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Some p-norm convergence results for Jacobi and Gauss-Seidel iterations

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Abstract. Let A be a matrix such that the diagonal matrix D with the same diagonal as A is invertible. It is well known that if (1) A satisfies the Sassenfeld condition then its Gauss-Seidel scheme is convergent, and (2) if $D^{-1}A$ certifies certain classical diagonal dominance conditions then the Jacobi iterations for A are convergent. In this paper we generalize the second result and extend the first result to irreducible matrices satisfying a weak Sassenfeld condition.

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1. Introduction

Consider the system of linear equations:

$$\sum_{j=1}^{n} a_{ij}u_j = b_i, \quad i = 1, \ldots, n. \tag{1.1}$$

If the n by n matrix $A = (a_{ij})$ has an invertible diagonal D =Diag$(a_{11}, \ldots, a_{nn})$ then we can express these equations in the equivalent form

$$u_i = (b_i - \sum_{j \neq i} a_{ij}u_j)/a_{ii}$$

which immediately suggests well-known iterative schemes (cf. [1–5; 7–8]):

$$u_i^{(m+1)} = (b_i - \sum_{j \neq i} a_{ij} u_j^{(m)})/a_{ii}, \quad i = 1, \ldots, n, \tag{1.2}$$

$$u_i^{(m+1)} = (b_i - \sum_{j<i} a_{ij} u_j^{(m+1)} - \sum_{j>i} a_{ij} u_j^{(m)})/a_{ii}, \quad i = 1, \ldots, n, \tag{1.3}$$

referred to, respectively, as the Jacobi method and the Gauss-Seidel method. In spite of their simplicity, these schemes are among the most popular iterative schemes for the solution of linear equations (cf. [1–5; 7-8]). It is not possible to say outright that one of these methods is better than the other, since there are situations in which each converges and the other does not (cf. [4, Chapter 4]). However, the Jacobi method is usually the preferred method for parallel computations, and the Gauss-Seidel is the usual choice for use on sequential computers.

Convergence results on the schemes (1.2) – (1.3) can be found in [1–5;7–8]. In particular, it is well known that if A satisfies the Sassenfeld condition then its Gauss-Seidel scheme is convergent, and if $D^{-1}A$ satisfies the certain diagonal dominance conditions (cf. (2.2) – (2.4) below) then the Jacobi iterations for A converge.

However, in this area – as in other parts of Numerical Analysis – there is still a wide gap between theory and practice, and there are many matrices like

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1.4}$$

for which one or both of the schemes (1.2) – (1.3) converge, and which are not covered by the available convergence theory. Our aim in the present paper is to reduce this gap a little bit, by generalizing the second result in the previous paragraph, and extending the first result to irreducible matrices satisfying a weak Sassenfeld condition (this includes the matrix in equation (1.4)).

2. The convergence results

In the sequel we will make use of the scalar product $(x, y) = \sum_{i=1}^{n} x_i y_i$ in $\mathbf{R}^n$ and the norms $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ and, when $1 \leq p < \infty$, $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$.

The following gives the basic estimate from which our first convergence result will be derived.

Theorem 2.1. *For a fixed* $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$, *let* $Jx = ((Jx)_1, \ldots, (Jx)_n)$, $\forall x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, *where* $(Jx)_i \equiv \mu_i(b_i - \sum_{j\neq i} a_{ij}x_j)$ *and* $\mu_i \equiv 1/a_{ii}$. *Then*

$$\|J(x) - J(y)\|_p \leq \alpha_p \|x - y\|_p, \quad \forall x, y \in \mathbf{R}^n,$$

where

$$\alpha_p = \begin{cases} \max_{1\leq j\leq n} \sum_{i\neq j} |a_{ij}||\mu_i|, & \text{if } p = 1 \\ [\sum_{i=1}^n (\sum_{j\neq i} |a_{ij}|^q)^{\frac{p}{q}} |\mu_i|^p]^{\frac{1}{p}}, & \text{if } 1 < p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \max_{1\leq i\leq n} \sum_{j\neq i} |a_{ij}||\mu_i|, & \text{if } p = \infty. \end{cases} \tag{2.1}$$

Proof. Since $|(Jx)_i - (Jy)_i| \leq |\mu_i| \sum_{j\neq i} |a_{ij}||x_j - y_j| \leq |\mu_i|(\sum_{j\neq i} |a_{ij}|)\|x - y\|_\infty$, $i = 1, \ldots, n$, we have

$$\|Jx - Jy\|_\infty \leq \max_{1\leq i\leq n} (\sum_{j\neq i} |a_{ij}||\mu_i|)\|x - y\|_\infty = \alpha_\infty \|x - y\|_\infty,$$

which proves the result when $p = \infty$.

The result also holds when $p = 1$ since:

$$\begin{aligned} \sum_{i=1}^n |(Jx)_i - (Jy)_i| &\leq \sum_{i=1}^n \sum_{j\neq i} |\mu_i||a_{ij}||x_j - y_j| = \sum_{j=1}^n \sum_{i\neq j} |\mu_i||a_{ij}||x_j - y_j| \\ &\leq (\max_{1\leq j\leq n} \sum_{i\neq j} |a_{ij}||\mu_i|) \sum_{j=1}^n |x_j - y_j| = \alpha_1 \|x - y\|_1. \end{aligned}$$

For the case $1 < p < \infty$ we set $(1/p) + (1/q) = 1$. Then it follows from Hölder's inequality (cf. [6, Chapter 9]) that $|(Jx)_i - (Jy)_i| \leq |\mu_i| \sum_{j\neq i} |a_{ij}||x_j - y_j| = (z^{(i)}, w) \leq \|z^{(i)}\|_q \|w\|_p = \|z^{(i)}\|_q \|x - y\|_p$, where $w_j \equiv |x_j - y_j|$, $z_j^{(i)} = |a_{ij}||\mu_i|$ if $j \neq i$ and $z_i^{(i)} = 0$. Hence $\sum_{i=1}^n |(Jx)_i - (Jy)_i|^p \leq \sum_{i=1}^n \|z^{(i)}\|_q^p \|x - y\|_p^p$. This implies that $\|Jx - Jy\|_p \leq (\sum_{i=1}^n \|z^{(i)}\|_q^p)^{1/p} \|x - y\|_p$, and proves the result when $1 < p < \infty$. ☑

Corollary 2.1. *If $\alpha_p < 1$, for any $1 \le p \le \infty$, then the Jacobi iterates in (1.2) converge in the $\|\cdot\|_p$ norm to the unique solution of problem (1.1).*

Proof. The results follows immediately from Theorem 2.1 and Banach's contraction mapping Theorem (cf. [6, Appendix 1]). ☑

Remark 2.1. If the matrix $D^{-1}A$ is diagonally dominant in any of the senses:

$$\sum_{j\neq i} |a_{ij}|/|a_{ii}| < 1, \quad i = 1, \dots, n, \tag{2.2}$$

$$\sum_{i\neq j} |a_{ij}|/|a_{ii}| < 1, \quad j = 1, \dots, n, \tag{2.3}$$

$$\sum_{1=1}^{n} \sum_{j\neq i} |a_{ij}|^2/|a_{ii}|^2 < 1, \tag{2.4}$$

then $\alpha_\infty < 1$ or $\alpha_1 < 1$ or $\alpha_2 < 1$, respectively. So it follows from Corollary 2.1 that the Jacobi iterates in (1.2) converge in the $\|\cdot\|_p$ norm – for some $p \in \{1, 2, \infty\}$ – to the unique solution of problem (1.1). This is a well known classical result (cf. [4, Theorem 4.1]). However, the case ($1 < p < \infty$ and $p \neq 2$) of Corollary 2.1 appears to be new.

As a simple illustration, we observe that for the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

we have $\alpha_1 = \alpha_2 = \alpha_\infty = 1$, so that the classical diagonal dominance results do not apply. However, it is easy to verify that when $2 < p < \infty$ we have

$$\alpha_p = (2 + 2^{p/q})^{1/p}/2 = (2 + 2^{p-1})^{1/p}/2 < 1,$$

so we can deduce the convergence of the Jacobi scheme in the $\|\cdot\|_p$ norm from Corollary 2.1.

Remark 2.2. An n by n matrix A is said to be irreducible if for any proper disjoint union $\{1, \dots, n\} = W \cup Z$ there exist $i_0 \in W$, $j_0 \in Z$ such that $a_{i_0,j_0} \neq 0$. It is well known (cf. [4, Theorem 4.7] and [6, Theorem 4.9]) that when A is irreducible and weakly rowwise diagonally dominant in the sense that

$$\sum_{j\neq i} |a_{ij}| \le |a_{ii}|, \quad i = 1, \dots, n, \tag{2.5}$$

with strict inequality for at least one index i, the Jacobi and Gauss-Seidel iterates in (1.2) – (1.3) converge.

It is also known (cf. [4, Theorem 4.3]) that whenever A satisfies the Sassenfeld conditions:

$$\begin{aligned} p_1 &\equiv (\sum_{j>1} |a_{1j}|)/|a_{11}| < 1, \\ p_i &\equiv (\sum_{j<i} |a_{ij}|p_j + \sum_{j>i} |a_{ij}|)/|a_{ii}| < 1, \quad i = 2, \ldots, n, \end{aligned} \tag{2.6}$$

the Gauss-Seidel iterates in (1.3) converge to the unique solution of problem (1.1). In the following (new) theorem we show that a weak Sassenfeld condition is sufficient for convergence if A is irreducible.

Theorem 2.2. *Suppose that A is irreducible and satisfies the weak Sassenfeld conditions:*

$$\begin{aligned} p_1 &\equiv |\mu_1|(\sum_{j>1} |a_{1j}|) \leq 1, \\ p_i &\equiv |\mu_i|(\sum_{j<i} |a_{ij}|p_j + \sum_{j>i} |a_{ij}|) \leq 1, \quad i = 2, \ldots, n, \\ p &= \min_{1 \leq i \leq n} p_i < 1, \end{aligned} \tag{2.7}$$

where $\mu_i \equiv 1/a_{ii}$. Then the Gauss-Seidel iterates in (1.3) converge.

Proof. The solution $u = (u_1, \ldots, u_n)$ of (1.1) is a fixed point of the operator defined, for $x \in \mathbf{R}^n$, by $Gx = ((Gx)_1, \ldots, (Gx)_n)$, where

$$(Gx)_i \equiv \mu_i(b_i - \sum_{j<i} a_{ij}(Gx)_j - \sum_{j>i} a_{ij}x_j).$$

Let $Cx = ((Cx)_1, \ldots, (Cx)_n)$, where

$$(Cx)_i \equiv \mu_i(\sum_{j<i}^{n} a_{ij}(Cx)_j + \sum_{j>i}^{n} a_{ij}x_j)$$

and let $r(C)$ designate the spectral radius of C.

We prove by induction that the estimate

$$|(Cx)_i| \leq p_i \|x\|_\infty, \quad \forall x \in \mathbf{R}^n \tag{2.8}$$

holds for all i. It is obviously true if $i = 1$, because of the definitions of C and p_1. Suppose $k > 1$ and that (2.8) holds for all $i < k$. Then

$$\begin{aligned} |(Cx)_k| &\leq |\mu_k|(\sum_{j<k} |a_{kj}||(Cx)_j| + \sum_{j>k} |a_{kj}||x_j|) \\ &\leq |\mu_k|(\sum_{j<k} p_j|a_{kj}| + \sum_{j>k} |a_{kj}|)\|x\|_\infty = p_k\|x\|_\infty. \end{aligned}$$

This proves, by induction, that (2.8) holds for i.

Since $0 \le p_i \le 1$ for all i, it follows from the hypotheses that $|(Cx)_i| \le p_i \le 1$ whenever $\|x\|_\infty = 1$, and hence, that $r(C) \le \|C\|_\infty \le 1$.

To see that $r(C) < 1$ (which implies the convergence of the Gauss-Seidel scheme), we suppose, by contradiction, that $Cv = sv$ for some v such that $\|v\|_\infty = 1 = |s|$. Then $|v_i| = |s||v_i| \le |\mu_i|(\sum_{j<i} |a_{ij}|p_j + \sum_{j<i} |a_{ij}|)\|v\|_\infty = |\mu_i|(\sum_{j<i} |a_{ij}|p_j + \sum_{j<i} |a_{ij}|) = p_i \le 1$, for $i = 1, \dots, n$. Let $W = \{i\colon |v_i| = 1\}$. Then $W \neq \emptyset$ since $\|v\|_\infty = 1$. For each $i \in W$, we have $1 = |\mu_i|(\sum_{j<i} |a_{ij}|p_j + \sum_{j>i} |a_{ij}|) = p_i \le 1$. Hence $p_i = 1$, so it follows from the weak Sassenfeld condition (2.7) that $Z \equiv \{1, \dots, n\} \setminus W \neq \emptyset$.

Since A is irreducible, there exist $i_0 \in W$, $j_0 \in Z$ such that $a_{i_0,j_0} \neq 0$. We have $|a_{i_0j_0}||v_{j_0}| < |a_{i_0j_0}|$, which implies that $1 = |v_{i_0}| = |s||v_{i_0}| = |\mu_{i_0}|(\sum_{j<i_0} p_j|a_{i_0j}||v_j| + \sum_{j>i_0} |a_{i_0j}||v_j|) < |\mu_{i_0}|(\sum_{j<i_0} p_j|a_{i_0j}| + \sum_{j>i_0} |a_{i_0j}|) = p_{i_0} \le 1$. This contradiction shows that we must have $r(C) < 1$. That concludes the proof. ☑

Remark 2.4. *The matrix in (1.4) does not satisfy the classical Sassenfeld condition (2.6). However, it is irreducible and satisfies the weak Sassenfeld condition (2.7), and it is easy to verify that its Gauss-Seidel iterations are convergent.*

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