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# A COMPLETENESS THEOREM FOR TWO-PARAMETER STOCHASTIC PROCESSES

by

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## §1. Introduction

In this short note we are going to prove a completeness theorem for a logic that is adequate for the study of stochastic processes with 2-parameters, a result that can be seen as a natural step within the development of the so called probability logics. It builds on previous definitions and results of Keisler, [K1] and [K2], and in order to avoid repetitions we are going to assume the reader is already familiar with sections 1 and 2 from [K1], so that here we just limit ourselves to add whatever is needed to handle the new concepts. In this section we present the basic definitions of the theory of 2-parameter stochastic processes, pointing out its differences with the one parameter case. A good introduction to the general theory of 2-parameter is [W] together with the collection of articles in [KMS].

DEFINITION 1. A 2-parameter adapted space is a structure of the form $\underline{\Omega} = (\Omega, (\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2}, P)$ where $(\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2}$, called 2-parameter filtration, is a family of $\sigma$-algebras on $\Omega$, satisfying the following properties:

(i) $(\Omega, \mathcal{F}_{(1,1)}, P)$ is a probability space.

(ii) Given (s, t) and (s', t') in $[0, 1]^2$ such that $s \leq s'$ and $t \leq t'$, -denote this relation by $(s, t) \leq (s', t')$- then $\mathcal{F}_{(s,t)} \subseteq \mathcal{F}_{(s',t')}$.

(iii) $\mathcal{F}_{(0,0)}$ is P-complete.

(iv) For each (s, t), $\mathcal{F}_{(s,t)} = \bigcap_{(s',t') \in ((s,t),(1,1)]} \mathcal{F}_{(s',t')}$, where the interval ((s, t), (1, 1)] denotes the set $\{(s', t') : s < s', t < t' \text{ and } (s', t') \leq (1, 1)\}$.

DEFINITION 2. A 2-parameter stochastic process on $\Omega$ is a function $X: \Omega \times [0, 1]^2 \to \mathbb{R}$, measurable with respect to $\mathcal{F}_{(1,1)} \times \mathcal{L}$, where $\mathcal{L}$ is the lebesgue measure on $[0, 1]^2$.

The point that makes the theory of 2-parameter stochastic processes different from the traditional theory of stochastic processes is the nonlinearity of the time set. This fact, in the context of the information contained in the filtration, changes the theory in many aspects, so that results about 2-parameter processes are not just trivial extensions of known cases. For this reason, new conditions are imposed on the filtrations so that a reasonable theory can be developed in two parameters (see [W]); the following one, known as the Cairoli-Walsh condition or condition (F4), introduced in [CW], is crucial for the development of the general theory of 2-parameter processes, in particular: martingales, stochastic analysis and related concepts.

DEFINITION 3. A 2-parameter filtrations $(\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2}$, satisfies condition (F4) if:
Given (s, t) and (s', t') such that $s \leq s'$ and $t \geq t'$, then $\mathcal{F}_{(s,t)}$ and $\mathcal{F}_{(s',t')}$ are conditionally independent given $\mathcal{F}_{(s,t')}$.

The above definition involves the notion of conditionally independent $\sigma$-algebras, which is simple but not very well known. We present it here so that the reader can see how it is axiomatized in the next section. In fact, the definition that we now give is one of the equivalent formulations of the original concept, see [L].

DEFINITION 4. Given σ-algebras $\mathfrak{F}$, $\mathfrak{G}$, $\mathcal{H}$ on a common probability space, we say that $\mathfrak{F}$ and $\mathfrak{G}$ are conditionally independent given $\mathcal{H}$, if:

for all X and Y positive integrable random variables measurable with respect to $\mathfrak{F}$ and $\mathfrak{G}$ respectively, we have

$$E(XY \mid \mathcal{H}) = E(X \mid \mathcal{H})\, E(Y \mid \mathcal{H}).$$

We are now ready to move on to the probability logic that allows us to study the theory of 2-parameter stochastic processes from the point of view of model theory. It is an extension of Keisler's adapted probability logic, $L_{ad}$, and we denote it by $L^2_{ad}$. Since it isn't our purpose here to deal with advanced probabilistic concepts, the probability notions we have introduced above are the only ones we are going to use in the next section.

## §2. The Completeness Theorem

As it has become customary, the first step that is taken once a new logic is introduced is to prove a completeness theorem for it. This has been the case for the different probability logics that have been studied so far and the new logic we are presenting here is no exception. Looking at the way previous completeness theorems have been proved in probability logic ([K1], [K2], [H], [F1], [F2]) we can see that in general, when possible, one uses a translation argument in order to reduce the completeness problem for the new logic to a property of a logic for which we already have proved its completeness theorem. We are going to follow this idea here. Taking into account the observations of section 1, we can modify in a more or less straightforward way the proof of the completeness theorem for $L_{ad}$ in [K1].

In [K1], page 69, Keisler explained that his proof, with the obvious modifications, also works for processes with several parameters. We don't bother to write down these obvious modifications, instead we leave them to the interested reader. As we explained in the previous section, the main new feature of $L^2_{ad}$, is the nonlinearity of time and the way it affects the information contained in the filtration. Therefore, from the point of view of the completeness theorem for the logic we are developing, all we have to do is to come up with an axiom that captures the Cairoli-Walsh condition F4. The only change that we have to make with respect to the presentation in [K1] of $L^2_{ad}$, is to take into account that in the definition of $L^2_{ad}$-formulas the conditional expectation operator works with a 2-parameter filtration, therefore the new formation rule is : If $\alpha$ is a term then the expression $E[\alpha \mid (s, t)]$ is a term. Consequently, we have to rewrite the conditional expectation axioms that correspond to A7 in page 66 of [K1] (we seem to need a slight change in Keisler's A7. (ii)). Here they are:

(i) For all $r \in \mathbb{R}$,

$$\int \chi_{[r,\infty)} (E[\alpha \mid (s, t)) \cdot \beta d\omega = \int \chi_{[r,\infty)} (E[\alpha \mid (s, t)) \cdot E[\beta \mid (s, t)] d\omega.$$

Observe that $\chi_{[r,\infty)}$ is the characteristic function of the interval $[r, \infty)$. Being rigorous, the above expression isn't a formula in our language since $\chi_{[r,\infty)}$ isn't a continuous function; but it can be seen as the abbreviation of a countable conjunction of true formulas as the reader can easily verify.

(ii) $s \leq t \wedge u \leq v \Rightarrow E[E[\alpha \mid (s, u)] \mid (t, v)] = E[\alpha \mid (s, u)]$.

The new axiom that handless condition (F4) is:

(iii) For all terms $\alpha$ and $\beta$, and time variables s, t, u, v such that $s \leq t$ and $u \geq v$ then

$$E[E[\alpha|(s, u)]\cdot E[\beta|(t, v)]|(s, v)] = E[E[\alpha|(s, u)]|(s, v)]\cdot E[E[\beta|(t, v)]|(s, v)].$$

We are ready to give the proof of the completeness theorem. As we indicated before, we present here what is really new with respect to [K1]. Theorem 3.2 of [K1], module the obvious modifications suggested by Keisler, remains the same. We concentrate in the analog of theorem 3.3 for $L^2_{ad}$ of [K1]. Assume L only contains a stochastic process symbol X. We include some details that were not explicitly presented in Keisler's paper.

**THEOREM.** $L^2_{ad}$ *is complete.*

**Proof.** Let ø be a consistent $L^2_{ad}$-sentence. We want to show that it has a model. As usual we work within an admissible set which contains ø and has enough on it for the construction we are carrying out.

Define as in [K1] (a similar argument appears in [F2]) a translation h of $L^2_{ad}$-formulas into $K_d$-formulas, where K is a language that has the following stochastic process symbols: one for X, the original symbol from L, and also one new symbol $X_\gamma$ for each $L^2_{ad}$-term γ of the form $E[\alpha \mid (s, u)]$. The translation h is then defined in the natural way, first for terms and then for formulas:

$h(\alpha) = \alpha$, for α atomic term,

$h(\gamma) = X_\gamma$, where γ is the term $E[\alpha \mid (s, u)]$,

h commutes with integrals and respects composition.

The extension to formulas is the obvious one.

Now let Σ be the set of $K_d$-formulas made up of the translations of the $L^2_{ad}$-axioms together with h(ø). The next step consists in showing that Σ has a $K_d$-model. This is the easy part. Simply observe, as in [K1], that the translation preserves consistency due to the fact the $L^2_{ad}$-axioms are sound and the inference rules are the same.

So let $\underline{\Gamma} = (\Gamma, X, X_\gamma)$ a $K_d$-model for Σ. From this model we will show how to construct an $L^2_{ad}$-model for ø. This is really the main new aspect of this proof, and here we are going to see how the axiom comes into the picture.

For each tuple $(a, b) \in [0, 1] \times [0, 1]$, let $\mathcal{F}_{(a, b)}$ be the σ-algebra generated by the sets of the form $\{\omega \in \Gamma : X_\gamma(\omega, (a, b), \vec{c}) \geq r\}$ with $r \in \mathbb{R}$, γ a term of the form $E[\alpha \mid (s, u)]$ and where we interpret the variable (s, u) by (a, b).

We now prove that this family of $\sigma$-algebras satisfies the following conditions:

(i) If $(a, b) \le (c, d)$ then $\mathcal{F}_{(a, b)} \subseteq \mathcal{F}_{(c, d)}$ (i.e. it is increasing).

(ii) If $a \le c$ and $b \ge d$ then $\mathcal{F}_{(a, b)}$ and $\mathcal{F}_{(c, d)}$ are conditionally independent given $\mathcal{F}_{(a, d)}$.

In order to verify condition (i) we check that any generator A of the $\sigma$-algebra $\mathcal{F}_{(a, b)}$ belongs to $\mathcal{F}_{(c, d)}$. This, of course, gives us (i). We use axiom (ii). First we need its translation:

If we let $\theta((s, u), (t, v))$ be the formula $E[E[\alpha \mid (s, u)] \mid (t, v)]$ and $\delta((s, u))$ the formula $E[\alpha \mid (s, u)]$, the translation says $X_\theta(\omega, (a, b), (c, d)) = X_\delta(\omega, (a, b))$.

From this we then have that for $r \in \mathbb{R}$,

$$X_\theta(\omega, (a, b), (c, d)) \ge r \text{ iff } X_\delta(\omega, (a, b)) \ge r.$$

Now observe that the event defined by the left hand side of this equation belongs to $\mathcal{F}_{(c, d)}$, taking (a, b) as a parameter, and the one on the other side of the equation belongs to $\mathcal{F}_{(a, b)}$. This is what is needed since the generators of $\mathcal{F}_{(a, b)}$ are all of this form.

Condition (ii) is the one that shows that the new axiom does its job. The following fact is the key step:

$$(*) \quad \text{For every } \beta,\ X_{E[\beta \mid (s, t)]}(\omega, (a, b)) = E[h(\beta) \mid \mathcal{F}_{(a, b)}](\omega) \text{ a.s.}$$

Since by definition, $X_{E[\beta \mid (s, t)]}(\omega, (a, b))$ is $\mathcal{F}_{(a, b)}$-measurable, and the generators of this $\sigma$-algebra are of the form $X_{E[\alpha \mid (a, b)]} \ge r$ then, in order to prove (*) we have to check for all $\alpha$ and r:

$$(**) \quad \int_{\{X_{E[\alpha \mid (a, b)]} \ge r\}} E[h(\beta) \mid \mathcal{F}_{(a, b)}]\, d\omega = \int_{\{X_{E[\alpha \mid (a, b)]} \ge r\}} X_{E[\beta \mid (a, b)]}\, d\omega .$$

For this we are going to need axiom (i). Its translation is :

$$\int \chi_{[r, \infty)}(X_{E[\alpha \mid (a, b)]}) \cdot h(\beta)\, d\omega = \int \chi_{[r, \infty)}(X_{E[\alpha \mid (a, b)]}) \cdot (X_{E[\beta \mid (a, b)]})\, d\omega.$$

In the model this equation means:

$$(***) \quad \int_{\{X_{E[\alpha \mid (a,b)]} \geq r\}} h(\beta)\, d\omega = \int_{\{X_{E[\alpha \mid (a,b)]} \geq r\}} X_{E[\beta \mid (a,b)]}\, d\omega .$$

By definition of conditional expectation, we have:

$$\int_{\{X_{E[\alpha \mid (a,b)]} \geq r\}} h(\beta)\, d\omega = \int_{\{X_{E[\alpha \mid (a,b)]} \geq r\}} E[h(\beta) \mid \mathcal{F}_{(a,b)}]\, d\omega .$$

And this last fact together with (***), gives us (**), as we wanted to verify.

With (*) we can easily see that condition F4 is satisfied. We need to translate axiom F4. Using (*) it becomes:

$$E[E[h(\alpha) \mid \mathcal{F}_{(a,b)}]\, E[h(\beta) \mid \mathcal{F}_{(c,d)}] \mid \mathcal{F}_{(a,d)}] = E[E[h(\alpha) \mid \mathcal{F}_{(a,b)}] \mid \mathcal{F}_{(a,d)}]\, E[E[h(\beta) \mid \mathcal{F}_{(c,d)}] \mid \mathcal{F}_{(a,d)}]$$

And this is exactly the definition of conditional independence of $\mathcal{F}_{(a,b)}$ and $\mathcal{F}_{(c,d)}$ with respect to $\mathcal{F}_{(a,d)}$, taking into account the way these $\sigma$-algebras were defined. The rest of the proof is as in [K1] and we don't bother to write it down. ■

## §3. Comments

As an anonymus referee has pointed out, we could view the above theorem in a different way: what the completeness theorem is proving is that a consistent theory closed under its consequences in the stochastic process logic has a model in which property the Cairoli-Walsh condition (F4) holds.

This paper is just a first step in the development of $L^2_{ad}$ and we haven't aimed for the most general results. There are several alternatives for continuing research in this area. As suggested by the referee, it could be interesting to prove, with methods similar to the ones presented here, a completeness theorem for the logic that one obtains replacing the Cairoli-Walsh condition by a weaker one, known as CQI, that in some contexts is useful (see [KS] and [D]); or even for stochastic processes indexed by a general class of partially ordered sets.

Another important line of work, of interest also for probabilists, is the development of the hyperfinite theory of 2-parameter stochastic processes using non-standard analysis. One should develop a theory of 2-parameter adapted distributions analog to the one in [HK], study the expressive power and then prove universality, homogeneity and saturation theorems for $L^2_{ad}$, similar to those that Keisler proved for $L_{ad}$ in [K3]. We have begun work on this elsewhere. These developments in the theory with one parameter have shown how useful the interaction of methods and ideas from model theory, nonstandard analysis and stochastic processes can be, and we expect that the same can happen in the context of two parameter stochastic processes.

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