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INTEGRAL GEOMETRY OF THE ACTION OF ST(3) ON THE SPACE E^3

by

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ABSTRACT. Some aspects of the integral geometry of the action on E^3 of the group ST(3) of upper triangular 3×3-matrices of determinant one are studied. Measurability for sets of linear surfaces is considered and it is shown that invariant measures do exist for sets of points or planes but not for sets of lines. Measurability for sets of couples point-line and point-plane is also discussed, and the existence of invariant measures is established in both cases. Explicit geometric formulae are given for the measures whenever they exist.

§1. Introduction

In [1] we have shown that

1. *The sets of linear subspaces of projective space* P_n *do not admit invariant measures for the action of the triangular group* $ST(n+1)$.
2. *The sets of couples* (P, H), *where* H *is* h*-plane of projective space* P_n *and* $P \notin H$, *has an invariant measure for the action of* $ST(n+1)$ *only if* $h = n - 1$.

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3. The set of couples (P, H), where H is a plane of projective space P_n and $P \in H$, has no invariant measure for the action of ST(n + 1).

In this paper we similarly study the action of the triangular group ST(3) on the space E^3. Some positive and negative answers are also obtainable in this situation, and geometric formulae for the invariant measures of the action are explicitly given when they exist.

§2. The action of ST(3) on E^3

The triangular group ST(3) acts on E^3 by means of the linear transformation

$$x'_k = \sum_{i=k}^{3} a_{ki} x_i \qquad (1)$$

where $A = (\alpha_{ki})$ is an upper triangular matrix of determinant one and (x_1, x_2, x_3), (x'_1, x'_2, x'_3) are point-coordinates in E^3. Observe that condition $\det(A) = 1$ restricts to five the number of parameters in the action.

§3. Integral geometry of the action. Maurer-Cartan forms

The Maurer-Cartan 1-forms, or left invariant 1-forms, of ST(3) (see [2]) are the linearly independent elements of the matrix $(\omega_{ij}) = A^{-1}\, dA$. Thus

$$\omega_{ij} = \sum^{j} A_{ki}\, da_{kj}, \; i,j = 1, 2, 3, \qquad (2)$$

with A_{ki} being the cofactor of a_{ki} in A. Since $\det(A) = 1$, the trace of (ω_{ij}) vanishes, i.e., $\omega_{11} + \omega_{22} + \omega_{33} = 0$. Equations (2) then allow to recover the da_{kj} in dA from the ω_{ij} by means of

$$da_{ij} = \sum_{k=i}^{j} a_{ik}\, \omega_{kj}, \; i,j = 1, 2, 3; \; \omega_{11} + \omega_{22} + \omega_{33} = 0. \qquad (3)$$

The structure equations for matrix groups are determined by the exterior product of the 1-forms matrices (see [2]). Thus, in the particular case of ST(3) we have

$$(d\omega_{ij}) = (\omega_{ij}) \wedge (\omega_{ij}), \quad i, j = 1, 2, 3,$$

i.e.,

$$d\omega_{ij} = \sum_{k=i}^{j} \omega_{ik} \wedge \omega_{kj}, \; i,j = 1, 2, 3. \tag{4}$$

Relations (4) are known as the Mauer- Cartan structure equations.

§4. Invariant measures with respect to ST(3) for sets of subspaces and of couples point-subspaces

4.1. Invariant measures for sets of points

M. S. Stoka [4] proves that a transitive subgroup of a measure group is itself measure and that the invariant measure it defines the same as that of the group. Since ST(3) is a subgroup of the homogeneous affin unimodular group, which is measure (see [4]), ST(3) is measurable, and the volume element

$$dP = dx \wedge dy \wedge dz \tag{5}$$

defines the ST(3)-invariant measure on the set of non-vanishing points of E^3.

4.2. Invariant measure for sets of lines

The group ST(3) acts transitively on the set of lines in E^3 not passing through the origin, the line x = y = z -1 being transformed into that G of equations

$$\begin{aligned} x' &= a_{33}^{-1} (a_{11} + a_{12} + a_{13})\, z' - (a_{11} + a_{12}) \\ y' &= a_{33}^{-1} (a_{22} + a_{23})\, z' - a_{22}, \end{aligned} \tag{6}$$

and invariance order the group action for this set of lines demanding that

$$d\left(\frac{a_{11}+a_{12}+a_{13}}{a_{33}}\right)=0, \quad d(a_{11}+a_{12})=0, \quad d\left(\frac{a_{22}+a_{23}}{a_{33}}\right)=0. \qquad (7)$$

In view of (2), (7) translates into

$$\omega_{22}=0, \omega_{11}+\omega_{13}=0, \omega_{11}+\omega_{23}=0, \omega_{11}+\omega_{12}=0.$$

Hence, if this set of lines has an invariant measure and

$$W=\omega_{22}\wedge(\omega_{11}+\omega_{13})\wedge(\omega_{11}+\omega_{23})\wedge(\omega_{11}+\omega_{12}), \qquad (8)$$

we should have dW = 0. But, from (4),

$$d\omega=\omega_{11}\wedge\omega_{12}\wedge\omega_{13}\wedge\omega_{22}\wedge\omega_{23}\neq 0$$

as the 1-forms involved are linearly independent. Hence, invariant measures for sets of lines fail to exist.

4.3. Invariant measures for sets of couples point-line

The group ST(3) acts transitively on the set of couples (P, G) where G is a line not through the origin and P is a point on G. Since ST(3) depends on five parameters, and five parameters are needed to define the set of couples (P,G), The invariant measure of the action is given by the kinematic density, i.e.,

$$d(P,G)=\omega_{11}\wedge\omega_{12}\wedge\omega_{13}\wedge\omega_{22}\wedge\omega_{23} \qquad (9)$$

which, in view of (2), is

$$d(P,G)=da_{11}\wedge da_{12}\wedge da_{13}\wedge da_{22}\wedge da_{23} \qquad (10)$$

in terms of the groups coordinates.

To give a geometrical interpretation of this measure, recall that if P(0, 0, 1) and G is the line x = z - 1 = y, ST(3) transforms (P, G) into ((a_{13}, a_{23}, a_{33}), G'), with G' determined by (7). More generally, if P = (x, y, z) and G is

$$x = z \cos\phi \tan\theta + p, \qquad y = z \operatorname{sen}\phi \tan\theta + q \tag{11}$$

where (p, q) are the coordinates of the point of intersection of G with the XY-plane, θ is the angle of G with the z-axis, and ϕ that which its projection on the XY-plane makes with the X-axis, the argument in 4.2, equations (1) and (6) shows that

$$\begin{gathered} x = a_{13}, \quad y = a_{23}, \quad z = a_{33} \\ \cos\theta \tan\theta = a_{11}\, a_{22}\, (a_{11} + a_{12} + a_{13}) \\ \operatorname{sen}\phi \tan\theta = a_{11}\, a_{22}\, (a_{22} + a_{23}) \\ p = -(a_{11} + a_{12}) \\ q = -a_{22} \end{gathered} \tag{12}$$

so that

$$\tan\theta \sec^2\theta\, dx \wedge dy \wedge dz \wedge d\phi \wedge d\theta = a_{22} da_{11} \wedge da_{12} \wedge da_{13} \wedge da_{22} \wedge da_{23}.$$

From (12) it also follows that $a_{22} = z \operatorname{sen}\phi \tan\theta - y$, and, from (10), the invariant measure for couples (P, G) as above is given by

$$d(P, G) = \frac{z^3 \operatorname{sen}\theta}{\cos^2\theta\, (z \operatorname{sen}\phi \operatorname{sen}\theta - y \cos\theta)} dP \wedge d\phi \wedge d\theta. \tag{13}$$

4.4. Invariant measures for sets of planes

As before, ST(3) acts transitively on the set of planes of E^3 not going through the origin, the plane x + y + z = 1 being transformed into that E of equation

$$a_{11}^{-1} x' + a_{33}(a_{11} - a_{12}) y' + (a_{12} a_{23} - a_{13} a_{22} - a_{23} a_{11} - a_{22} a_{11}) z' = 1, \tag{14}$$

and the variance condition demanding that

$$da_{11}^{-1} = 0, \quad d[a_{33}(a_{11} - a_{12})] = 0, \quad d(a_{12} a_{23} - a_{13} a_{22} - a_{23} a_{11} - a_{22} a_{11}) = 0.$$

Relations (2) imply that the above relationships are equivalent to

$$\omega_{11} = 0, \quad \omega_{12} + \omega_{22} = 0, \quad \omega_{13} + \omega_{23} - \omega_{22} = 0$$

and structure equations (4) insure that if

$$\omega = \omega_{11} \wedge (\omega_{12} + \omega_{22}) \wedge (\omega_{13} + \omega_{23} - \omega_{22})$$

then dw = 0. Thus, an invariant measure for sets of planes exists and its invariant density dE is

$$dE = \omega_{11} \wedge (\omega_{12} + \omega_{22}) \wedge (\omega_{13} + \omega_{23} - \omega_{32}). \tag{15}$$

To obtain a generical interpretation of this measure we write (14) in spherical coordinates as

$$x \cos\phi \operatorname{sen}\theta + y \operatorname{sen}\phi \operatorname{sen}\theta + z \cos\theta = p, \tag{16}$$

so that

$$a_{11}^{-1} = \frac{\cos\phi \operatorname{sen}\theta}{p}, \quad a_{33}(a_{11} - a_{12}) = \frac{\operatorname{sen}\phi \operatorname{sen}\theta}{\rho}$$
$$a_{12}a_{23} - a_{13}a_{22} - a_{23}a_{11} - a_{22}a_{11} = \frac{\cos\theta}{\rho} \tag{17}$$

which together with (15) yield

$$dE = \frac{\operatorname{sen}\theta \, d\theta \wedge d\phi \wedge d\rho}{p^4}.$$

This can also be written as

$$dE = \frac{d\sigma \wedge d\rho}{p^4} \tag{18}$$

with $d\sigma = \operatorname{sen}\theta \, d\theta \wedge d\phi$ being the area element of the unit sphere whose radius is the unit vector which is perpendicular to the plane.

4.5. Invariant measures for sets of couples point-plane

The set of couples (P, E), where E is a plane avoiding the origin and P is a point on E, is a five parameters family. Thus, as before, ST(3) acts transitively on this set, and the invariant measure of the action is given by the kinematic density of the group:

$$d(P, E) = \omega_{11} \wedge \omega_{12} \wedge \omega_{13} \wedge \omega_{22} \wedge \omega_{23} = da_{11} \wedge da_{12} \wedge da_{13} \wedge da_{22} \wedge da_{23} \quad (19)$$

To give a geometrical interpretation of this density assume P = (x, y, z) and E is given by (16). Recalling (14), which determines the transform of ((0, 0, 1), x + y + z = 1) under the action, we obtain that

$$a_{13} = x, \quad a_{23} = y, \quad a_{33} = z,$$

$$a_{11}^{-1} = \frac{\cos\varnothing \ \mathrm{sen}\theta}{p}, \quad a_{33}(a_{11} - a_{12}) = \frac{\mathrm{sen}\varnothing \ \mathrm{sen}\theta}{\rho} \quad (20)$$

$$a_{12}a_{23} - a_{13}a_{22} - a_{23}a_{11} - a_{22}a_{11} = \frac{\cos\theta}{p}$$

Passing to the differentials of the left hand side terms and taking their exterior product we get

$$\left(\frac{a_{33}}{a_{11}}\right) da_{11} \wedge da_{12} \wedge da_{13} \wedge da_{22} \wedge da_{23} = \frac{\mathrm{sen}\theta\,(x\cos\varnothing + z)\, d\theta \wedge d\varnothing \wedge dP}{(\cos\theta(x\cos\varnothing + z) + y\,\mathrm{sen}\varnothing\,\mathrm{sen}\theta)^2} \quad (21)$$

where dP = dx ∧ dy ∧ dz. From (20) we further obtain that

$$\frac{a_{11}}{a_{33}} = \frac{P\ \mathrm{sen}\theta}{\cos\varnothing\ \cos\theta}, \quad P = \cos\theta\,(\cos\varnothing + z) + y\ \mathrm{sen}\ \varnothing\ \mathrm{sen}\theta,$$

and, with $d\sigma = \mathrm{sen}\theta \wedge d\theta\, d\varnothing$ denoting the area element of the unit sphere through the extreme of the unit vector orthogonal to E, we may write (19) in the form

$$d(P, E) = \frac{(x\cos\varnothing + z)\, d\sigma \wedge dP}{\cos\varnothing\ \cos\theta\,(\cos\theta\,(x\cos\varnothing + z) + y\ \mathrm{sen}\varnothing\ \mathrm{sen}\ \theta)^2}. \quad (22)$$

Summing up, we can state the following results, which have been proved above:

THEOREM 1. *The sets of non zero points and of planes not passing through the origin in* E^3 *admit invariant measures for the action of* ST(3) *on these sets. The density of the measures are given by* (5) *and* (18), *respectively.*

THEOREM 2. *The set of lines on* E^3 *which avoid the origin fails to have an invariant measure for the action of* ST(3) *on this set.*

THEOREM 3. *The sets of couples* (P, G), *where* P *is a point on* G *and* G *is either a line or a plane not passing through the origin, admit invariant measures for the actions of* ST(3) *on these sets. Geometric expressions for the densities of these measures are given respectively by* (13) *and* (22).

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