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Nontrivial solitary waves of GKP equation in multi-dimensional spaces

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ABSTRACT. In this paper, using the Mountain Pass Lemma without (PS) condition due to Ambrosetti and Rabinowitz, we obtain the existence of the nontrivial solitary waves of Generalized Kadomtsev-Petviashvili equation in multi-dimensional spaces and for superlinear nonlinear term $f(u)$ which satisfies some growth condition. By the Pohozaev type variational identity, we obtain the nonexistence of the nontrivial solitary waves for power function nonlinear case, i.e. $f(u) = u^p$ where $p \geq 2(2n-1)/(2n-3)$.

Keywords and phrases. Mountain Pass Lemma, Solitary wave, Generalized Kadomtsev-Petviashvili equation.

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1. Introduction

In this paper, we shall investigate the existence and nonexistence of the nontrivial solitary waves of Generalized Kadomtsev-Petviashvili equation in multi-dimensional spaces

$$w_t + w_{xxx} + (f(w))_x = D_x^{-1}\Delta_y w, \tag{1.1}$$

where $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{n-1}$, $n \geq 2$, $D_x^{-1}h(x,y) = \int_{-\infty}^{x} h(s,y)ds$ and $\Delta_y := \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}$.

Kadomtsev-Petviashvili equation and its generalization appear in many Physic progress (cf. [3], [4], [5], [6], [7] and the references therein). A solitary wave

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is a solution of the form

$$w(t,x,y)=u(x-ct,y),$$

where $c>0$ is fixed. Substituting in (1.1), there holds

$$-cu_x+u_{xxx}+(f(u))_x=D_x^{-1}\Delta_y u,$$

or

$$\big(-u_{xx}+D_x^{-2}\Delta_y u+cu-f(u)\big)_x=0. \tag{1.2}$$

In [4] and [5], using constrained minimization, De Bouard and Saut obtained the existence and nonexistence of solitary waves in the case where power nonlinearities $f(u)=u^p$, $p=m/n$, m,n are relatively prime, n is odd. In Chapter 7 of [7], Willem extended the results of [4] to the case where $n=2$, $f(u)$ is a continuous function satisfying some structure conditions.

In this paper we mainly deal with the case where $n\geq 2$ and $f(u)$ is a continuous function. The rest of this paper is organized as: §2 gives the functional setting of the problem and some embedding theorems which will be used latter; §3 deals with the existence of the nontrivial solitary waves. In §4, first we derive a variational identity and then use this identity to prove the nonexistence of the nontrivial solitary waves.

2. Preliminaries

In order to attack the existence and nonexistence of the nontrivial solitary waves of problem (1.1) we apply the following functional setting:

Definition 2.1. On $Y:=\{g_x\,|\,g\in\mathcal{D}(\mathbb{R}^n)\}$, we define the inner product

$$(u,v):=\int_{R^n}\big[u_xv_x+D_x^{-1}\nabla_y u\cdot D_x^{-1}\nabla_y v+cuv\big]\,dV, \tag{2.1}$$

where $\nabla_y=(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_{n-1}})$, $dV=dxdy$, and the corresponding norm

$$\|u\|:=\Big(\int_{R^n}\big[u_x^2+|D_x^{-1}\nabla_y u|^2+cu^2\big]\,dV\Big)^{1/2}. \tag{2.2}$$

A function $u:\mathbb{R}^n\to\mathbb{R}$ belongs to X if there exists $\{u_m\}_{m=1}^{+\infty}\subset Y$ such that:

(a) $u_m\to u$ a.e. on $\mathbb{R}^n$;
(b) $\|u_j-u_k\|\to 0$ as $j,k\to\infty$.

Note that the space X with inner product (2.1) and norm (2.2) is a Hilbert space.

We will show that if estimate

$$\|u\|_{L^q(\mathbb{R}^n)}\leq C\Big(\int_{\mathbb{R}^n}\big[u_x^2+|D_x^{-1}\nabla_y u|^2\big]\,dV\Big)^{1/2} \tag{2.3}$$

holds for a certain constant $C > 0$ and all functions $u \in Y$, there is only one possibility: $q = \bar{p} = \frac{2(2n-1)}{2n-3}$. In fact, let $u \in Y, u \not\equiv 0$, and define for $\lambda > 0$ the rescaled function

$$u_\lambda(x,y) = u(\lambda x, \lambda^2 y),\ (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Applying (2.3) to u_λ, there holds

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\Big(\int_{\mathbb{R}^n} \left[(u_\lambda)_x^2 + |D_x^{-1}\nabla_y u_\lambda|^2\right] dV\Big)^{1/2}. \tag{2.4}$$

But simple computation implies

$$\int_{\mathbb{R}^n} |u_\lambda|^q\, dV = \frac{1}{\lambda^{2n-1}} \int_{\mathbb{R}^n} |u|^q\, dV, \tag{2.5}$$

$$\int_{\mathbb{R}^n} (u_\lambda)_x^2\, dV = \frac{1}{\lambda^{2n-3}} \int_{\mathbb{R}^n} u_x^2\, dV, \tag{2.6}$$

and

$$\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u_\lambda|^2\, dV = \frac{1}{\lambda^{2n-3}} \int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV. \tag{2.7}$$

Inserting these equalities into (2.4), there holds

$$\frac{1}{\lambda^{(2n-1)/q}}\|u\|_{L^q(\mathbb{R}^n)} \le C\frac{1}{\lambda^{(2n-3)/2}}\Big(\int_{\mathbb{R}^n} \left[u_x^2 + |D_x^{-1}\nabla_y u|^2\right] dV\Big)^{1/2}.$$

That is

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{\frac{2n-1}{q}-\frac{2n-3}{2}}\Big(\int_{\mathbb{R}^n} \left[u_x^2 + |D_x^{-1}\nabla_y u|^2\right] dV\Big)^{1/2} \tag{2.8}$$

But then if $\frac{2n-1}{q} - \frac{2n-3}{2} \neq 0$, upon sending λ to either 0 or ∞ in (2.8), we can obtain a contradiction. Thus the only possibility is that $\frac{2n-1}{q} - \frac{2n-3}{2} = 0$, i.e, $q = \bar{p} = \frac{2(2n-1)}{2n-3}$.

Actually, from the embedding theorems for anisotropic Sobolev spaces(cf. [2], p. 323), the following lemma asserts that (2.3) holds if and only if $q = \bar{p}$.

Lemma 2.2. *If $q = \bar{p} = \frac{2(2n-1)}{2n-3}$, there exists a constant $C > 0$ such that (2.3) holds for all functions $u \in X$.*

From the interpolation theorem and estimate (2.3), there is an embedding theorem about X as follows:

Lemma 2.3. *The following embeddings are continuous:*

$$X \hookrightarrow L^p(\mathbb{R}^n), 2 \le p \le \bar{p}.$$

Lemma 2.4. *The following embeddings are compact:*

$$X \hookrightarrow\hookrightarrow L^p_{loc}(\mathbb{R}^n), 2 \le p < \bar{p}.$$

Proof. Suppose that $\{u_m\}_{m=1}^{\infty} \subset X$ is bounded in norm (2.2). Without loss of generality, assume that there exists $\{g_m\}_{m=1}^{\infty} \subset L^2_{loc}(R^n)$ such that $u_m = \partial_x g_m$. Let $v_m = (v_{m,1},\ v_{m,2},\ \cdots,\ v_{m,n-1}) = \nabla_y g_m \in (L^2(R^n))^{n-1}$.

Multiplying g_m by $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $B(0,R)$ and $\operatorname{supp}\psi \subset B(0,2R)$, we may assume that $\operatorname{supp} g_m \subset B(0,2R)$. Selecting if necessary to a subsequence, we may assume that $u_m \rightharpoonup u = \partial_x g$ in X and replacing g_m by $g_m - g$, we may assume that $g = 0$. Denote by $F[u](r,s)$ the Fourier transform of $u(x,y)$.

Let

$$Q_{-1} = \{(r,s) \in \mathbb{R}^n \mid |r| \le \rho,\ |s_i| \le \rho^2, i = 1,2,\cdots,n-1\},$$
$$Q_0 = \{(r,s) \in \mathbb{R}^n \mid |r| > \rho\},\ Q_1 = \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| > \rho^2\},$$
$$\vdots$$
$$Q_i = \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{i-1}| < \rho^2,\ |s_i| > \rho^2\},$$
$$\vdots$$
$$Q_{n-1} = \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{n-2}| < \rho^2,\ |s_{n-1}| > \rho^2\}.$$

Then $\mathbb{R}^n = \bigcup_{i=-1}^{n-1} Q_i$ and $Q_i \cap Q_j = \emptyset,\ i \ne j$.

For $\rho > 0$, there holds

$$\int_{B(0,2R)} |u_m|^2\, dV = \int_{\mathbb{R}^n} |F[u_m]|^2\, drds = \sum_{i=-1}^{n-1} \int_{Q_i} \left|F[u_m]\right|^2 drds. \qquad (2.9)$$

It is clear that

$$\int_{Q_0} \left|F[u_m]\right|^2 drds = \int_{Q_0} \frac{1}{4\pi^2 r^2} \left|F[\partial_x u_m]\right|^2 drds \le \frac{1}{4\pi^2\rho^2} |\partial_x u_m|_2^2,$$

and for $i = 1, \cdots, n-1$, there holds

$$\int_{Q_i} \left|F[u_m]\right|^2 dxdy = \int_{Q_i} \frac{r^2}{|s_i|^2} \left|F[v_{m,i}]\right|^2 drds \le \frac{1}{\rho^2} |v_m|_2^2.$$

For any $\varepsilon > 0$, there exists $\rho > 0$ large enough, such that

$$\sum_{i=0}^{n-1} \int_{Q_i} \left|F[u_m]\right|^2 drds \le \varepsilon/2.$$

Since $u_m \rightharpoonup 0$ in $L^2(R^n)$, there holds

$$F[u_m](r,s) = \int_{B(0,2R)} u_m(x,y) e^{-2i\pi(xr + y\cdot s)}\, dV \to 0, \text{ as } m \to \infty$$

and

$$\left|F[u_m](r,s)\right| \le c_0 |u_m|_2 \le c_1.$$

Lebesgue's dominated convergence theorem implies that

$$\int_{Q_{-1}} \left|F[u_m]\right|^2 drds \to 0, \text{ as } m \to \infty.$$

Thus we have proved that $u_m \to 0$ in $L^2_{loc}(\mathbb{R}^n)$. By Lemma 2.3 and interpolation theorem, there holds $u_m \to 0$ in $L^p_{loc}(\mathbb{R}^n)$ if $2 \le p < \bar{p}$. ☑

Lemma 2.5. *If $\{u_m\}_{m=1}^{+\infty}$ is bounded in X and if*

$$\sup_{(x,y)\in R^n} \int_{B(x,y;r)} |u_m|^2 dV \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Then $u_m \to 0$ in $L^p(\mathbb{R}^n)$ for $2 < p < \bar{p}$.

Proof. Let $2 < s < \bar{p}$ and $u \in X$. By Hölder inequality and Lemma 2.3, there holds

$$\begin{aligned} |u|_{L^s(B(x,y;r))} &\le |u|^{1-\lambda}_{L^2(B(x,y;r))} |u|^{\lambda}_{L^p(B(x,y;r))} \\ &\le c_0 |u|^{1-\lambda}_{L^2(B(x,y;r))} \Big(\int_{B(x,y;r)} [u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2]\, dV \Big)^{\frac{\lambda}{2}}, \end{aligned} \tag{2.11}$$

where $\frac{1}{s} = \frac{1-\lambda}{2} + \frac{\lambda}{\bar{p}}$. Choosing s such that $\frac{\lambda s}{2} = 1$, i.e., $s = \frac{2(2n+1)}{2n-1}$, there holds

$$\int_{B(x,y;r)} |u|^s \, dV \le c_0^s |u|^{(1-\lambda)s}_{L^2(B(x,y;r))} \int_{B(x,y;r)} [u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2]\, dV. \tag{2.12}$$

Now, covering $\mathbb{R}^n$ by balls of radius r in such a way that each point of $\mathbb{R}^n$ is contained in at most 3 balls, then there holds

$$\int_{\mathbb{R}^n} |u|^s \, dV \le 3c_0^s \sup_{(x,y)\in R^n} |u|^{(1-\lambda)s}_{L^2(B(x,y;r))} \int_{\mathbb{R}^n} [u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2]\, dV. \tag{2.13}$$

Under assumption (2.10), (2.13) implies $u_m \to 0$ in $L^s(\mathbb{R}^n)$. By Hölder inequality and Lemma 2.3, there holds $u_m \to 0$ in $L^p(\mathbb{R}^n)$ for all $2 < p < \bar{p}$. ☑

We recall the following Mountain Pass Lemma without (PS) condition as our Lemma 2.6 (cf. [1]).

Lemma 2.6 (Mountain Pass Lemma). *Suppose X is a Banach space and $E \in C^1(X, R)$ satisfies the following geometrical properties:*

(1) *$E(0) = 0$, and there exists $\rho > 0$, such that $E\Big|_{\partial B_\rho(0)} \ge \alpha > 0$;*

(2) *There exists $e \in X \setminus \overline{B_\rho(0)}$, such that $E(e) \le 0$.*

Let Γ be the set of all passes which connects 0 and e, i.e.,

$$\Gamma = \{g \in C([0,1], E) \,|\, g(0) = 0, g(1) = e\}, \tag{2.14}$$

and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} E(g(t)). \tag{2.15}$$

Then $c \geq \alpha$ and E possesses a $(PS)_c$ sequence at level c defined by (2.15), i.e., there exists a sequence $\{u_m\}_{m=1}^{+\infty}$ such that $E(u_m) \to c$ and $DE(u_m) \to 0$ as $m \to \infty$.

3. Existence of nontrivial solitary waves

The solitary waves of problem (1.1) satisfies:

$$\begin{cases} \left(-u_{xx} + D_x^{-2}\Delta_y u + cu - f(u)\right)_x = 0, \\ u \in X, \end{cases} \tag{3.1}$$

where $c > 0$. The weak solutions of (3.1) are the critical points of the functional E defined on X as

$$E(u) := \int_{\mathbb{R}^n} \left(\frac{1}{2}[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2] - F(u)\right) dV,$$

where $F(u) = \int_0^u f(s)\,ds$. Assume:

(f$_1$) $f \in C^0(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and for some $2 < p < \bar{p} = \frac{2(2n-1)}{2n-3}$, $0 < c_0 < c$, $c_1 > 0$, there holds

$$|f(u)| \leq c_0|u| + c_1|u|^{p-1};$$

(f$_2$) There exists $v \in X$ such that

$$\frac{f(\lambda v)}{\lambda} \to +\infty, \text{ as } \lambda \to +\infty;$$

(f$_3$) There exists $\alpha > 2$ such that, for $u \in \mathbb{R}$, there holds

$$\alpha F(u) \leq u f(u).$$

By assumption (f$_1$) and Lemma 2.3, $E \in C^1(X, \mathbb{R})$.

Lemma 3.1. *Under assumptions (f$_1$) and (f$_2$), there exists $e \in X$ and $r > 0$ such that $\|e\| \geq r$ and*

$$b := \inf_{\|u\|=r} E(u) > E(0) = 0 \geq E(e).$$

Proof. From (f$_1$), there holds

$$|F(u)| = \left|\int_0^u f(s)\,ds\right| \leq c_0\frac{|u|^2}{2} + \frac{c_1}{p}|u|^p.$$

Then from the definition of the norm (2.2) in X, there holds

$$E(u) \geq \frac{\|u\|^2}{2} - \int_{R^n} \left(\frac{c_0}{2}|u|^2 + \frac{c_1}{p}|u|^p\right) dV \geq \left(\frac{1}{2} - \frac{c_0}{2c}\right)\|u\|^2 - c_1|u|_p^p.$$

By Lemma 2.3, there exists $r > 0$ such that

$$b := \inf_{\|u\|=r} E(u) > E(0) = 0.$$

It follows from assumption (f_2) that

$$E(\lambda v) \to -\infty, \text{ as } \lambda \to +\infty.$$

Hence there exists $\lambda_0 > 0$ such that $e = \lambda_0 v$ satisfies $\|e\| > r,\ E(e) \le 0.$ ☑

Define

$$d := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)),$$

$$\Gamma := \{\gamma \in C([0,1];X) : \ \gamma(0) = 0,\ \gamma(1) = e\}.$$

Clearly, $d \ge b > 0$. Applying Lemma 2.6, there exists a $(PS)_c$ sequence $\{u_m\}_{m=1}^{+\infty}$ at level $c = d$ such that

$$E(u_m) \to d \text{ and } DE(u_m) \to 0 \text{ as } m \to \infty.$$

Theorem 3.2. *Under assumptions (f_1)–(f_3), problem* (3.1) *possesses a nontrivial solution.*

Proof. **1.** Boundness of $(PS)_c$ sequence.

Let $\{u_m\}_{m=1}^{+\infty}$ be the sequence derived by Lemma 2.6, i.e., $E(u_m) \to d$ and $DE(u_m) \to 0$ as $m \to \infty$. As $m \to \infty$, from assumption (f_3), there holds

$$\begin{aligned} d + o(1) + o(1)\|u_m\| &\ge E(u_m) - \alpha^{-1}(DE(u_m), u_m) \\ &= (\tfrac{1}{2} - \tfrac{1}{\alpha})\|u_m\|^2 + \int_{\mathbf{R}^n} [\alpha^{-1}u_m f(u_m) - F(u_m)]\, dV \\ &\ge (\tfrac{1}{2} - \tfrac{1}{\alpha})\|u_m\|^2. \end{aligned}$$

Hence $\{u_m\}_{m=1}^{+\infty}$ is bounded in X.

2. $\delta := \overline{\lim_{m\to\infty}} \sup_{(x,y)\in R^n} \int_{B(x,y;1)} |u_m|^2\, dV \ne 0.$

Otherwise, by Lemma 2.5, there holds $u_m \to 0$ in $L^s(R^n)$ for $2 < s < \frac{2(2n-1)}{2n-3}$. It follows that

$$\begin{aligned} 0 < d &= E(u_m) - \tfrac{1}{2}(DE(u_m), u_m) + o(1) \\ &= \int_{\mathbf{R}^n} [\frac{1}{2}u_m f(u_m) - F(u_m)]\, dV + o(1) = 0(1), \end{aligned}$$

which is a contradiction.

3. Existence of a nontrivial solution of problem (3.1).

Selecting if necessary a subsequence, we can assume that there existes a sequence $(x_m, y_m) \subset R^n$ such that

$$\int_{B(x_m,y_m;1)} |u_m|^2\, dV > \delta/2.$$

Define $v_m(x,y) := u_m(x + x_m, y + y_m)$ so that

$$\int_{B(0;1)} |v_m|^2\, dV > \delta/2.$$

Selecting if necessary a subsequence, we can assume that there existes a $v \in X$ such that

$$v_m \rightharpoonup v \text{ in } X, \text{ as } m \to \infty.$$

By Lemma 2.4, $v_m \to v$ in $L^2_{loc}(\mathbb{R}^n)$ and so $v \neq 0$, and for every $w \in X$, there holds

$$\int_{\mathbb{R}^n} (f(v_m) - f(v))w\,dV = \int_{B(0,R)} (f(v_m) - f(v))w\,dV + \int_{\mathbb{R}^n \setminus B(0,R)} (f(v_m) - f(v))w\,dV.$$

Since $w \in X$, then $w \in L^p(\mathbb{R}^n)$ and $\{v_m\}$ is bounded in X, hence $\{v_m\}$ is bounded in $L^p(\mathbb{R}^n)$, thus for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ large enough and independent on m such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} (f(v_m) - f(v))w\,dV < \varepsilon, \ \forall m$$

On the other hand, for this $R > 0$, from Lemma 2.4, there holds

$$\int_{B(0,R)} (f(v_m) - f(v))w\,dV \to 0, \text{ as } m \to \infty.$$

Thus, there holds

$$\int_{\mathbb{R}^n} f(v_m)w\,dV \to \int_{\mathbb{R}^n} f(v)w\,dV, \text{ as } m \to \infty,$$

which implies

$$(DE(v), w) = \lim_{m \to \infty} (DE(v_m), w) = 0$$

Hence $DE(v) = 0$ and v is a nontrivial solution of problem (3.1). ☑

4. Nonexistence of nontrivial solitary waves

In this section, we derive a Pohozaev type variational identity of the solitary wave of problem:

$$\big(- u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)_x = 0,$$

where $g \in C^1(\mathbb{R}, \mathbb{R})$ such that $g(0) = 0$ and define $G(u) := \int_0^u g(s)\,ds$.

First, we give a formal argument explaining the variational identity. For any $\lambda > 0$, define a transformation $T(\lambda) : X \to X$ as

$$T(\lambda)u(x, y) := u(x/\lambda, y/\lambda^2), \ (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Then $T(1) = \mathrm{id}_X$. If $u \in X$ is a critical point of functional $E(u)$, we conjecture that

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda=1} E(T(\lambda)u) = 0. \tag{4.1}$$

A simple computation shows that

$$E(T(\lambda)u) = \frac{\lambda^{2n-3}}{2} \int_{\mathbb{R}^n} (u_x^2 + |D_x^{-1}\nabla_y u|^2)\, dV - \lambda^{2n-1} \int_{\mathbb{R}^n} G(u)\, dV. \tag{4.2}$$

and

$$\begin{aligned} &\frac{\partial}{\partial \lambda}\Big|_{\lambda=1} E(T(\lambda)u) = \\ &\qquad \frac{2n-3}{2} \int_{\mathbb{R}^n} (u_x^2 + |D_x^{-1}\nabla_y u|^2)\, dV - (2n-1) \int_{\mathbb{R}^n} G(u)\, dV, \end{aligned} \tag{4.3}$$

which implies that

$$\int_{\mathbb{R}^n} (u_x^2 + |D_x^{-1}\nabla_y u|^2)\, dV = \frac{2(2n-1)}{2n-3} \int_{\mathbb{R}^n} G(u)\, dV. \tag{4.4}$$

In fact, we have the following Theorem:

Theorem 4.1. *Any solution of*

$$\begin{cases} \big(- u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)_x = 0, \\ u \in X \cap H^2_{loc}(\mathbb{R}^n), \\ G(u), g(u)u \in L^1(\mathbb{R}^n),\ g(u)D_x^{-1}\nabla_y u \in (L^1(\mathbb{R}^n))^{n-1}, \end{cases} \tag{4.5}$$

satisfies (4.4).

Proof. **1.** Let

$$J(u) := \int_{\mathbb{R}^n} \big(\frac{1}{2}[u_x^2 + |D_x^{-1}\nabla_y u|^2] - G(u)\big)\, dV.$$

Then a weak solution of problem (4.5) is a critical point of operator J. Let $\psi \in \mathcal{D}(\mathbb{R})$ be such that $0 \le \psi \le 1, \psi(r) = 1$ for $r = 1$ and $\psi(r) = 0$ for $r \ge 2$, $|\psi'(r)| \le 2$, $|\psi''(r)| \le 4$. Define a sequence of functions on $\mathbb{R}^n$ as:

$$\psi_m(x, y) := \psi(\frac{x^2 + |y|^2}{m^2}),\ \forall (x, y) \in \mathbb{R}^n.$$

2. For any solution of problem (4.5), there holds

$$\frac{3}{2} \int_{\mathbb{R}^n} u_x^2\, dV - \frac{1}{2} \int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + \int_{\mathbb{R}^n} (G(u) - g(u)u)\, dV = 0. \tag{4.6}$$

For every integer m, there holds

$$\int_{\mathbb{R}^n} \big(- u_{xx} + D_x^{-2}\Delta_y u - g(u)\big) \big(\psi_m x u\big)_x\, dV = 0. \tag{4.7}$$

Integrating by parts, there holds

$$-\int_{\mathbb{R}^n} u_{xx}\big(\psi_m x u\big)_x\, dV = -\int_{\mathbb{R}^n} u_{xx}\big(\psi_{m,x} x u + \psi_m u + \psi_m x u_x\big)\, dV$$
$$= \int_{\mathbb{R}^n} \left[\frac{3}{2}u_x^2(\psi_{m,x}x + \psi_m) + 2\psi_{m,x} u u_x + \psi_{m,xx} x u u_x\right] dV.$$

Lebesgue dominated convergence theorem implies that, as $m \to \infty$, there holds

$$-\int_{\mathbb{R}^n} u_{xx}\big(\psi_m x u\big)_x\, dV = \frac{3}{2}\int_{\mathbb{R}^n} u_x^2\, dV + o(1). \tag{4.8}$$

Similarly, there hold

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_x^{-2}\Delta_y u\big(\psi_m x u\big)_x\, dV \\
&= -\int_{\mathbb{R}^n} \big(D_x^{-1}\Delta_y u\big)\big(\psi_m x u\big)\, dV \\
&= -\int_{\mathbb{R}^n} \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i}\big(D_x^{-1}u_{y_i}\big)\big(\psi_m x u\big)\, dV \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\frac{\partial}{\partial y_i}\big(\psi_m x u\big)\, dV \\
&= \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_{m,y_i} x u + \sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_m x \frac{\partial}{\partial x} D_x^{-1}u_{y_i}\Big)\, dV \\
&= \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_{m,y_i} x u - \frac{1}{2}\sum_{i=1}^{n-1} |D_x^{-1}u_{y_i}|^2(\psi_{m,x}x + \psi_m)\Big)\, dV \\
&= -\frac{1}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + o(1),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^n} g(u)\big(\psi_m x u\big)_x\, dV \\
&= -\int_{\mathbb{R}^n} g(u)\big(\psi_{m,x} x u + \psi_m u + \psi_m x u_x\big)\, dV \\
&= -\int_{\mathbb{R}^n} \Big(g(u)\psi_m u + g(u)\psi_{m,x} x u + \frac{dG(u)}{dx}\psi_m x\Big)\, dV \\
&= \int_{\mathbb{R}^n} \big(G(u) - g(u)u\big)\, dV + o(1).
\end{aligned} \tag{4.10}$$

Substituting (4.8)–(4.10) into (4.7) yields (4.6)

3. On the other hand, since u is a weak solution of problem (4.5), i.e., $DJ(u) = 0$, then from $(DJ(u), u) = 0$, there holds

$$\int_{\mathbb{R}^n} \left(u_x^2 + |D_x^{-1}\nabla_y u|^2\right) dV = \int_{\mathbb{R}^n} g(u)u\, dV. \tag{4.11}$$

4. For any solution of problem (4.5), there holds

$$-\frac{n-1}{2}\int_{\mathbb{R}^n} u_x^2\, dV - \frac{n-3}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + (n-1)\int_{\mathbb{R}^n} G(u)\, dV = 0. \tag{4.12}$$

For every integer m, there also holds

$$\int_{\mathbb{R}^n} \left(-u_{xx} + D_x^{-2}\Delta_y u - g(u)\right)\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right)_x dV = 0. \tag{4.13}$$

Integrating by parts and applying Lebesgue dominated convergence theorem imply that, as $m \to \infty$, there hold

$$\begin{aligned}
&-\int_{\mathbb{R}^n} u_{xx}\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right)_x dV \\
&= -\int_{\mathbb{R}^n} u_{xx}\left(\psi_{m,x} y \cdot D_x^{-1}\nabla_y u + \psi_m y \cdot \nabla_y u\right) dV \\
&= \int_{\mathbb{R}^n} u_x\left(\psi_{m,x} y \cdot D_x^{-1}\nabla_y u + \psi_m y \cdot \nabla_y u\right)_x dV \\
&= \int_{\mathbb{R}^n} u_x\left(\psi_{m,xx} y \cdot D_x^{-1}\nabla_y u + 2\psi_{m,x} y \cdot \nabla_y u + \psi_m y \cdot \nabla_y u_x\right) dV \\
&= -\frac{n-1}{2}\int_{\mathbb{R}^n} u_x^2\, dV + o(1),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(D_x^{-2}\Delta_y u\right)\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right)_x dV \\
&= -\int_{\mathbb{R}^n} \left(D_x^{-1}\Delta_y u\right)\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right) dV \\
&= -\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1} \frac{\partial}{\partial y_i}(D_x^{-1} u_{y_i})\Big)\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right) dV \\
&= \int_{\mathbb{R}^n} \sum_{i=1}^{n-1}(D_x^{-1} u_{y_i})\left(\psi_m y \cdot D_x^{-1}\nabla_y u\right)_{y_i} dV \\
&= -\frac{n-3}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + o(1)
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^n} g(u)\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)_x\,dV\\
&\quad=-\int_{\mathbb{R}^n} g(u)\big(\psi_{m,x}y\cdot D_x^{-1}\nabla_y u+\psi_m y\cdot\nabla_y u\big)\,dV\\
&\quad=-\int_{\mathbb{R}^n}\big(g(u)\psi_{m,x}y\cdot D_x^{-1}\nabla_y u+\sum_{i=1}^{n-1}\frac{dG(u)}{dy_i}y_i\psi_m\big)\,dV\\
&\quad=(n-1)\int_{\mathbb{R}^n} G(u)\,dV+o(1).
\end{aligned}\tag{4.16}$$

Thus, from equations (4.13)–(4.16) (4.12) holds. Equations (4.6), (4.11) and (4.12) imply equation (4.4). ☑

Theorem 4.2. *(Nonexistence of nontrivial solitary wave) If $g\in C^1(\mathbb{R};\mathbb{R})$ satisfies $g(0)=0$ and*

$$\frac{2(2n-1)}{2n-3}G(u)-g(u)u<0,\ \forall u\neq 0,\tag{4.17}$$

then 0 is the only solution of problem (4.5).

Proof. If $u\not\equiv 0$ is a solution of problem (4.5), then (4.4)-(4.11), there holds

$$\int_{\mathbb{R}^n}\big[\frac{2(2n-1)}{2n-3}G(u)-g(u)u\big]\,dV=0$$

which contradicts (4.17). ☑

Corollary 4.3. *Let $c>0$, and $p\geq\frac{2(2n-1)}{2n-3}$, then 0 is the only solution of problem:*

$$\begin{cases}\big(-u_{xx}+D_x^{-2}\Delta_y u+cu-|u|^{p-2}u\big)_x=0,\\ u\in X\cap H^2_{loc}(\mathbb{R}^n),\\ |u|^{p-2}uD_x^{-1}\nabla_y u\in(L^1(\mathbb{R}^n))^{n-1}.\end{cases}\tag{4.18}$$

Proof. Since $g(u)=|u|^{p-2}u-cu$, then $G(u)=\frac{1}{p}|u|^p-\frac{c}{2}u^2$, thus (4.17) holds. ☑

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