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On the normality of operators

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ABSTRACT. In this paper we will investigate the normality in (WN) and $(\mathcal{Y})$ classes.

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RESUMEN. En este artículo nosotros investigaremos la normalidad en clases (WN) y $(\mathcal{Y})$.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle,\rangle$ and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. For A in $\mathcal{B}(H)$ the adjoint of A is denoted by A^*. For any operator A in $\mathcal{B}(H)$ set, as usual, $| A |= (A^*A)^{\frac{1}{2}}$ and $[A^*,A] = A^*A - AA^* = | A |^2 - | A^* |^2$ (the self commutator of A), and consider the following standard definitions: A is hyponormal if $| A^* |^2 \leq | A |^2$ (i.e., if $[A^*,A]$ is nonnegative or, equivalently, if $\|A^*x\| \leq \|Ax\|$ for every x in H), normal if $A^*A = AA^*$, quasinormal if A^*A commutes with A, and m-hyponormal if there exists a positive number m, such that

$$m^2(A-\lambda I)^*(A-\lambda I) - (A-\lambda I)(A-\lambda I)^* \geq 0, \text{ for all } \lambda \in \mathbb{C}.$$

Let $(N), (QN), (H)$, and $(m-H)$ denote the classes constituting of normal, quasinormal, hyponormal, and m-hyponormal operators. Then

$$(N) \subset (QN) \subset (H) \subset (m-H).$$

An operator T in $\mathcal{B}(H)$ is said to be hermitian if $T = T^*$. It is well known that hermitian operators can be characterized in the following way: an operator T in $\mathcal{B}(H)$ is hermitian if and only if $\langle Tx, x\rangle$ is real. In [3] the authors gave an other characterization involving inequalities. We denote by (WN) the class of operators in $\mathcal{B}(H)$ satisfying the following inequality

$$(Re\,T)^2 \leq |T|^2,$$

where $ReT = \frac{(T+T^*)}{2}$ is the real part of T and we will write $ImT = \frac{(T-T^*)}{2}$ for the imaginary part of T. This class has been introduced by Fong and Istratescu [3] who conjectured:

If $T \in (WN)$ and $\sigma(T)$(the spectrum of T) is real, then T is hermitian. It is known that if in addition T is hyponormal or $T+T^*$ commutes with TT^* or if H is finite-dimensionel the conjecture holds [3, Corollary 2.2]. Notice that T is hermitian if and only if

$$(ReT)^2 \geq |T|^2. \tag{1.1}$$

By reversing the inequality (1.1) we obtain the class (WN). This class contains the class of hyponormal operators. Indeed, T hyponormal implies that

$$(ReT)^2 + (ImT)^2 \leq |T|^2.$$

Since $(ImT)^2$ is a positive operator,

$$(ReT)^2 \leq |T|^2,$$

that is, $T \in (WN)$. In this note we will investigate the normality in (WN).

A is said to be of class $\mathcal{Y}_\alpha$ for $\alpha \geq 1$ if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A-\lambda)^*(A-\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1\leq\alpha}\mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e., there exists a positive number M such that

$$(A-\lambda I)(A-\lambda I)^* \leq M^2(A-\lambda I)^*(A-\lambda I) \quad \text{for all} \quad \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ (see[13]).

In [15] Weber shows that every compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$ is quasinilpotent, where $\{A\}'$ denotes the commutant of A, δ_A stands for the derivation operator generated by A, i.e., $\delta_A(X) = AX - XA$, $R(\delta_A)$ is the range (image) of δ_A and $\overline{R(\delta_A)}^w$ is its weak closure. In [7] we showed that Weber's result is a consequence of a more general result. A reasonable conjecture is the following:

(S) Let T be a compact operator. If $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then T is quasinilpotent, where $\{A^*\}'$ is the commutant of the adjoint of A.

In [7], [8] it is proved that if A is normal, subnormal, dominant or m-hyponormal, then the conjecture holds. In this paper we shall see that the conjecture holds for an interesting class of operators which includes hyponormal operators. It is clear that the class $(\mathcal{Y})$ contains the class of normal operators. In the following we will denote the spectrum, the point spectrum, the approximate reduced spectrum, and the approximate spectrum of an operator $A \in \mathcal{B}(H)$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ar}(A)$, and $\sigma_a(A)$ respectively (see [7]). In this paper we will investigate the normality in $(\mathcal{Y})$.

2. Preliminaries

Let us list some spectral properties of operators in (WN).

Proposition 2.1. [3] *Suppose that $T \in B(H)$ is in (WN). Then we have*

(i) *If λ is a real number, then $T - \lambda$ is also in (WN).*
(ii) $\|(T-\lambda)^*x\| \leq 3\,\|(T-\lambda)x\|$, for all $x \in H$ and all real number λ.
(iii) *If M is an invariant supspace of T, then $T\mid_M$ is also in (WN); if furthermore, $T\mid_M$ is hermitian, then M reduces T.*
(iv) *If λ is real eigenvalue of T, then the eigenspace $\ker(T-\lambda)$ reduces T.*
(v) *If λ is real and $(T-\lambda)^n x = 0$ for some $n \geq 1$, then $(T-\lambda)x = 0$.*

Remark 2.1. *As a consequence of the previous proposition, if $(\lambda_n)_{n\in\mathbb{N}}$ is a countable sequence of real eigenvalues of T in (WN), then $H = M \oplus M^{\perp}$ and $T\mid_M$ is hermitian where $M = \oplus_{n\geq 0}\ker(T-\lambda_n I)$.*

3. Normality in $(\mathcal{Y})$-classes

Let us begin by the following Berberian techniques [2]: Let H be a complex Hilbert space. Then there exists an Hilbert space $H^\circ \supset H$, and an isometric $*$–homomorphism

$$\varphi : \mathcal{B}(H) \mapsto \mathcal{B}(H^\circ)\ (A \mapsto A^\circ)$$

preserving order, i.e., for all $A, B \in \mathcal{B}(H)$ and for all $\alpha, \beta \in \mathbb{C}$ we have:

(1) $\varphi(A^*) = \varphi(A)^*$;
(2) $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$;
(3) $\varphi(I_{\mathcal{H}}) = I_{\mathcal{H}^\circ}$;
(4) $\varphi(AB) = \varphi(A)\varphi(B)$;
(5) $\| \varphi(A) \| = \| A \|$;
(6) $\varphi(A) \leq \varphi(B)$ if $A \leq B$;
(7) $\sigma(\varphi(A)) = \sigma(A)$, $\sigma_a(A) = \sigma_a(\varphi(A)) = \sigma_p(\varphi(A))$;
(8) if A is a positive operator, then $\varphi(A^\alpha) = \mid \varphi(A) \mid^\alpha$ for all $\alpha > 0$.

Lemma 3.1. *If $S \in (\mathcal{Y})$, then $\varphi(S) \in (\mathcal{Y})$.*

Proof. If $S \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$\mid TT^* - T^*T \mid^\alpha \leq k_\alpha^2 (T-\lambda I)^*(T-\lambda I), \text{ for all } \lambda \in \mathbb{C}.$$

It follows from the properties of the map φ that

$$\varphi(\mid TT^* - T^*T \mid^\alpha) \leq \varphi(k_\alpha^2 (T-\lambda I)^*(T-\lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

By the condition (8) above we have

$$\varphi(\mid TT^* - T^*T \mid^\alpha) = \mid \varphi(TT^* - T^*T) \mid^\alpha,$$

for all $\alpha > 0$. Therefore

$$|\varphi(T)\varphi(T^*) - \varphi(T^*)\varphi(T)|^\alpha \leq \varphi(k_\alpha^2 (T-\lambda I)^*(T-\lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

Hence $\varphi(T) \in (\mathcal{Y})$. ☑

Now we will present some spectral properties of the class $(\mathcal{Y})$.

Theorem 3.1. *Let $S \in (\mathcal{Y})$.*

(i) *If $\lambda \in \sigma_p(S)$, then $\overline{\lambda} \in \sigma_p(S^*)$, furthermore if $\lambda \neq \mu$, then M_λ (the proper subspace associated with λ) is orthogonal to M_μ;*
(ii) *If $\lambda \in \sigma_a(S)$, then $\overline{\lambda} \in \sigma_a(S^*)$;*
(iii) *$SS^* - S^*S$ is not invertible;*
(iv) *If M is an invariant subspace for S and $S\mid_M$ is normal, then M reduces S;*
(v) *If there exists a reducing subspace M , then $S\mid_M \in (\mathcal{Y})$.*

Proof. For (i) and (iv) see [13].

(ii) Let $\mu \in \sigma_a(S)$ from the condition (7) above, we have

$$\sigma_a(S) = \sigma_a(\varphi(S)) = \sigma_p(\varphi(S).$$

Therefore $\mu \in \sigma_p(\varphi(S))$. By applying Lemma 3.1 and the above condition (i), we get

$$\overline{\mu} \in \sigma_p(\varphi(S)^*) = \sigma_p(\varphi(S^*)).$$

Hence $\overline{\mu} \in \sigma_a(\varphi(S^*))$.

(iii) Let $S \in (\mathcal{Y})$. Then there exists an integer $n \geq 1$ and $k_n > 0$ such that

$$\| \mid SS^* - S^*S \mid^{2^{n-1}} x \| \leq k_n^2 \| (S - \lambda I)x \|$$

for all $x \in H$, and for all $\lambda \in \mathbb{C}$.
It is known that $\sigma_a(S) \neq \phi$. If $\lambda \in \sigma_a(S)$, then there exists a normed sequence (x_n) in $\mathcal{H}$ such that $\| (S - \lambda I)x_n \| \to 0$. Then $(SS^* - S^*S)x_n \to 0$ and so, $(SS^* - S^*S)$ is not invertible.

☑

Let $\mathcal{A}$ denote a complex Banach Algebra with identity e. A state on $\mathcal{A}$ is a functional $f \in \mathcal{A}^*$ such that $f(e) = 1 = \|f\|$. For $x \in \mathcal{A}$ let

$$W_0(x) = \{f(x) : f \text{ is a state on } \mathcal{A}\}$$

be the numerical range of x [14]. $W_0(x)$ is a compact convex set containing $co\sigma(x)$ (the convex hull of the spectrum of x) [1].

For the case $\mathcal{A} = \mathcal{B}(H)$, if $A \in \mathcal{B}(H)$ then $W_0(A) = \overline{W(A)}$, where

$$W(A) = \{(Ah, h) : h \in H, \|h\| = 1\}$$

is the special numerical range of $\mathcal{A}$. An element $a \in \mathcal{A}$ is finite if $0 \in W_0(ax - xa)$ for each x in $\mathcal{A}$; $\mathcal{F}(A)$ (or $\mathcal{F}$) denotes the set of all finite elements of $\mathcal{A}$. It is known that $\mathcal{F}$ contains every normal, hyponormal and dominant operator (see [7], [14]). In [10] the author initiated the study of a more general class of finite operators called generalized pair of finite operators defined by

$$\mathcal{G}F = \{(A, B) \in B(H) \times B(H) : \|AX - XB - I\| \geq 1, \forall X \in B(H)\}.$$

Now we will prove that the class $(\mathcal{Y})$ of operators is included in the class of finite operators. For this we need the following lemma.

Lemma 3.2. *If $S \in (\mathcal{Y})$, then $\sigma_{ar}(S) \neq \phi$.*

Proof. It is known that $\sigma_{ar}(S) \subset \sigma_a(S)$. Since $\sigma_a(S) \neq 0$, it suffices to prove that $\sigma_a(S) \subset \sigma_{ar}(S)$. If $S \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$\| \mid SS^* - S^*S \mid^{\frac{\alpha}{2}} x \| \leq k_\alpha^2 \| (S - \lambda I)x \| \text{ for all } x \in H \text{ and for all } \lambda \in \mathbb{C}. \quad (2.1)$$

Since

$$(S - \mu I)(S - \mu I)^* = SS^* - S^*S + (S - \mu I)^*(S - \mu I) \text{ for all } \mu \in \mathbb{C},$$

then

$$| \langle (SS^* - S^*S)x, x \rangle | \leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2, \text{ for all } x \in H.$$

Indeed, consider the polar decomposition of the operator $SS^* - S^*S = VD$, where $D = |SS^* - S^*S|$. Then V is a Hermitian partial isometry which commutes with D because $SS^* - S^*S$ is Hermitian. Hence, for any $x \in H$ such that $\|x\| = 1$

$$\begin{aligned} |\langle |SS^* - S^*S| \, x, x \rangle| &\leq \left| \left\langle |SS^* - S^*S|^{\frac{1}{2}} x, |SS^* - S^*S|^{\frac{1}{2} V^* x} \right\rangle \right| \\ &\leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\| \left\| |SS^* - S^*S|^{\frac{1}{2}} V^* x \right\| \\ &= \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\| \left\| V^* |SS^* - S^*S|^{\frac{1}{2}} x \right\| \\ &\leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2. \end{aligned}$$

Consequently

$$\|(S - \mu I)^* x\|^2 \leq \|(S - \mu I)x\|^2 + \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2, \quad (2.2)$$

for all $\mu \in \mathbb{C}$ and for all $x \in \mathcal{H}$. Let $\lambda \in \sigma_a(S)$, then there exists a normed sequence $(x_n)_n \subset \mathcal{H}$ such that $\|(S - \lambda I)x_n\| \to 0$. Therefore for $\lambda = \mu$, $x_n = x$, and for all n we get

$$\| \mid SS^* - S^*S \mid^{\frac{\alpha}{2}} x_n \| \leq k_\alpha^2 \| (S - \mu I)x_n \| . \quad (2.3)$$

By applying (2.2) and (2.3) we deduce that

$$\|(S - \mu I)^* x\|^2 \leq (1 + k_\alpha^2) \|(S - \mu I)x\|, \text{ for all } n.$$

Therefore $\|(S - \mu I)^* x\| \to 0$ and $\lambda \in \sigma_{ar}(S)$, that is, $\sigma_{ar}(S) \neq \phi$. ☑

Now we are ready to show that $(\mathcal{Y}) \subset \mathcal{F}$.

Theorem 3.2. *The class $(\mathcal{Y})$ of operators is included in the class of finite operators.*

Proof. It is shown in [9] that if $\sigma_a(A) \neq \phi$, then A is finite. It suffices to apply the above lemma. ☑

It is shown in [13] that:

(1) If $T \in (\mathcal{Y})$ and $T^* \in (\mathcal{Y})$, then T is normal.
(2) If $T \in (\mathcal{Y})$is a compact operator, then T is normal.
(3) if $T \in (\mathcal{Y})$ is similar to a normal operator, then T is normal.

In this section we will continue this study.

We begin by the following lemma. Note that concerning this lemma more general results in this direction can be found in [11] and [4]. Recall that an operator T is said to be algebraic if $P(T) = 0$ for a certain polynomial P.

Lemma 3.3. *If T is an algebraic operator, then $\sigma(T) = \sigma_p(T)$ (point spectrum of T).*

Proof. It is well known that an operator T is algebraic if and only if its spectrum consists of poles only. But a pole of an operator is always an eigenvalue. Hence for an algebraic operator the spectrum and the point spectrum coincide. ☑

In the following propositions, theorems and corollaries we will show other cases for the normality in $(\mathcal{Y})$-classes.

Theorem 3.3. *Let $T \in (\mathcal{Y})$. If T is an algebraic operator, then T is normal.*

Proof. According to [13] the nilpotent operators of order n in $(\mathcal{Y})$ are null. Since $T \in (\mathcal{Y})$ implies the existence of a number $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \mu I)^* (T - \mu I), \forall \mu \in \mathbb{C},$$

it follows that for all $\mu \in \mathbb{C}$, that

$$\left|(T - \mu I)(T - \mu I)^* - (T - \mu I)^* (T - \mu I)\right|^\alpha = |TT^* - T^*T|^\alpha \leq 1$$

$$k_\alpha^2 (T - \mu I)^* (T - \mu I) \leq k_\alpha^2 [(T - \mu I) - (T - \mu I)]^* [(T - \mu I) - (T - \mu I)].$$

Therefore,

$$(T - \mu I)(T - \mu I)^* - (T - \mu I)^* (T - \mu I) \leq k_\alpha^2 [(T - \mu I) - \gamma I]^* [(T - \mu I) - \gamma I],$$

for all $\gamma \in \mathbb{C}$. Hence $(T - \mu I) \in (\mathcal{Y})$, for all $\mu \in \mathbb{C}$. By applying the above lemma we get $\sigma(T) = \sigma_p(T)$. Finally Theorem 3.1(i) ensures that T is normal. ☑

Theorem 3.4. *Let $T \in (\mathcal{Y})$. If there exists a polynomial P such that $P(T)$ is normal, then T is normal.*

Proof. According to [6] there exist reducing subspaces (H_n) for T such that $H = \oplus_n H_n$, where $T_0 = T\ |_{H_0}$ is algebraic and $T_n = T \mid H_n$ $(n \geq 1)$ is similar to a normal operator. Hence from Theorem 3.1 and [13] we complete the proof. ☑

4. Normality in (WN)-classes

Proposition 4.1. [3, Theorem 2.3] *Let $T \in (WN)$. If T is similar to a normal operator, then T is normal.*

Recall that an operator T is said to be isometric if $T^*T = I$ and co-isometric if $TT^* = I$.

Proposition 4.2. *Let $T \in (WN)$. If T is co-isometric, then T is unitary.*

Proof. It is known [5] that if an operator in (WN) is right invertible, then it is invertible. ☑

Remark 4.1. *It is evident that any polynomial of a normal operator is normal. but the converse is not true. As an example take $T \in B(H)$ such that $T^2 = 0$. It is shown in [20] that:*

Theorem 4.1. [13] *Let $T \in (WN)$. If T^p and T^q are normal operators for certain coprime integers p, q, then T is normal.*

Remark 4.2. *The normality of T^2 for a certain operator $T \in (WN)$ it is not sufficient to ensure the normality of T. As an example take*

$$\dim H = 2, \qquad T = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix},$$

$T \in (WN)$. It is clear that T is normal but

$$T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is not normal.

Theorem 4.2. *Let $T \in (WN)$. If T is a partial isometry and $0 \notin W(T)$, then T is normal.*

Proof. It is known [5] that If T is a partial isometry in (WN), then T is quasinormal. Therefore

$$[T, T^*]T = 0.$$

If there exists a vector $x \in H$ such that $x \notin \ker[T, T^*]$, set

$$y = \frac{[T, T^*]x}{\|[T, T^*]\|},$$

hence $\langle Ty, y\rangle \in W(T)$. Since $0 \notin W(T)$, it results that $\langle Ty, y\rangle$ is not null. This contradicts the fact that $[T, T^*]T = 0$. Consequently $\ker[T, T^*] = H$, i.e., T is normal. ☑

Now we are ready to prove that the conjecture holds for operators in (WN).

Theorem 4.3. *If A or $A^* \in (WN)$, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

Proof. We start with the second assumption. Suppose that $A^* \in (WN)$ and $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$. Let $\lambda \in \sigma_p(T)$ such that $E = \ker(T - \lambda)$ be finite dimensional, then the subspace E is invariant under T and A^*. Since $A^* \in (WN)$, E reduces A^*. Let $H = E \oplus E^\perp$, hence we can write

$$A^* = \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix}, \qquad T = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}.$$

Since $T \in \overline{R(\delta_A)}^w$, $\lambda I_E \in R(\delta_{A_1})$ and this implies that $\lambda = 0$. Since T is a compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$, it results that $\sigma(T) = \{0\}$ which implies that T is quasinilpotent. This completes the proof of the second assymption.

Remark that if $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then

$$T^* \in \overline{R(\delta_{A^*})}^w \cap \{A\}'.$$

Then the first assumption of the theorem follows in exactly the same way as the second. ☑

By the same arguments as in the above proof we prove the following theorem

Theorem 4.4. *If A or A^* is of class $(\mathcal{Y})$, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

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