1 & \end{matrix}\,\middle|\, \beta^2\right).$$

A simple calculation shows that

$$\operatorname{Res}(F, 0) = \lim_{x \to 0} xF(x) = \frac{1}{2\lambda}\frac{a}{b}\left(1 - \frac{b}{a}\right)^{2\lambda+1}$$

Since

$$G(0) = c + \frac{1}{2\lambda}\frac{a}{b}\left(1 - \frac{b}{a}\right)^{2\lambda+1}, \tag{4.30}$$

0 is a simple pole of $p(x)$, and

$$\operatorname{Res}\big(p(x)\big) = \frac{1}{1 + 2\lambda c \frac{b}{a}\left(1 - \frac{b}{a}\right)^{-2\lambda - 1}}. \tag{4.31}$$

Hence, 0 is a mass point of μ, and $\mu\{0\}$ is given by (4.31). This proves the theorem. ■

Now we determine the set D where G(x) vanishes. We observe that $D = D_0 \cup D_1$ where $D_0 \subseteq (-z_0, z_0)$ and $D_1 \subseteq [-M, -z_1) \cup (z, M]$. Furthermore $0 \notin D_0$, and if $D_0' = (0, z_0) \cap D_0$ then $D_0 = D_0' \cup (-D_0')$. Also $D_1 = D_1' \cup (-D_1')$, where $D_1' = D_1 \cap (z_1, M]$.

We will show that D is infinite countable with $\pm z_1$ as its only limit points, whereas D_0 is finite with at most two elements if $a < b$ and infinite countable with $\pm z_0$ as its only limit points when $b < a$. Obviously $D_0 = \emptyset$ if $a = b$. The next two results are easily established. The following formula, which allows to determine where B is increasing or decreasing, will be useful:

$$\frac{dB}{dx}(x) = -\frac{4\lambda\beta^2 x}{\sqrt{ab}} \, \frac{\left(\sqrt{\frac{b}{a}} + \beta\right) + \beta^2\left(\sqrt{\frac{b}{a}}\ \alpha\right)}{(1 - \beta^2)^3}, \tag{4.32}$$

where $\alpha = \alpha(\omega)$, $\beta = \beta(\omega)$, $B = B(\omega)$, $\omega = (x^2 - a - b)/2\sqrt{ab}$.

LEMMA 4.2. *As a function of* x, B *is decreasing in* $(z_1, +\infty)$ *and tends to* 0 *as* $x \to +\infty$ *and to* $+\infty$ *when* $x \to z_1+$. *Hence* B *is positive in this interval, and there are* $M \geq y_1 > y_2 > \dots y_n > \dots > z_1$ *such that* $B(y_n) = n$, $n = 1, 2, \dots$.

LEMMA 4.3. *As a function of* x, B *is negative on* $(0, z_0)$ *if* $a < b$ *and positive and increasing in this interval if* $b < a$. *In the latter case,* B *tends to* 0 *as* $x \to 0+$ *and to* $+\infty$ *if* $x \to z_0-$. *Hence, there are* $0 < x_1 < x_2 < \dots < x_n < \dots < z_0$ *such that* $B(x_n) = n$, $n = 1, 2, \dots$.

Somewhat more delicate is the following

LEMMA 4.4. *Let* $\{x_n\}$, $\{y_n\}$ *be as in Lemmas 4.2 and 4.3. Then* $\{x_n\}$, $\{y_n\}$ *are poles of* $G(x)$, *and for each* $n = 1, 2, \ldots$ *there are* $\bar{x}_0, \bar{x}_n, \bar{y}_n$, *unique, such that* $0 < \bar{x}_0 < x_1 < x_n < \bar{x}_n < x_{n+1}, y_n > \bar{y}_n > y_{n+1}$, *and* $G(\bar{x}_0) = G(\bar{x}_n) = G(\bar{y}_n) = 0$. *Furthermore, these are the only points in* $[-M, M] - \bar{L}$ *where* G *vanishes.*

Proof. Clearly x_n, y_n are poles of $G(x)$, and a simple calculation shows that if $\varepsilon_n = x_n, y_n$ then

$$\operatorname{Res}\left(G, \varepsilon_n\right) = -\frac{(2\lambda)_n}{\sqrt{ab}\; n!}\beta^{2n-1}\varepsilon_n^2 (1 - \beta_n^2)^{2\lambda - 1}\left(\frac{dB}{dx}(\varepsilon_n)\right)^{-1}, n \geq 1, \quad (4.33)$$

where $\beta_n = \beta(\omega_n)$, $\omega_n = \omega(\varepsilon_n)$, and $(dB/dx)(\varepsilon_n)$ is given by (4.32) at $x = \varepsilon_n$. Hence, $\operatorname{Res}(G, \varepsilon_n) > 0$, $n \geq 1$. It follows that

$$\lim_{x \to x_n^+} G(x) = +\infty, \qquad \lim_{x \to x_{n+1}^-} G(x) = -\infty, \; n \geq 1. \quad (4.34)$$

Also

$$\lim_{x \to y_{n+1}^+} G(x) = +\infty, \qquad \lim_{x \to y_n^-} G(x) = -\infty, \; n \geq 1. \quad (4.35)$$

Since G is continuous (in fact, analytic) in each interval (x_n, x_{n+1}) and (y_n, y_{n-1}), there are $\bar{x}_n, \bar{y}_n$ in these intervals such that $G(\bar{x}_n) = G(\bar{y}_n) = 0$. Since $G(0) > 0$ (see (4.29)), also $G(\bar{x}_0) = 0$ for some $\bar{x}_0 \in (0, x_1)$. Clearly, $\bar{x}_n, \bar{y}_n$ are simple poles of $p(x)$, and $\operatorname{Res}(p(x), \bar{\varepsilon}_n) = F(\bar{\varepsilon}_n)/G'(\bar{\varepsilon}_n)$, $\bar{\varepsilon}_n = \bar{x}_n, \bar{y}_n$, $n \geq 1$. Now assume there are other points $\bar{x}$ in (x_n, x_{n+1}) (or in$(0, x_1)$) where G vanishes. These can not be infinite in number; so, we can choose two of them, $\bar{x}_n', \bar{x}_n''$, with $\bar{x}_n' < \bar{x}_n''$ and G non-vanishing in $(\bar{x}_n', \bar{x}_n'')$. We have $G'(\bar{x}_n') \neq 0 \neq G'(\bar{x}_n'')$, as $\bar{x}_n', \bar{x}_n''$ are simple poles of $p(x)$. Hence, $G'(\bar{x}_n')$ and $G'(\bar{x}_n'')$ carry opposite sings. But $F(\bar{x}_n') = -c/\bar{x}_n' < 0$, $F(\bar{x}_n'') = -c/\bar{x}_n'' < 0$. Hence, $\operatorname{Res}(p, x)$ carries opposite sings at $x = \bar{x}_n', \bar{x}_n''$. This is impossible, since $\operatorname{Res}(p, x) = \mu(\{x\}) \geq 0$. The same argument applies to (y_{n+1}, y_n). Since G does not vanish at x_n, y_n, this argument also shows that G only vanishes at $\bar{x} = \bar{x}_0, \bar{x}_n, \bar{y}_n$, $n \geq 1$. ■

LEMMA 4.5. *Assume* $b > a$. *Then, if* $c \geq 1/2\lambda$, $D_0 = \emptyset$. *If on the contrary* $c < 1/2\lambda$, *there is* $N > 0$ *such that if* $b/a > N$ *then* $D_0 = \{-x_0^*, x_0^*\}$, *where* $x_0^* \in (0, z_0)$.

Proof. Under the assumptions, $z_0^* < 0$ and $B < 0$ on $(0, z_0)$. Moreover, B is decreasing in this interval, and $B \to -\infty$ when $x \to z_0-$. Hence, if

$$G(x) = c + \frac{x^2}{\sqrt{ab}} R(\omega)$$

then

$$G(x) = c + \frac{x^2}{\sqrt{ab}} \int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1} du,$$

the integral being proper. An application of L'Hospital's rule and Levi's theorem ([27], p.32) shows that

$$\lim_{x \to z_0^-} \int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1} du$$

$$= \exp(-z_0^*) \int_0^1 (1-u)^{2\lambda-1} \exp\left(\frac{z_0^*}{1-u}\right) du$$

$$= \exp(-z_0^*) \int_1^{+\infty} u^{2\lambda} \exp(z_0^* u)\, du$$

$$\leq \exp(-z_0^*) \int_1^{+\infty} \exp(z_0^* u)\, du$$

$$= -\frac{1}{z_0^*}$$

Since $\beta \to -1$ when $x \to z_0^-$, and $z_0 = (\sqrt{b} - \sqrt{a})$, $z_0^* = 2\lambda(1 - \sqrt{b/a})$, then

$$\lim_{x \to z_0} G(x) \geq c + \frac{z_0^2}{z_0^* \sqrt{ab}} = c - \frac{1}{2\lambda} + \sqrt{\frac{a}{b}} \geq \sqrt{\frac{a}{b}} > 0,$$

provided that $c \geq 1/2\lambda$. This and $G(0) = c > 0$ imply that $G(x) > 0$ in $(0, z_0)$. (Recall the proof of Lemma 4.4 to conclude that $(0, z_0)$ can not hold more than one zero of $G(x)$; and can not hold one, since $G'(x) \neq 0$ for $x \in (0, z_0)$; otherwise, x would be a double pole of $p(x)$).

Now we prove that if $c < 1/2\lambda$ and $b >> a$ then $G(x_0^*) = 0$ for one value $x_0^* \in (0, z_0)$ and, therefore, exactly for one. Let $\varepsilon = -z_0^* = 2\lambda(\sqrt{b/a} - 1)$. Then $\sqrt{b/a} = (\varepsilon/2\lambda) + 1$ and $\sqrt{a/b} = 2\lambda/(\varepsilon + 2\lambda)$. Now

$$\lim_{x \to z_0^-} \int_0^1 (1 - \beta^2 u)^{-A-1} (1-u)^{-B-1} du = e^{\varepsilon} \varepsilon^{2\lambda - 1} \int_{\varepsilon}^{+\infty} u^{-2\lambda} e^{-u} du.$$

Hence

$$\lim_{x \to z_0^-} G(x) = c - e^{\varepsilon} \frac{e^{2\lambda} (\varepsilon + 4\lambda)}{2\lambda (\varepsilon + 2\lambda)} \int_{\varepsilon}^{+\infty} u^{-2\lambda} e^{-u} du.$$

Let $G(z_0) = \lim_{x \to z_0^-} G(x)$. Then, an application of L'Hospital's rule shows that $\lim_{\varepsilon \to +\infty} G(z_0) = c - (1/2\lambda) < 0$. Since $G(0) > 0$, the assertion follows . ■

With the notations in Theorem 4.6 and in Lemmas 4.4 and 4.5 we then have the following consequence of them:

THEOREM 4.7. *The support* Supp μ *of the orthogonality measure* μ *of* $\{p_n(x)\}$ *is for non critical* c *as follows:*

1) $b > a, c \geq 1/2\lambda$. *Then*

$$\mathrm{Supp}\mu = D_1 \cup \overline{L}, \quad P_\mu = D_\mu = D_1, \quad C_\mu = \overline{L}, \tag{4.36}$$

where

$$D_1 = \{\pm \overline{y}_n | n \geq 1\}, L = (-\sqrt{a} - \sqrt{b}, \sqrt{a} - \sqrt{b}) \cup (\sqrt{b} - \sqrt{a}, \sqrt{a} + \sqrt{b}), \tag{4.37}$$

2) $b >> a, c < 1/2\lambda$. *Then*

$$\mathrm{Supp}\,\mu = D_1 \cup \overline{L} \cup \left\{-x_0^*, x_0^*\right\},\ P_\mu = D_\mu = D_1 \cup \left\{-x_0^*, x_0^*\right\}, C_\mu = \overline{L}, \quad (4.38)$$

with D_1 *and* $\overline{L}$ *as in* (4.37) *and* $x_0^* \in (0, \sqrt{b} - \sqrt{a})$ *such that* $G(x_0^*) = 0$.

3) a = b. *Then*

$$\mathrm{Supp}\,\mu = D_1 \cup \left[-2\sqrt{a},\ 2\sqrt{a}\right],\ P_\mu = D_\mu = D_1,\ C_\mu = \left[-2\sqrt{a}, 2\sqrt{a}\right], \quad (4.39)$$

where D_1 *is as in* (4.37).

4) a > b. *Then*

$$\mathrm{Supp}\,\mu = D_0 \cup D_1 \cup \overline{L} \cup \{0\},\ P_\mu = D_\mu = D_0 \cup D_1 \cup \{0\},\ C_\mu = \overline{L}, \quad (4.40)$$

where $D_0 = \{\pm \overline{x}_n \mid n \geq 0\}$, $D_1 = \{\pm \overline{y}_n \mid n \geq 1\}$, *and* $L = (-\sqrt{a} - \sqrt{b}, \sqrt{b} - \sqrt{a}) \cup (\sqrt{a} - \sqrt{b}, \sqrt{a} + \sqrt{b})$.

The absolutely continuous part μ_c *of the orthogonality measure* μ *is in all cases*

$$d\mu_c(x) = \frac{c|x|\left|\Gamma(-B)\right|^2}{\sqrt{ab}\ \Gamma(2\lambda)\,\pi} \cdot \frac{\sqrt{1-\omega^2}\ \left|(1-\beta^2)^{-A-1}\right|^2 \chi(x)}{\left|c - \frac{x^2\beta}{\sqrt{ab}} \frac{1}{B}\, {}_2F_1\left(\begin{matrix} A+1 & 1 \\ -B+1 & \end{matrix} \middle| \beta^2\right)\right|^2} dx, \quad (4.41)$$

where $\chi(x)$ *is the characteristic function of* L, *and its support is* $\overline{L}$; *the singular continous part* μ_s *vanishes; and* μ_p *is purely discrete and has at each point* $\pm z, z = \overline{x}_0, x_0^*, \overline{x}_n, \overline{y}_n, n \geq 1$, *a mass whose value is*

$$\mu(\{z\}) = \frac{F(z)}{G'(z)} = \frac{1}{1 - \frac{1}{c} zF'(z)}, \quad (4.42)$$

where

$$F(z) = \frac{z}{\sqrt{ab}} R(\omega(z)),\quad G(z) = \frac{z^2}{\sqrt{ab}} R(\omega(z)) + c, \quad (4.43)$$

and, when b < a, *a mass at* x = 0, *whose value is*

$$\mu(\{0\}) = \frac{1}{1 + 2\lambda c \frac{b}{a}\left(1 - \frac{b}{a}\right)^{-2\lambda-1}}. \tag{4.44}$$

Remark 4.3. We observe that $z = \overline{x}_0, \overline{x}_n, \overline{y}_n, x_0^*$, and therefore $\mu(\{z\})$, can not be exactly determined, as $G(x) = 0$ can not be solved in closed form. However, x_n, y_n can be obtained by solving the equation $B(x) = n$.

Remark 4.4. We remark that when $b > a$, $\lambda > 1/2$ and $c = c_0(\lambda, a, b)$ (given by (4.20) or (4.26)), then $z_0 = \sqrt{b} - \sqrt{a}$ carries a mass of μ. As for $\mu(\{z_0\})$, only estimations based on (2.23) can be given.

§5. The system $\{p_n^{(1)}(x)\}$

Now we consider the system $\{p_n^{(1)}(x)\}$ of first associates of $\{p_n(x)\}$. From (4.17) we have

$$p_{2n+1}^{(1)}(x) = (n+1)\left(\sqrt{ab}\right)^n x\, R_n^{(1)}(\omega),\ n \geq 0, \tag{5.1}$$

where $\{R_n^{(1)}(\omega)\}$ is the system of first associates of $\{R_n(\omega)\}$, $R_n(x)$ given by (3.26), and ω is as in (4.1). To determine the continued fraction limit $p^{(1)}(x)$ of $\{p_n^{(1)}(x)\}$, we need to consider the system $\{p_n^{(2)}(x)\}$ of second associates of $\{p_n(x)\}$. These are given by

$$\begin{aligned} x\, p_{2n}^{(2)}(x) &= p_{2n+1}^{(2)}(x) + a_{n+1}^{(0)}\, p_{2n-1}^{(2)}(x),\ n \geq 0 \\ x\, p_{2n+1}^{(2)}(x) &= p_{2n+2}^{(2)}(x) + a_{n+1}^{(1)}\, p_{2n+1}^{(2)}(x),\ n \geq 0, \end{aligned} \tag{5.2}$$

and the initial conditions $p_{-1}^{(2)}(x) = 0$, $p_0^{(2)}(x) = 1$, where $a_n^{(0)}$, $a_n^{(1)}$ are as in (1.3). Results in [10] and Theorem 3.2 yield

$$p_{2n}^{(2)}(x) = (n+1)\left(\sqrt{ab}\right)^n\left[R_n^{(1)}(\omega) + \frac{1}{2}\sqrt{\frac{a}{b}}\, R_{n-1}^{(2)}(\omega)\right],\ \ n \geq 0, \tag{5.3}$$

and $p^{(1)}(x)$ is

$$p^{(1)}(x) = \frac{1}{x}\left[1 + \frac{1}{2}\sqrt{\frac{a}{b}}\, R^{(1)}(\omega)\right], \quad x \notin [-M', M'], \tag{5.4}$$

where

$$M' = \sqrt{(1 + 2\lambda)\, b} + \sqrt{a}, \tag{5.5}$$

as follows from (2.14). If μ_1 is the normalized orthogonality measure of $\{p_n^{(1)}(x)\}$ then $\operatorname{Supp}\mu_1 \subseteq [-M', M']$, and (2,5), (2,7) yield

$$\int_{-\infty}^{+\infty} p_n^{(1)}(x)\, p_m^{(1)}(x)\, d\mu_1(x) = \lambda_n' \,\delta_{mn},\ m, n \geq 0, \tag{5.6}$$

with

$$\lambda_{2n}' = \frac{(1 + 2\lambda)_n}{n!}(ab)^n,\ \lambda_{2n+1}' = \frac{(1 + 2\lambda)_n}{n!}\, a\,(ab)^n, \quad n \geq 0. \tag{5.7}$$

Now, $R^{(1)}(\omega)$ is analytic for $x \notin \overline{L}$, L given by (4.8), except possibly for simple poles on $[-M', M'] - \overline{L}$. These poles are located at those points x such that $\omega(x)$ is a pole of $R^{(1)}(\omega)$ and, perhaps, at $x = 0$ (when $a \neq b$).

We also observe that the absolutely continuous part of μ_1, μ_{1c}, is given by

$$d\mu_{1c}(x) = \frac{1}{\pi|x|}\sqrt{\frac{a}{b}}\, \frac{|\Gamma(-B+1)|^2}{\Gamma(2\lambda+1)}\sqrt{1-\omega^2}\cdot \frac{\left|(1-\beta^2)^{-A-1}\right|^2 \chi(x)\, dx}{\left|{}_2F_1\left(\begin{matrix} A+1 & 1 \\ -B+1 & \end{matrix}\middle|\, \beta^2\right)\right|}, \tag{5.8}$$

where $\chi(x)$ is the characteristic function of L and A, B, β are functions of ω. This follows from (3.18) and (5.4), and implies that $L \subseteq C_{\mu_1}$. We now prove that $\overline{L} \subseteq C_{\mu}$ and, since $([-M', M'] - \overline{L}) \cap \operatorname{Supp}\mu_1 = D_{\mu_1}$, that $\overline{L} = C_{\mu_1}$ and $P_{\mu_1} = D_{\mu_1}$. To do so, observe that at $x = z_0 = |\sqrt{a} - \sqrt{b}|$, (5.1) becomes

$$p_{2n+1}^{(1)}(z_0) = (-1)^n (n + 1)\left(\sqrt{ab}\right)^n z_0\, L_n^{(2\lambda-1)}(z_0^*; 1), \quad n \geq 0, \tag{5.9}$$

with z_0^* given by (4.17) Since

$$\ell_n^{(2\lambda-1)}(x;1) = \sqrt{\frac{\Gamma(2\lambda+1)\ (n+1)!}{(2\lambda+1)_n}}\ L_n^{(2\lambda-1)}(x;1), \quad n \geq 0, \qquad (5.10)$$

is the orthogonal system of $\{L_n^{(2\lambda-1)}(x;1)\}$, it follows from (5.9) that

$$\frac{\left(p_{2n+1}^{(1)}(z_0)\right)^2}{\lambda_{2n+1}^{*}} = z_0^2\ \frac{\Gamma(2\lambda+1)\ (n+1)}{a} \left(\ell_n^{(2\lambda-1)}(z_0^*;1)\right)^2, \quad n \geq 0. \qquad (5.11)$$

In the same way we prove that

$$\frac{\left(p_{2n+1}^{(1)}(z_1)\right)^2}{\lambda_{2n+1}^{*}} = z_1^2\ \frac{\Gamma(2\lambda+1)\ (n+1)}{a} \left(\ell_n^{(2\lambda-1)}(z_1^*;1)\right)^2, \quad n \geq 0. \qquad (5.11')$$

Since $\{\ell_n^{(2\lambda-1)}(x;1)\}$ has no mass points (see [4]),

$$\sum_{n=0}^{\infty} \frac{\left(p_{2n+1}^{(1)}(x)\right)^2}{\lambda_{2n+1}^{*}}$$

is divergent for $x = z_1$, and this also holds for $z = z_0$ if $a \neq b$. Hence

THEOREM 5.1. *The points* $z = \pm(\sqrt{a} + \sqrt{b})$ *do not carry masses of* μ_1, *and the same is true of* $z = \pm(\sqrt{a} - \sqrt{b})$ *when* $a \neq b$. *Under this assumption,* $\overline{L} = C_{\mu_1}$ *and* $P_{\mu_1} = D_{\mu_1}$.

From (3.15) it follows that

$$R^{(1)}(\omega) = \lim_{n\to\infty} \frac{R_{n-1}^{(2)}(\omega)}{R_n^{(1)}(\omega)} = 2\beta\, \frac{\displaystyle\int_0^1 u(1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du}{\displaystyle\int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\,du} \qquad (5.12)$$

where $\alpha = \alpha(\omega)$, $\beta = \beta(\omega)$, and A, B are given by (4.7).

To determine P_{μ_1}, we first prove

THEOREM 5.2. *For* $a \neq b$ *the point* $x = 0$ *is in* P_{μ_1} *if and only if* $b > a$. *In theses circumstances,*

$$\mu_1(\{0\}) = \frac{1}{{}_2F_1\left(\begin{matrix} 1 & 1 \\ 2\lambda+1 & \end{matrix}\middle| \frac{a}{b}\right)}. \tag{5.13}$$

Provided $\lambda > 1/2$, (5.13) *still holds if* $a = b$, *and* $\mu_1(\{0\}) = 2\lambda\text{-}1/2\lambda$.

Proof. Assume first $a \neq b$, so that 0 is an isolated singularity of $p^{(1)}(x)$. We have

$$\operatorname{Res}\left(p^{(1)}(x), 0\right) = \lim_{x \to 0} x\, p^{(1)}(x) = \lim_{x \to 0}\left[1 + \frac{1}{2}\sqrt{\frac{a}{b}}\, R^{(1)}(\omega)\right].$$

Now, when $b < a$ and $x \to 0$, we have that $\omega \to -(a+b)/2\sqrt{ab}$, $\beta(\omega) \to -\sqrt{b/a}$, $\alpha(\omega) \to -\sqrt{a/b}$, $A(\omega) \to -2\lambda$, $B(\omega) \to 0$. Then, using (5.12), we get

$$\lim_{x \to 0}\left[1 + \frac{1}{2}\sqrt{\frac{a}{b}}\, R^{(1)}(\omega)\right] = 0.$$

Therefore, $p^{(1)}(x)$ is analytic at $x = 0$. On the other hand, when $b > a$ and $x \to 0$ then $\alpha(\omega) \to -\sqrt{b/a}$, $\beta(\omega) \to -\sqrt{a/b}$, $B(\omega) \to -2\lambda$, $A(\omega) \to 0$, so that

$$\lim_{x \to 0}\left[1 + \frac{1}{2}\sqrt{\frac{a}{b}}\, R^{(1)}(\omega)\right] = \frac{1}{{}_2F_1\left(\begin{matrix} 1 & 1 \\ 2\lambda+1 & \end{matrix}\middle| \frac{a}{b}\right)}.$$

Now assume $a = b$. From (1.1), (1.3) and (5.1) it follows that

$$p_{2n}^{(1)}(x) = (n+1)\, a^n \left[R_n^{(1)}(\omega) + \frac{2\lambda + n}{n+1} R_{n-1}^{(1)}(\omega)\right]$$

which in view of (5.9) gives

$$p_{2n}^{(1)}(0) = (-1)^n (n+1)\, a^n \left[L_n^{(2\lambda-1)}(0; 1) - \frac{2\lambda + n}{n+1} \widehat{L}_{n-1}^{(2\lambda-1)}(0; 1)\right].$$

A calculation based on (3.28) shows, on the other hand, that

$$L_n^{(2\lambda-1)}(0;1) = \frac{1}{2\lambda+1}\left[\frac{(2\lambda)_{n+1}}{(n+1)!} - 1\right],$$

so that $p_{2n}^{(1)}(0) = (-1)^n\ a^n$. Hence

$$\frac{\left(p_{2n}^{(1)}(0)\right)^2}{\lambda'_{2n}} = \frac{n!}{(2\lambda+1)_n},$$

and

$$\sum_{n=0}^{\infty} \frac{\left(p_{2n}^{(1)}(0)\right)^2}{\lambda'_{2n}} = {}_2\mathbb{F}_1\left(\begin{matrix} 1 & 1 \\ 2\lambda+1 & \end{matrix}\middle| 1\right) = \frac{2\lambda}{2\lambda-1}. \tag{5.14}$$

Since $p_{2n+1}^{(1)}(0) = 0$, the assertion follows from (2.23). This completes the proof. ■

Now numerator and denominator in (5.12) have , outside of $\bar{L}$, simple poles as their only singularities, and these are located at the same points, i.e., at the points x where $B(\omega(x)) = 0, 1, 2, \ldots$; hence, these are not singularities of $R^{(1)}(\omega)$. The poles of $R^{(1)}(\omega)$, if any, will then be located at those points x where

$$\int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du = 0, \tag{5.15}$$

provided that

$$\int_0^1 (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du \neq 0 \tag{5.16}$$

at those points. We need:

LEMMA 5.1 *Let* $x \in \mathbb{R}$ *be such that* (5.15) *holds and assume that there is* $n \geq 1$ *such that* $A + k \neq 0$, $k = 1, 2, \ldots, n$ *and that* $-B + n > -1$. *Then* (5.16) *holds.*

Proof. Since the integral in (5.15) is $\neq 0$ if $x \in L$, i. e., if $\alpha \in (-1, 1)$, as follows from (4.12) and (4.13), for example, we may assume that $x \notin \overline{L}$, so that $\beta^2 < 1$. Now assume that the integrals in both (5.15) and (5.16) vanish. Then

$$\int_0^{1]} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B}\, du$$

$$= \int_0^{1]} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du - \int_0^{1]} u(1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du = 0.$$

Since

$$(A + 1)\, \beta^2 \int_0^{1]} u(1 - \beta^2 u)^{-A-2} (1 - u)^{-B}\, du$$

$$+ B \int_0^{1]} u(1 - \beta^2 u)^{-A-1} (1 - u)^{-B-1}\, du + \int_0^{1]} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B}\, du = 0$$

when the integrals are all convergent (integration by parts), the same is true of the Hadamard integrals (which are analytic continuations of the proper integrals). Then, taking into account that $A + 1 \neq 0$, we conclude that

$$\int_0^{1]} u(1 - \beta^2 u)^{-A-2} (1 - u)^{-B}\, du = 0$$

and, from

$$\int_0^{1]} (1 - \beta^2 u)^{-A-1} (1 - u)^{-B}\, du$$

$$= \int_0^{1]} (1 - \beta^2 u)^{-A-2} (1 - u)^{-B}\, du - \beta^2 \int_0^{1]} u(1 - \beta^2 u)^{-A-2} (1 - u)^{-B}\, du,$$

that

$$\int_0^{1]} (1 - \beta^2 u)^{-A-2} (1 - u)^{-B}\, du = 0.$$

Hence

$$\int_0^{1} (1 - \beta^2 u)^{-A-2}(1 - u)^{-B+1}\, du = 0.$$

Iteration of these arguments, using that $A + k \neq 0$, $k = 1, 2, \ldots, n$, shows that

$$\int_0^{1} (1 - \beta^2 u)^{-A-n-1}(1 - u)^{-B+n}\, du = 0.$$

But this is impossible, since the integral is convergent and A is real. ■

Remark 5.1. Observe that the integrals in (5.15) and (5.16) can not vanish simultaneously if $B < 1$, as in that case also

$$\int_0^{1} (1 - \beta^2 u)^{-A-1}(1 - u)^{-B}\, du = 0$$

even if $A + 1 = 0$, and this is absurd.

Let

$$\overline{F}(x) = \int_0^{1} u(1 - \beta^2 u)^{-A-1}(1 - u)^{-B-1}\, du,$$
$$\overline{G}(x) = \int_0^{1} (1 - \beta^2 u)^{-A-1}(1 - u)^{-B-1}\, du. \tag{5.17}$$

The poles of $\overline{G}(x)$ are located at those points $\pm y_k$, $y_1 > y_2 > \ldots > y_n > \ldots > \sqrt{a} + \sqrt{b}$ and $y_k \to \sqrt{a} + \sqrt{b}$, such that $B(\omega(y_k)) = k \geq 1$. When $a < b$, these are the only poles of $G(x)$, since $G(x)$ is then continuous and non-vanishing in $(\sqrt{a} - \sqrt{b}, \sqrt{b} - \sqrt{a})$. When $a > b$, $G(x)$ has also poles at those points $\pm x_k$, $0 < x_0 < x_1 < \ldots < x_n < \ldots < \sqrt{a} - \sqrt{b}$ and $x_k \to \sqrt{a} - \sqrt{b}$, such that $B(w(x_k)) = k \geq 0$. This was established in Section 4. Since

$$\operatorname{Res}\left(\overline{G}, x\right) = -\frac{(2\lambda)_n}{n!}\beta^{2n}\,(1-\beta^2)^{2\lambda-1}\left(\frac{dB}{dx}(x)\right)^{-1} > 0 \qquad (5.18)$$

at $x = x_n, y_n$, we can prove that

LEMMA 5.2. *There are* $y_n > \overline{y}_n > y_{n+1}$, $n \geq 1$, *and* $x_m < \overline{x}_m < x_{m+1}$, $m \geq 0$, *unique, such that* $\overline{G}(\overline{x}_m) = \overline{G}(\overline{y}_n) = 0$. *Furthermore,* $\overline{F}(\overline{x}_m) \neq 0 \neq \overline{F}(\overline{y}_n)$ *for* $m \geq 0$, $n \geq 1$, *and* $\overline{x}_m, \overline{y}_n$ *are poles of* $R^{(1)}(\omega(x))$. *Moreover*

$$\operatorname{Res}\left(R^{(1)}(\omega(x)),\, x\right) = \frac{\overline{F}(x)}{\overline{G}'(x)}\,\beta(\omega(x)),\ x = \overline{x}_m, \overline{y}_n. \qquad (5.19)$$

Proof. Since $\operatorname{Res}(\overline{G}, x) > 0$ at $x = x_m, y_n$, $\overline{G}(x)$ vanishes in (x_m, x_{m+1}), $m \geq 0$, and in (y_{n+1}, y_n), $n \geq 1$, at least once. The same argument as in Lemma 4.3. then shows that exactly once. Relation (5.19) is a simple consequence of $\overline{x}_m, \overline{y}_n$ being simple zeros of $\overline{G}$. Now we prove that $\overline{F}(\overline{x}_m) \neq 0$, $m \geq 0$, so that $\operatorname{Res}(R^{(1)}(\omega(x)), x_m) \neq 0$. In (x_0, x_1) we have that $0 < B < 1$. Since $\overline{G}(\overline{x}_0) = 0$, Remark 5.1 ensures that $\overline{F}(\overline{x}_0) \neq 0$. On the other hand, in (x_m, x_{m+1}) $m \geq 1$, we have $m < B < m + 1$, so that $-B + m > -1$. Since $A + k = -B + k - 2\lambda < -m + k - 2\lambda \leq -2\lambda$, $k = 1, 2, \ldots, m$, Lemma 5.1 then implies, as $\overline{G}(\overline{x}_m) = 0$, that $\overline{F}(\overline{x}_m) \neq 0$. The proof of the assertion $\overline{F}(\overline{y}_n) \neq 0$ follows essentially the same argument. ■

Hence

THEOREM 5.3. *Assume* $a \neq b$. *Then the support* $\operatorname{Supp}\mu_1$ *of* μ_1 *is* $\overline{L} \cup D_{\mu_1}$, *with* L *given by* (4.8) *and* $D_{\mu_1} = D_0 \cup D_1$, *where* $D_1 \subseteq (-|\sqrt{a} - \sqrt{b}\,|, |\sqrt{a} - \sqrt{b}\,|)$ *and* $D_1 \subseteq ([-M', -\sqrt{a} - \sqrt{b}\,) \cup (\sqrt{a} + \sqrt{b}\,, M']$ *are as follows:*

i) *If* $a < b$, *then* $D_0 = \{0\}$, $D_1 = \{\pm \overline{y}_n \mid n \geq 1\}$.

ii) *If* $a > b$, *then* $D_1 = \{\pm \overline{x}_n \mid n \geq 0\}$, $D_1 = \{\pm \overline{y}_n \mid n \geq 1\}$.

Futhermore, the absolutely continous part μ_{1c} *of* μ_1 *is supported by* $\overline{L}$ *and given by* (5.8), *and the discrete part,* μ_{1p}, *has masses in* D_{μ_1} *as follows:*

$$\mu(\{0\}) = \frac{1}{F_1\left(\begin{matrix} 1 & 1 \\ 2\lambda+1 & \end{matrix} \middle| \frac{a}{b}\right)}, \quad a<b \tag{5.20}$$

$$\mu(\{\pm \overline{x}_n\}) = \frac{1}{\overline{x}_n} \sqrt{\frac{a}{b}} \frac{\overline{F}(\overline{x}_n)}{\overline{G}'(\overline{x}_n)} \beta(\omega(\overline{x}_n)), \quad n \geq 0. \tag{5.21}$$

$$\mu(\{\pm \overline{y}_n\}) = \frac{1}{\overline{y}_n} \sqrt{\frac{a}{b}} \frac{\overline{F}(\overline{y}_n)}{\overline{G}'(\overline{y}_n)} \beta(\omega(\overline{y}_n)), \quad n \geq 1. \tag{5.22}$$

Proof. All that remains to be proved is relations (5.21) and (5.22). But (5.21) is a consequence of (5.19), taking into account that

$$\operatorname{Res}(p^{(1)}(x), \overline{x}_n) = \frac{1}{\overline{x}_n} \operatorname{Res}(R^{(1)}(\omega(x)), \overline{x}_n).$$

The proof of (5.22) is similar. ■

Remark 5.2. If $a = b$, $\operatorname{Supp}\mu_1 = \overline{L} \cup D_1$, with D_1 as in Theorem 5.3., but $C_{\mu_1} = \overline{L} - \{0\}$, $P_{\mu_1} = D_1 \cup \{0\}$, so that 0 is an embedded mass point of μ_1; i.e., if $\mathbb{J}$ is the Jacobi matrix (2.29) with

$$a_n = 0, \; b_{2n+1} = \sqrt{a}, \; b_{2n+2} = \sqrt{\frac{2\lambda+n+1}{n+1} b}, \; n \geq 0, \tag{5.23}$$

its Jacobi operator $\hat{\mathbb{J}}$ has 0 as an embedded eigenvalue. This is the interesting feature of the system $\{p_n^{(1)}(x)\}$.

§ 6. Final observations

As mentioned in the introduction and in Section 2, the systems of polynomials considered in this paper originated in some models in solid state physical-chemistry. The models are usually handled numerically or by simulation, and conclusions are thus obtained. In the case of our polynomials, for example, there is evidence supporting the infinite number of discrete energy levels and of resonances suggesting end-point or embedded masses. The available data also support the existence of end-point masses when $a \neq b$ and the end-points are limit points of discrete masses, a situation we have been unable to deal with.

From the mathematical point of view, our models disprove what intuition says about embedded mass-points being obtained though a sieving process when disjoint intervals, one of which at least bears an end-point mass, are brought together, or when an isolated mass in between two intervals is trapped when the intervals weld. In the case of the polynomials in section 5, for example, the intervals of L weld when $a = b$ and an embedded mass shows up, but the end-points of theses intervals are free of masses and no isolated mass is trapped when $a \to b+$. On the contrary, the polynomials in Section 4 have masses at end-points of intervals, that vanish when the intervals are brought together, and no discrete mass in $(\sqrt{b} - \sqrt{a}, \sqrt{a} - \sqrt{b})$ is trapped when $b \to a-$.

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