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Hölder-continuous solution for a nonlinear elasticity system

Gilberto Pérez P.
Universidad Pedagógica y Tecnológica de Colombia, Tunja
Leonardo Rendón A.
Universidad Nacional de Colombia, Bogotá

Abstract. In this paper the Cauchy problem for a non-linear elasticity system in Lagrange coordinates is considered. Using the method of null viscosity we prove the existence of Hölder-Continuous solutions for the non-linear elasticity system $v_t - u_x = 0$, $u_t - \sigma(v)_x = 0$.

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1. Introduction

We consider the hyperbolic system of conservation laws

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v)_x &= 0, \end{aligned} \tag{1}$$

with $(x,t) \in \mathbb{R} \times (0,\infty)$ and the initial conditions

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)), \tag{2}$$

where $v_0(x)$ and $u_0(x)$ are measurable bounded functions, u represents the speed, v represents the tension and $\sigma(v)$ the force in transverse sections (compression).

We assume the following consistence conditions:

$$\sigma(v) \in C^2(-\infty,\infty), \tag{3}$$

$$\sigma'(v) > k > 0, \sigma''(v) > 0. \tag{4}$$

The system (1) can be written in the general form

$$\mathbf{u}_t + F(\mathbf{u})_x = 0 \tag{5}$$

where $\mathbf{u}(x,t) = (v(x,t), u(x,t))$ and $F(v,u) = (-u, -\sigma(v))$.

A weak solution to (5) is a measurable function $\mathbf{u}(x,t) = (v(x,t), u(x,t))$ such that $\mathbf{u}$ and $F(\mathbf{u})$ are in $L^1_{loc}(\mathbb{R}\times[0,\infty);\mathbb{R}^2)$ and

$$\int_{t>0}\int\left[\mathbf{u}\frac{\partial\phi}{\partial t} + F(\mathbf{u})\frac{\partial\phi}{\partial x}\right]dxdt + \int_{t=0}\mathbf{u}_0\phi dx = 0,\ \forall\phi\in C^1_0(\mathbb{R}\times[0,\infty);\mathbb{R}).$$

The eigenvalues of the system (1) are

$$\lambda_2 = \sqrt{\sigma'(v)}, \quad \lambda_1 = -\sqrt{\sigma'(v)} \tag{6}$$

and

$$z = u + \int_{v_1}^{v}\sqrt{\sigma'(s)}ds, \quad w = u - \int_{v_1}^{v}\sqrt{\sigma'(s)}ds \tag{7}$$

are the corresponding Riemman invariants, (see [12]) and v_1 is a constant . From the assumptions on $\sigma(v)$, we have

$$u_w = \frac{1}{2}, \quad u_z = \frac{1}{2}, \quad v_w = -\frac{1}{2\sqrt{\sigma'(v)}}, \quad v_z = \frac{1}{2\sqrt{\sigma'(v)}}, \tag{8}$$

$$\lambda_{1w} = \lambda_{2z} = \frac{\sigma''(v)}{4\sigma'(v)} > 0, \quad \lambda_{1z} = \lambda_{2w} = -\frac{\sigma''(v)}{4\sigma'(v)} < 0. \tag{9}$$

Then the system (1) is genuinely non linear and strictly hyperbolic.

For studies of (1) using Young measure and compensated compactness the reader is referred to [4], [9] and [11]. Here we use a variant of the standard viscosity method introduced in [7].

The smooth solution is obtained by replacing the system

$$\begin{aligned} v_t - u_x &= \varepsilon v_{xx} \\ u_t - \sigma(v)_x &= \varepsilon u_{xx} \end{aligned} \tag{10}$$

by

$$\begin{aligned} w_t + \lambda_2 w_x &= \varepsilon w_{xx} \\ z_t + \lambda_1 z_x &= \varepsilon u_{xx}, \end{aligned} \tag{11}$$

with initial values

$$(w(x,0), z(x,0)) = (w_0(x), z_0(x)). \tag{12}$$

We will show that for $\epsilon > 0$ the problem (11), (12) has a bounded solution (w^ϵ, z^ϵ) and we prove that there is a subsequence $(v^{\epsilon_k}, u^{\epsilon_k})$ converging uniformly to a pair of functions (v,u). Then we show that (v,u) is a Hölder-continuous solution of the problem (1), (2).

Differentiating (11) with respect to x and taking $w_x = r$ y $z_x = s$, we obtain the following identities

$$\begin{aligned} r_t + \lambda_2 r_x + (\lambda_{2w} r + \lambda_{2z} s)\, r &= \varepsilon r_{xx}, \\ s_t + \lambda_1 s_x + (\lambda_{1w} r + \lambda_{1z} s)\, s &= \varepsilon s_{xx} \end{aligned} \tag{13}$$

This paper is organized in the following way: in Section 2 we find the local solution and the *a priori* approximations for the solution the Cauchy problem (11), (12). In Section 3 we find the global solution for problem (11), (12) and in section 4 we find the Hölder-continuous solution of the problem (1), (2).

2. Solutions by the viscosity method

2.1. Local Solution. We consider the set

$$\mathbf{U} = \{ (w, z) : c_1 - \gamma \le w \le c_2 + \gamma, c_3 - \gamma \le z \le c_4 + \gamma, |w_x| \le 2M, |z_x| \le 2M \}$$

where $\gamma = \int_{v_1-\delta}^{0} \sqrt{\sigma'(v)}dv$, δ is a small positive constant. We choose the norm $\|f\| = \|f\|_\infty + \|f_x\|_\infty$ for $f \in C^1(\mathbb{R})$. For $(w, z) \in \mathbf{U}$ we define

$$T\begin{pmatrix} w(x,t) \\ z(x,t) \end{pmatrix} = \begin{pmatrix} w^0(x,t) + \int_0^t \int_{-\infty}^{\infty} -\lambda_2 w_y(y,s)\, G(x-y, t-s)\, dy\, ds \\ z^0(x,t) + \int_0^t \int_{-\infty}^{\infty} -\lambda_1 z_y(y,s)\, G(x-y, t-s)\, dy\, ds \end{pmatrix}$$

where $G(x,t)$ is the heat kernel and

$$w^0(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, w_0(y)\, dy, \quad z^0(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, z_0(y)\, dy.$$

It can be shown that $T(\mathbf{U}) \subseteq \mathbf{U}$ and that T is a contraction in some strip $R_\tau = (-\infty, \infty) \times [0, \tau]$.

Lemma 1. *If* $\sigma'(v) > k > 0$, $\sigma''(v) > 0$ *and* $w_0(x), z_0(x)$ *are* $C^1(\mathbb{R})$ *functions satisfying:*

$$c_1 \le w_0(x) \le c_2, \quad c_3 \le z_0(x) \le c_4,$$

$$|w_{0x}(x)| \le M, \quad |z_{0x}(x)| \le M.$$

Then there exists a smooth solution for the Cauchy problem (11), (12) *in some region* $R_\tau = (-\infty, \infty) \times [0, \tau]$. *This solution satisfies*

$$c_1 - \gamma \le w \le c_2 + \gamma, \quad c_3 - \gamma \le z \le c_4 + \gamma,$$

$$|w_x(x,t)| \le 2M, \quad |z_x(x,t)| \le 2M.$$

Proof. For $(w,z)\in \mathbf{U}$ we have that

$$c_1-\gamma\le w\le c_2+\gamma,\quad c_3-\gamma\le z\le c_4+\gamma,\quad -c_2-\gamma\le -w\le -c_1+\gamma$$

where $c_3-c_2-2\gamma\le z-w\le c_4-c_1+2\gamma$, that is,

$$c_3-c_2-2\gamma\le 2\int_{v_1}^{v}\sqrt{\sigma'(s)}ds\le c_4-c_1+2\gamma.$$

Hence, there exists $c>0$ such that

$$|\lambda_2|,|\lambda_1|\le c.$$

Let us call W and Z the first and the second components of $T\begin{pmatrix} w(x,t)\\ z(x,t)\end{pmatrix}$ respectively. Thus $c_1-\gamma\le W\le c_2+\gamma$, $c_3-\gamma\le Z\le c_4+\gamma$.

To show that $|W_x|\le 2M$, $|Z_x|\le 2M$ in a strip $(-\infty,\infty)\times[0,\tau]$ for some appropriate τ, we observe that

$$\int_{-\infty}^{\infty}|G_x(x-y,t-s)|\,dy=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t-s}}$$

and that

$$\int_{-\infty}^{\infty}G_x(x-y,t)\,w_0(y)\,dy=-\int_{-\infty}^{\infty}G(x-y,t)\,w_{0y}(y)\,dy.$$

Hence

$$|W_x|=\left|\int_{-\infty}^{\infty}G_x(x-y,t)\,w_0(y)\,dy+\int_0^t\int_{-\infty}^{\infty}-\lambda_2 w_y(y,s)\,G_x(x-y,t-s)\,dyds\right|$$

$$\le\left|\int_{-\infty}^{\infty}G_x(x-y,t)\,w_0(y)\,dy\right|+\int_0^t\int_{-\infty}^{\infty}|-\lambda_2 w_y(y,s)|\,|G_x(x-y,t-s)|\,dyds$$

$$\le\int_{-\infty}^{\infty}|G_x(x-y,t)|\,|w_{0y}(y)|\,dy+2Mc\int_0^t|G_x(x-y,t-s)|\,ds\le M+\tfrac{4Mc}{\sqrt{\pi}}\sqrt{t}.$$

It follows that $|W_x|\le 2M$ in $(-\infty,\infty)\times[0,\tau]$ with $\tau=\frac{\pi}{(4Mc)^2}$. The inequality $|Z_x|\le 2M$ is obtained in a similar way. ☑

2.2. ***A priori* estimates.** The next lemma contains the *a priori* approximation that we need to establish the global existence of smooth solutions for the Cauchy problem (11), (12).

Lemma 2. *Let $\sigma(v)$ satisfy* (3) *and* (4)*. Let $w_0(x)$ and $z_0(x)$ be bounded in the $C^1(\mathbb{R})$ space and satisfying*

$$\begin{aligned} c_1\le w_0(x)\le c_2,\quad & c_3\le z_0(x)\le c_4,\\ 0\le w_{0x}(x)\le \bar{M},\quad & 0\le z_{0x}(x)\le \bar{M}.\end{aligned}\tag{14}$$

If $(w(x,t),z(x,t))$ is a smooth solution of (11), (12) *defined in the strip $(-\infty,\infty)\times[0,T]$, with $0<T<\infty$, and $\lambda_{1w},\lambda_{1z},\lambda_{1w},\lambda_{2z}$ are bounded in*

$(-\infty,\infty)\times[0,T]$ *then*

$$c_1 \le w(x,t) \le c_2, \quad c_3 \le z(x,t) \le c_4, \tag{15}$$

$$0 \le w_x(x,t) \le \bar{M}, \quad 0 \le z_x(x,t) \le \bar{M}. \tag{16}$$

Proof. The proof of (15) is similar to that of the linear case presented in [10]. To show that $0 \le w_x(x,t) \le \bar{M}$, let us introduce the transformation,

$$r = \left(\bar{r} - \frac{M}{L^2}\left(x^2 + cLe^t\right)\right)e^{\beta t} \tag{17}$$

where c, β, L are positive constants and M is a bound for r and s over the set $\mathbb{R}\times(0,t)$. (M can be obtained from the local existence solution).

From (17), we obtain

$$r_t = \left(\bar{r}_t - \frac{M}{L^2}cLe^t\right)e^{\beta t} + \left(\bar{r}_t - \frac{M}{L^2}\left(x^2 + cLe^t\right)\right)\beta e^{\beta t},$$

$$\lambda_2 r_x = \left(\lambda_2 \bar{r}_x - \frac{2Mx}{L^2}\right)e^{\beta t},$$

$$(\lambda_{2w} r + \lambda_{2z} s)\, r = (\lambda_{2w} r + \lambda_{2z} s)\left(\bar{r} - \frac{M}{L^2}\left(x^2 + cLe^t\right)\right)e^{\beta t},$$

$$\varepsilon r_{xx} = \varepsilon\left(\bar{r}_{xx} - \frac{2M}{L^2}\right)e^{\beta t}.$$

Replacing these identities in the first equation of (13), *we have that*

$$\varepsilon\left(\bar{r}_{xx} - \frac{2M}{L^2}\right) = \bar{r}_t - \frac{M}{L^2}cLe^t + \left(\bar{r} - \frac{M}{L^2}\left(x^2 + cLe^t\right)\right)\beta$$
$$+ \lambda_2\left(\bar{r}_x - \frac{2Mx}{L^2}\right) + (\lambda_{2w} r + \lambda_{2z} s)\left(\bar{r} - \frac{M}{L^2}\left(x^2 + cLe^t\right)\right)$$

or equivalently

$$\varepsilon\bar{r}_{xx} - \frac{2M}{L^2}\varepsilon = \bar{r}_t - \frac{M}{L^2}cLe^t + \beta\bar{r} - \frac{M}{L^2}\left(x^2 + cLe^t\right)\beta + \lambda_2\bar{r}_x$$
$$- \frac{2Mx}{L^2}\lambda_2 + (\lambda_{2w} r + \lambda_{2z} s)\,\bar{r} - \frac{M}{L^2}(\lambda_{2w} r + \lambda_{2z} s)\left(x^2 + cLe^t\right),$$

$$\bar{r}_t + \lambda_2\bar{r}_x - \varepsilon\bar{r}_{xx} + (\beta + \lambda_{2w} r + \lambda_{2z} s)\,\bar{r}$$
$$= \frac{M}{L^2}\left(cLe^t + 2x\lambda_2 - 2\varepsilon\right) + \frac{M}{L^2}(\beta + \lambda_{2w} r + \lambda_{2z} s)\left(x^2 + cLe^t\right). \tag{18}$$

Also from (17)

$$\bar{r}(x,0) = r(x,0) + \frac{M}{L^2}\left(x^2 + cL\right) > 0. \tag{19}$$

Since $w_{0x} = r(0,x)$ and $0 \le w_{0x} \le \bar{M}$ then

$$\bar{r}(\pm L,t) = r(\pm L,t)\,e^{-\beta t} + \frac{M}{L^2}\left(L^2 + cLe^t\right) > 0, \tag{20}$$

because M is a bound of r over $\mathbb{R}\times(0,T)$.

From (18), (19) and (20) we have that

$$\bar{r}(x,t) > 0 \text{ over } (-L,L)\times(0,T). \tag{21}$$

In fact, if (21) does not hold at a point (x,t) in $(-L,L)\times(0,T)$, let $\bar{t}$ be the minimum upper bound of the t values of t such that $\bar{r} > 0$; then due to continuity we see that $\bar{r} = 0$ at some points $(\bar{x},\bar{t})\times(-L,L)$.

Thus $\bar{r}_t \leq 0$, $\bar{r}_x = 0$ and $-\varepsilon r_{xx} \leq 0$ in $(\bar{x},\bar{t})$; then

$$\bar{r}_t + \lambda_2\bar{r}_x - \varepsilon\bar{r}_{xx} \leq 0 \text{ in } (\bar{x},\bar{t}). \tag{22}$$

But, if we choose β, c large enough such that

$$\beta + \lambda_{2w}r + \lambda_{2z}s > 0, \quad cLe^t + 2x\lambda_2 - 2\varepsilon > 0, \tag{23}$$

over $(-L,L)\times(0,T)$, equation (22) contradicts (18). Therefore (21) is proved. Then for all points (x_0,t_0) in $(-L,L)\times(0,T)$,

$$r(x_0,t_0) > -\frac{M}{L^2}\left(x_0^2 + cLe^{t_0}\right)e^{\beta t_0}, \tag{24}$$

which gives the required estimate for r when $L\to\infty$. We can obtain the estimate $s \geq 0$ in a similar way.

From $r \geq 0$, $s \geq 0$ and using (13) we deduce that

$$\begin{aligned} r_t + \lambda_2 r_x &\leq \varepsilon r_{xx}, \\ s_t + \lambda_1 s_x &\leq \varepsilon s_{xx}. \end{aligned} \tag{25}$$

Then $w_x(x,t) \leq \bar{M}$, $z_x(x,t) \leq \bar{M}$ are obtained from both of the inequalities above. ☑

3. Global Solutions

Choosing $w(x,T)$, $z(x,T)$ as initial data at time $t = T$ and using the *a priori* approximations, we see that the Cauchy problem (11), (12) has a smooth solution in $(-\infty,\infty)\times[T,2T]$. Repeating this process, we have a global solution. Thus , the local existence in Lemma 1, and the *a priori* approximation of Lemma 2, yield the next global existence result.

Theorem 3. *Let $\sigma(v)$ satisfy* (3), (4) *and let $w_0(x)$, and $z_0(x) \in C^1(\mathbb{R})$ satisfy*

$$\begin{aligned} c_1 &\leq w_0(x) \leq c_2, \quad c_3 \leq z_0(x) \leq c_4, \\ 0 &\leq w_{0x}(x) \leq M, \quad 0 \leq z_{0x}(x) \leq M, \end{aligned}$$

then the Cauchy problem (11), (12) *has a unique global smooth solution that satisfies (15) and (16).*

Now, we will give the w_t and z_t approximations. Let $X = w_t$, $Y = z_t$ then

$$\begin{aligned} X|_{t=0} &= w_t|_{t=0} = \varepsilon w_{xx} - \lambda_2\, w_x|_{t=0}, \\ Y|_{t=0} &= z_t|_{t=0} = \varepsilon z_{xx} - \lambda_1\, z_x|_{t=0}. \end{aligned} \tag{26}$$

Differentiating the first equation of (11) with respect to t, we have

$$\frac{\partial w_t}{\partial t}+\frac{\partial(\lambda_2 w_x)}{\partial t}=\frac{\partial w_{xx}}{\partial t},$$

$$X_t+\frac{\partial\lambda_2}{\partial t}.w_x+\lambda_2\frac{\partial w_x}{\partial_t}=\varepsilon X_{xx},$$

$$X_t+\left(\frac{\partial\lambda_2}{\partial w}\cdot\frac{\partial w}{\partial t}+\frac{\partial\lambda_2}{\partial z}\cdot\frac{\partial z}{\partial t}\right)w_x+\lambda_2 X_x=\varepsilon X_{xx},$$

$$X_t+(\lambda_{2w}X+\lambda_{2z}Y)\,w_x+\lambda_2 X_x=\varepsilon X_{xx}. \tag{27a}$$

Similarly

$$Y_t+\lambda_1 Y_x+(\lambda_{1z}Y+\lambda_{1w}X)\,z_x=\varepsilon Y_{xx}. \tag{27}$$

Lemma 4. *If $(w_0(x), z_0(x))$ satisfy the hypotheses in Theorem 3, $w_0(x), z_0(x)$ are of class $C^2(\mathbb{R})$ and*

$$|X_0(x)|\le M \text{ and} |Y_0(x)|\le M \tag{28}$$

then

$$|X(x,t)|\le Me^{\lambda T} \text{ and} |Y(x,t)|\le Me^{\lambda T} \tag{29}$$

where $\lambda=\max\{0, \sup\limits_{\mathbb{R}\times[0,T]}(\lambda_{2z}-\lambda_{2w})\,w_x, \sup\limits_{\mathbb{R}\times[0,T]}(\lambda_{1w}-\lambda_{1z})\,z_x\}.$

Proof. We consider the transformation

$$\begin{aligned}-X&=\left(\bar{X}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)e^{\lambda t},\\ Y&=\left(\bar{Y}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)e^{\lambda t},\end{aligned} \tag{30}$$

with c, N positive constants and N is the $(w_t,\ z_t)$ upper bound over $\mathbb{R}\times[0,T]$. Then we have,

$$-X_t=\left(\bar{X}_t+\frac{NcLe^t}{L^2}\right)e^{\lambda t}+\left(\bar{X}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\lambda e^{\lambda t},$$

$$-X_x=\left(\bar{X}_x+\frac{2xN}{L^2}\right)e^{\lambda t},\quad -X_{xx}=\left(\bar{X}_{xx}+\frac{2N}{L^2}\right)e^{\lambda t},$$

$$\lambda_{2w}X+\lambda_{2z}Y=$$

$$\left[\lambda_{2w}\left(\bar{X}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)+\lambda_{2z}\left(\bar{Y}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\right]e^{\lambda t}.$$

Replacing these identities in the equation in (27a), it is obtained that

$$\varepsilon\overline{X}_{xx}+\frac{2N}{L^2}\varepsilon = \bar{X}_t+\frac{NcLe^t}{L^2}+\left(\bar{X}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\lambda+$$
$$\left[\lambda_{2w}\left(\bar{X}+M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)+\lambda_{2z}\left(-\bar{Y}-M-\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\right]w_x$$
$$+\lambda_2\left(\bar{X}_x+\frac{2xN}{L^2}\right)$$

or

$$\begin{aligned}\varepsilon X_{xx}+\frac{2N}{L^2}\varepsilon &= \bar{X}_t+\lambda_2\bar{X}_x+\frac{NcLe^t}{L^2}+\frac{2xN}{L^2}\lambda_2+\left(\lambda_{2w}\bar{X}-\lambda_{2z}\bar{Y}\right)w_x\\ &+\lambda\overline{X}+\left[\lambda_{2w}\left(M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)-\lambda_{2z}\left(M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\right]w_x\\ &+\left(M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)\lambda.\end{aligned}$$

Thus

$$\begin{aligned}\varepsilon X_{xx}=\bar{X}_t+\lambda_2\bar{X}_x+\left(\lambda_{2w}\bar{X}-\lambda_{2z}\bar{Y}\right)w_x&\\ +\lambda\overline{X}+\left(cLe^t+2\lambda_2x-2\varepsilon\right)\frac{N}{L^2}&\\ +\left[\lambda+\left(\lambda_{2w}-\lambda_{2z}\right)w_x\right]\left(M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right)&\end{aligned}\tag{31}$$

Similarly, the second equation in (27) is transformed into

$$\begin{aligned}\varepsilon Y_{xx}=\bar{Y}_t+\lambda_1\bar{Y}+\left(\lambda_{1z}\bar{Y}-\lambda_{1w}\bar{X}\right)z_x&\\ +\lambda\overline{Y}+\left(cLe^t+2\lambda_1x-2\varepsilon\right)\frac{N}{L^2}&\\ +\left[\lambda+\left(\lambda_{1z}-\lambda_{1w}\right)z_x\right]\left(M+\frac{N\left(x^2+cLe^t\right)}{L^2}\right).&\end{aligned}\tag{32}$$

Also

$$\bar{X}_0(x)=-X_0(x)-M-\frac{cLN}{L^2}<0,\quad \bar{Y}_0(x)=-Y_0(x)-M-\frac{cLN}{L^2}<0,\tag{33}$$

$$\overline{X}(\pm L,t)<0,\quad \bar{Y}(\pm L,t)<0,\tag{34}$$

provided we choose c large enough such that $cL+2\lambda_2x-2\varepsilon>0$ and $cL+2\lambda_1x-2\varepsilon>0$ over $(-L,L)\times[0,T]$, then in a similar way to the proof in Lemma 2, we can obtain from (31) - (34) that

$$\bar{X}(x,t) < 0, \quad \bar{Y}(x,t) < 0 \text{ over } (-L,L)\times[0,T].$$

If $L \to \infty$ in (30), we have

$$X \geq -e^{\lambda t}M, \text{ and } Y \leq e^{\lambda t}M \text{ in } (-\infty,\infty)\times[0,T]$$

To obtain inequalities

$$X \leq e^{\lambda t}M, \quad Y \geq -e^{\lambda t}M$$

the transformations given in (27) are changed by the transformations

$$X = \left(\bar{\bar{X}} + M + \frac{N\left(x^2 + cLe^t\right)}{L^2}\right)e^{\lambda t}, \quad -Y = \left(\bar{\bar{Y}} + M + \frac{N\left(x^2 + cLe^t\right)}{L^2}\right)e^{\lambda t}.$$

☑

4. A Hölder Continuous solution

Lemma 5. *If the hypotheses of lemma* (4) *are satisfied, then*

$$|u(x,t)| \leq M, \quad |v(x,t)| \leq M, \tag{35}$$

$$|u_x(x,t)| \leq M, \quad |v_x(x,t)| \leq M, \tag{36}$$

$$|u_t(x,t)| \leq M(T), \quad |v_t(x,t)| \leq M(T) \tag{37}$$

where M, $M(T)$ *are independent of* ε.

Proof. This proof is based on the estimates (15), (16), (29) and the assumptions (3) and (4) on $\sigma(v)$.

To prove (35), we add the inequalities in (15) getting, $c_1 + c_2 \leq 2u \leq c_3 + c_4$. Then there exists M such that $|u(x,t)| \leq M$.

Using again (15), we have $c_3 - c_2 \leq z - w \leq c_4 - c_1$. Thus

$$c_3 - c_2 \leq 2\int_{v_1}^{v} \sqrt{\sigma'(s)}ds \leq c_3 - c_1,$$

then there exists M such that $|v(x,t)| \leq M$.

To show (36) we see that $w_x + z_x = 2u_x$ and using (16) it is obtained that $0 \leq u_x \leq \bar{M}$. This gives $|u_x(x,t)| \leq M$.

Also, from (16), $-\bar{M} \leq -w_x \leq 0$ and $0 \leq z_x \leq \bar{M}$, which implies $-\overline{M} \leq z_x - w_x \leq \bar{M}$, from where

$$\frac{-\bar{M}}{2\sqrt{\sigma'(v)}} \leq \frac{\partial v}{\partial x} \leq \frac{\bar{M}}{2\sqrt{\sigma'(v)}},$$

but $\sigma'(v) > k > 0$, then $|v_x| \leq M$. To prove (37) we use (29) and argue as we did in the proof of (36), which implies the existence of $M(T)$ such that $|u_t(x,t)| \leq M(T)$, $|v_t(x,t)| \leq M(T)$. ☑

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. Let $(v^{\varepsilon_n}, u^{\varepsilon_n})$ be the corresponding solution to (10) given by Theorem (3). By Lemma (5) $(v^{\varepsilon_n}, u^{\varepsilon_n})$ is bounded in $W^{1,\infty}((-\infty,\infty)\times[0,T))$. By the compactness of the inmbededing immersion $W^{1,\infty}(\Omega) \to C\left(\bar{\Omega}\right)$, it has a subsequence $\{v^{\varepsilon_n}, u^{\varepsilon_n}\}$ over each bounded region Ω of $\mathbb{R}\times\mathbb{R}^+$ that converges uniformly to a pair of Hölder-continuous functions $(v(x,t), u(x,t))$.

Let us see that the limit function (v,u) is the solution for the Cauchy problem (1), (2).

Multiplying both sides of (11) by

$$A = \begin{pmatrix} w_v & w_u \\ z_v & z_u \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2\sqrt{\sigma'(v)}} & \frac{1}{2\sqrt{\sigma'(v)}} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

it follows that

$$\begin{aligned} v_t - u_x &= \frac{\varepsilon}{2\sqrt{\sigma'(v)}}\left(z_{xx} - w_{xx}\right), \\ u_t - \sigma(v)_x &= \frac{\varepsilon}{2}\left(z_{xx} + w_{xx}\right). \end{aligned} \tag{38}$$

Multiplying both sides of (38) by ϕ, $\phi \in C_0^1(\mathbb{R}\times[0,\infty);\mathbb{R})$, and integrating over the set $(\mathbb{R}\times[0,\infty)$ we have

$$\begin{aligned} \int_{t\geq 0}\int (v\phi_t - u\phi_x)\,dxdt + \int_{t=0} v_0\phi dx = 0, \\ \int_{t\geq 0}\int (u\phi_t - \sigma(v)\,\phi_x)\,dxdt + \int_{t=0} u_0\phi dx = 0. \end{aligned} \tag{39}$$

If we assume the data to be smooth, we get:

Theorem 6. *Let $\sigma(v)$ be such that $\sigma'(v) > k > 0$, $\sigma''(v) > 0$, and let $(w_0(x), z_0(x))$ be defined by (7) which satisfy $c_1 \leq w_0(x) \leq c_2$, $c_3 \leq z_0(x) \leq c_4$. If $w_0(x)$, $z_0(x)$ are bounded in $W^{1,\infty}(\mathbb{R})$ and non-decreasing, then the Cauchy problem (1), (2) has a global Hölder-continuous solution (v,u). That is a (v,u) satisfies (39).*

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DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD PEDAGÓGICA Y TECNOLÓGICA DE COLOMBIA
TUNJA, COLOMBIA

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL DE COLOMBIA
BOGOTÁ, COLOMBIA