Reduction to normal form of a self-adjoint linear transformation with respect to a pseudo-unitary or a pseudo-euclidean inner product

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Abstract. We provide a self-contained and constructive approach to reduce a self-adjoint linear transformation defined on a pseudo-unitary (resp., pseudo-euclidean) space to a canonical form.

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Resumen. Nosotros damos una aproximación auto-contenida y constructiva para reducir una transformación lineal auto-adjunta definida sobre un espacio pseudo-unitario (resp. pseudo-euclidiano) a una forma canónica.

1. Introduction

Using pencils of matrices the simultaneous reduction of a pair of real symmetric matrices to a normal form was first developed by Weierstrass [9] and Kronecker [4]. Since then several authors, e.g., [2], [6], and [8], have studied the canonical forms for a pair of structured matrices. In this paper, we show that a complex (resp. real) self-adjoint linear transformation on a pseudo-unitary (resp. pseudo-euclidean) space can be brought to a canonical form via a basis that is orthonormal with respective to the inner product. Following a different approach than the authors mentioned above, we provide an explicit construction that produces an orthonormal basis relative to which the given linear transformation has a canonical form. In the complex case, the orthonormal basis is a modification of a Jordan basis, whereas in the real case, our basis will be a modification of a basis associated with the real Jordan form.

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2. Definitions and preliminaries

A complex linear space E of dimension n is called *pseudo-unitary* if there is a non-degenerate, indefinite, Hermitian form defined on it. This form is called an *inner product* and is denoted by $\langle , \rangle$. When the inner product is positive definite then the space E is simply a *unitary space*. We denote $\|x\| = \langle x, x \rangle$ and call it the *norm of x*. Two vectors x and y in E are called *orthogonal* if $\langle x, y \rangle = 0$. A basis $\{e_i\}, (1 \le i \le n)$ is called an *orthonormal basis* if for some $p, 1 \le p \le n$,

$$\langle e_i, e_i \rangle = \begin{cases} 1 & \text{for } 1 \le i \le p\,, \\ -1 & \text{for } p+1 \le i \le n\,, \end{cases}$$

and $\langle e_i, e_j \rangle = 0$ for $i \ne j$.

Given a pseudo-unitary space, it is well known that an orthonormal basis $\{e_i\}$ can always be constructed and that p is a fixed number, ([3]; 9.8, pp. 267-8). In this basis, if we write $v = \sum_{i=1}^{n} v_i e_i$ and $w = \sum_{i=1}^{n} w_i e_i$ then

$$\langle v, w \rangle = v_1 \overline{w}_1 + v_2 \overline{w}_2 + \cdots + v_p \overline{w}_p - v_{p+1} \overline{w}_{p+1} - \cdots - v_n \overline{w}_n.$$

A linear transformation, T, on E is called *self-adjoint* if $\langle Tx, y \rangle = \langle x, Ty \rangle$. When E is a unitary space, T is called *symmetric.*

We now make analogous definitions for a *real* vector space. A real n-dimensional vector space, V, is called *pseudo-euclidean* if there is a non-degenerative, indefinite, symmetric bilinear form defined on it. As before, we call this bilinear form an *inner product* and denote it by $\langle , \rangle$. When the inner product is positive definite the space is a *euclidean space. Orthogonal vectors, norm, orthonormal basis,* and *self-adjoint linear transformations* are defined as above. Again, it is well known that an orthonormal basis exists ([3]; 9.8, pp. 267-8). If $\{e_i\}$ is an orthonormal basis of V, and if $x = \sum_{i=1}^{n} x_i e_i$, $y = \sum_{i=1}^{n} y_i e_i$, then

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_n y_n.$$

In Section 3, we consider a self-adjoint, linear transformation T acting on a pseudo-unitary space E. Our goal is to construct an orthonormal basis of E such that the matrix of T with respect to this basis is of a normal form. We begin by showing that E is a direct sum of invariant, mutually orthogonal subspaces where the restriction of T to each subspace has either a single real or a single pair of complex conjugate eigenvalues. This will essentially reduce our problem to two cases: (1) T has a single real eigenvalue, and (2) T has a single pair of complex conjugate eigenvalues. In each of these cases, a modification of the standard Jordan basis produces the desired orthonormal basis.

In Section 4 we consider a self-adjoint linear transformation acting on a pseudo-euclidean space. Here again, we show that our vector space is a direct

sum of invariant, mutually orthogonal subspaces where the restriction of T to each subspace has either a single real or a single pair of complex conjugate eigenvalues, and this reduces the problem to the two cases mentioned above. If an eigenvalue is real we use the usual Jordan basis. Otherwise, we use the basis corresponding to the real Jordan form for a complex eigenvalue.

Below are a number of types of matrices that will be used throughout the paper. An omitted entry in a matrix is meant to be the entry '0'. The superscript "F" refers either to $r =$ the field of real numbers or to $c =$ the field of complex numbers. For a matrix A, A^* will denote its negative transpose. The matrix I_n is the identity matrix of dimension n.

The following are square matrices:

$$(A_y)^F(x) = \begin{bmatrix} x & \frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & x & \frac{1}{2} \\ & & \frac{1}{2} & x + \frac{y}{2} \end{bmatrix} \qquad (\widehat{A}_y)^F(x) = \begin{bmatrix} x & -\frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & x & -\frac{1}{2} \\ & & \frac{1}{2} & x + \frac{y}{2} \end{bmatrix}$$

$$B^F(x) = \begin{bmatrix} x & \frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & x & \frac{1}{\sqrt{2}} \\ & & \frac{1}{\sqrt{2}} & x \end{bmatrix} \qquad (^yA)^F(x) = \begin{bmatrix} x + \frac{y}{2} & \frac{1}{2} & & \\ \frac{1}{2} & x & \ddots & \\ & \ddots & \ddots & \frac{1}{2} \\ & & \frac{1}{2} & x \end{bmatrix}$$

$$_y\widehat{D}^F = \begin{bmatrix} & & -\frac{1}{2} \\ & \cdot^{\cdot^{\cdot}} & \frac{1}{2} \\ -\frac{1}{2} & & \\ \frac{y}{2} & \frac{1}{2} & \end{bmatrix} \qquad L^F(x) = \begin{bmatrix} x & & & & \frac{1}{\sqrt{2}} \\ & \ddots & & \frac{1}{2} & \\ & \frac{1}{2} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & & & \frac{1}{2} & x \end{bmatrix}$$

$$M^r(\alpha) = \begin{bmatrix} x & \frac{1}{\sqrt{2}} & & & \\ \frac{1}{\sqrt{2}} & \ddots & \frac{1}{2} & & \\ & \frac{1}{2} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2} \\ & & & \frac{1}{2} & x \end{bmatrix}$$

When $x = 0$ or $y = 0$ we simply drop the zero, e.g., we write $(A_y)^F$ for $(A_y)^F(0)$ and $A^F(x)$ for $(A_0)^F(x)$.

The following matrices are of dimension $l \times (l+1)$:

$$\widehat{K}^F = \begin{bmatrix} & -\frac{1}{2} & & \\ \frac{1}{2} & & \ddots & \\ & \ddots & & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & & \frac{1}{2} & \end{bmatrix} \qquad \widehat{E} = \begin{bmatrix} & & -\frac{1}{2} & \\ & \iddots & & \frac{1}{2} \\ & -\frac{1}{2} & \iddots & \\ -\frac{1}{\sqrt{2}} & & \frac{1}{2} & \end{bmatrix}.$$

The following matrices are of dimension $(l+1) \times l$:

$$G^F(x) = \begin{bmatrix} \frac{1}{\sqrt{2}} & & \\ x & \frac{1}{2} & \\ \frac{1}{2} & \ddots & \ddots \\ & \ddots & \ddots & \frac{1}{2} \\ & & \frac{1}{2} & x \end{bmatrix} \qquad \widehat{C}^F = \begin{bmatrix} -\frac{1}{2} & & \\ \frac{1}{2} & \ddots & \\ \ddots & & -\frac{1}{2} \\ & \frac{1}{2} & \\ & & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\widehat{N}^r = \begin{bmatrix} & & \frac{1}{2} & \\ & \iddots & & \frac{1}{2} \\ -\frac{1}{2} & & \iddots & \\ & \frac{1}{2} & & \\ \frac{1}{\sqrt{2}} & & & \end{bmatrix}.$$

3. Self-adjoint linear transformations on a pseudo-unitary Space

Let E be a pseudo-unitary space and let T be a self-adjoint linear transformation on E. It is known (See [1], Theorem 3.3, pp. 35) if $\alpha \neq \bar{\beta}$ are eigenvalues of T then the corresponding eigenspaces E_α and E_β are perpendicular. This along with non-degeneracy of the inner product implies that complex eigenvalues of T appear in complex conjugate pairs, and we can decompose E into a direct sum of invariant and pairwise orthogonal subspaces

$$E = E_1 + \cdots + E_s,$$

where each E_i is either an eigenspace of T corresponding to a real eigenvalue or is the sum of eigenspaces corresponding to a single pair of complex conjugate eigenvalues. Therefore, our study of the transformation T, is reduced to its action on each subspace E_p. We consider two cases.

Case I: T has a single real eigenvalue.
The next lemma shows that a particular kind of Jordan basis can be chosen on each subspace E_p.

Lemma 3.1. *Let T be a self-adjoint, linear transformation on the pseudo-unitary space E, with a single real eigenvalue α. Then, E can be decomposed into the direct sum of pairwise orthogonal, invariant subspaces, $E = E_1 + E_2 + \cdots + E_t$, where the restriction of T to each subspace produces a Jordan block. The Jordan basis*

$$v_1 = v, \quad v_2 = (T - \alpha I)v_1, \quad \cdots, \quad v_k = (T - \alpha I)v_{k-1},$$

for an orthogonal subspace E_p can be chosen such that

$$\langle v_i, v_j \rangle = \langle v_1, v_k \rangle = \epsilon \quad \text{if} \quad i + j = k + 1 \,,$$

where $\epsilon = \pm 1$ and

$$\langle v_i, v_j \rangle = 0 \quad \text{if} \quad i + j \neq k + 1.$$

(Note: The sign of $\langle v_1, v_k \rangle$, known as the sign characteristic of the inner product on E_p is determined by the transformation T and cannot be changed.)

Proof. See [5] page 244. ☑

Next we construct an orthonormal basis for each E_p.

Proposition 3.1. *Let E_p be an invariant subspace of E with associated real eigenvalue α as in Lemma 3.1. Then the Jordan basis given in this lemma can be modified to get an orthonormal basis β of E_p such that the matrix of T with respect to β is*

a) *for $k = 2l$ even*

$$\begin{bmatrix} (A_\epsilon)^r(\alpha) & \left(\widehat{A_\epsilon}\right)^r \\ \left(\left(\widehat{A_\epsilon}\right)^r\right)^* & (A_{-\epsilon})^r(\alpha) \end{bmatrix}.$$

(Each block is of dimension l).

b) *for $k = 2l + 1$ odd and the sign characteristic is positive*

$$\begin{bmatrix} B^r(\alpha) & \widehat{C}^r \\ \left(\widehat{C}^r\right)^* & A^r(\alpha) \end{bmatrix}.$$

($B^r(\alpha)$ is of dimension $l + 1, A^r(\alpha)$ is of dimension l).

c) *for $k = 2l + 1$ odd and the sign characteristic is negative*

$$\begin{bmatrix} A^r(\alpha) & \widehat{K}^r \\ \left(\widehat{K}^r\right)^* & L^r(\alpha) \end{bmatrix}.$$

($A^r(\alpha)$ is of dimension $l, L^r(\alpha)$ is of dimension $l + 1$).

Proof. Suppose $k = 2l$ is even and the sign characteristic is positive. Then the following is an orthonormal basis

$$a_1 = \frac{1}{\sqrt{2}}(v_1 + v_{2l}), \quad a_2 = \frac{1}{\sqrt{2}}(v_2 + v_{2l-1}), \cdots, a_l = \frac{1}{\sqrt{2}}(v_l + v_{l+1}),$$

$$a_{l+1} = \frac{1}{\sqrt{2}}(v_1 - v_{2l}), \quad a_{l+2} = \frac{1}{\sqrt{2}}(v_2 - v_{2l-1}), \cdots, a_{2l} = \frac{1}{\sqrt{2}}(v_l - v_{l+1}).$$

(If the sign characteristic is negative, replace a_i by a_{l+i}, and viceversa for $1 \le i \le l$).

If $k = 2l + 1$ is odd and the sign characteristic is positive, then the orthonormal basis is

$$a_1 = \frac{1}{\sqrt{2}}(v_1 + v_{2l+1}), \quad a_2 = \frac{1}{\sqrt{2}}(v_2 + v_{2l}), \cdots, a_l = \frac{1}{\sqrt{2}}(v_l + v_{l+2}),$$

$$a_{l+1} = v_{l+1}$$

$$a_{l+2} = \frac{1}{\sqrt{2}}(v_1 - v_{2l+1}), \quad a_{l+3} = \frac{1}{\sqrt{2}}(v_2 - v_{2l}), \cdots, a_{2l+1} = \frac{1}{\sqrt{2}}(v_l - v_{l+2}).$$

(If the sign characteristic is negative, replace a_i by a_{l+1+i}, and viceversa for $1 \le i \le l$).   ☑

Case II: *T* **has a single pair of complex conjugate eigenvalues.**

Lemma 3.2. *Let T be a self-adjoint, linear transformation on the pseudo-unitary space E with a single pair of complex conjugate eigenvalues λ and $\overline{\lambda}$. E can be decomposed into a direct sum of orthogonal invariant subspaces, $E = E_1 + \cdots + E_r$, where the restriction of T to each subspace produces a pair of Jordan blocks. Jordan bases for a subspace E_p can be chosen such that*

$$v_1 = v, \quad v_2 = (T - \lambda I)v_1, \cdots, v_k = (T - \lambda I)v_{k-1};$$

$$w_1 = w, \quad w_2 = (T - \overline{\lambda} I)w_1, \cdots, w_k = (T - \overline{\lambda} I)w_{k-1},$$

where

$$\langle v_i, w_j \rangle = \langle v_k, w_1 \rangle = 1 \quad if \quad i + j = k + 1,$$

and

$$\langle v_i, w_j \rangle = 0 \quad if \quad i + j \ne k + 1.$$

Proof. See [5] page 242.   ☑

Proposition 3.2. *Let E_p be an invariant orthogonal component with a complex eigenvalue λ as in Lemma 3.2 and let $\lambda = a + bi, b > 0$. The Jordan basis defined above can be modified to obtain an orthonormal basis of E_p such that T has the following matrix form*

$$\begin{bmatrix} A^c(a) & \widehat{A}^c(bi) \\ \widehat{A}^c(bi) & A^c(a) \end{bmatrix}.$$

Proof. Let $\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_k\}$ be a pair of Jordan bases as in Lemma 3.2. Since $\langle v_i, w_j \rangle = \langle v_k, w_1 \rangle = 1$ for $i + j = k + 1$, the following form an orthonormal basis of E_p

$$a_1 = \frac{1}{2}(v_1 + w_k), \quad a_2 = \frac{1}{2}(v_2 + w_{k-1}), \cdots$$

$$a_{k+1} = \frac{1}{2}(v_1 - w_k), \quad a_{k+2} = \frac{1}{2}(v_2 - w_{k-1}), \cdots$$

and the matrix of T is given as above. (We note that $||a_1|| = \cdots = ||a_l|| = 1$ and $||a_{l+1}|| = \cdots = ||a_{2l}|| = -1$). ☑

Finally in the most general case we obtain an orthonormal basis and construct the normal form of T as follows: First decompose E into the direct sum of pairwise orthogonal subspaces where each subspace is invariant under T and the restriction of T to a given subspace is a Jordan block. For each subspace form the orthonormal basis given by Propositions 3.1 or 3.2 accordingly, then rearrange these basis elements to construct an orthonormal basis by first listing the basis elements of norm one corresponding to the real eigenvalues followed by the basis elements of norm one corresponding to the complex eigenvalues. Then follow these by the basis elements of norm minus one corresponding first to the real and then to the complex eigenvalues. This result can be stated as

Theorem 3.3. *Let E be a pseudo-unitary space, and let T be a self-adjoint linear transformation on E with real eigenvalues $\alpha_1, \cdots, \alpha_p$ and complex eigenvalues $\lambda_1, \cdots, \lambda_s$ with $\lambda_j = a_j + b_j i$ and $b_j > 0$. Then, there is an orthonormal basis β for E such that the transformation is represented by the following matrix with respect to β:*

$$\begin{bmatrix} e^r & & h^r & \\ & a^c & & (\widehat{a})^c \\ (h^r)^* & & g^r & \\ & (\widehat{a})^c & & a^c \end{bmatrix}.$$

Blocks with the superscript c are associated with the complex eigenvalues, and blocks with the superscript r are obtained from the real eigenvalues. Furthermore, using the notation of Propositions 3.1 and 3.2 each superscripted lower case letter is a block diagonal matrix whose structure is given by

(1) $e^r = (E^r(\alpha_1), \ldots, E^r(\alpha_p))$ where:

$$E^r(\alpha_j) = \begin{cases} (A_\epsilon)^r(\alpha_j) & \text{if } k \text{ is even;} \\ B^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ A^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(2) $h^r = (H^r(\alpha_1), \ldots, H^r(\alpha_p))$ where:

$$H^r(\alpha_j) = \begin{cases} (\widehat{A_\epsilon})^r & \text{if } k \text{ is even;} \\ \widehat{C}^r & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ \widehat{K}^r & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(3) $g^r = (G^r(\alpha_1), \ldots, G^r(\alpha_p))$ where:

$$G^r(\alpha_j) = \begin{cases} (A_{-\epsilon})^r(\alpha_j) & \text{if } k \text{ is even;} \\ A^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ L^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(4) $a^c = (A^c(a_1), \ldots, A^c(a_s))$.
(5) $(\widehat{a})^c = \left(\widehat{A}^c(b_1 i), \ldots, \widehat{A}^c(b_s i) \right)$.

4. Self-adjoint linear transformations
on a pseudo-euclidean space

In this section we consider a self-adjoint, linear transformation T, on a pseudo-euclidean space V. As in the complex case above, V can be decomposed into the direct sum of invariant and pairwise orthogonal subspaces where the restriction of T to each subspace has either a single real eigenvalue or a single pair of complex, conjugate eigenvalues. This again allows us to reduce the problem to two cases and to further decompose each subspace into the direct sum of invariant and pairwise orthogonal subspaces where the restriction of T to each subspace is now either a real or a *generalized* real Jordan block. In the case of a complex eigenvalue, we first complexify the subspace and extend T to the complexified subspace. We then use the Jordan basis constructed in Section 3 to obtain a real Jordan basis for the real subspace. We next modify this basis to obtain an orthonormal basis for this subspace. Finally, to obtain the orthonormal basis in the general case, we combine the basis elements for all the subspaces. Here, as in the complex case, we list all basis elements of norm-1 after the basis elements of norm 1, and obtain our normal form.

Similar to the complex case we have

Lemma 4.1. *If V_1 and V_2 are eigenspaces of T corresponding to real eigenvalues $\alpha_1 \neq \alpha_2$, then V_1 and V_2 are orthogonal.*

The next lemma shows that the null space corresponding to a complex eigenvalue of T is orthogonal to the eigenspace corresponding to a real eigenvalue of T.

Lemma 4.2. *Let $V_1 = \{x | (T - \alpha I)^m x = 0\}$ be the eigenspace of T corresponding to the real eigenvalue α, and let λ and be a complex eigenvalue of T, and suppose that λ is a root of the polynomial: $x^2 + ax + b$. Let $V_2 = \{y | (T^2 + aT + b)^n y = 0\}$ be the null space of $(T^2 + aT + b)^n$. Then V_1 and V_2 are orthogonal.*

Proof. Choose vectors $x \in V_1$ and $y \in V_2$. Then $(T - \alpha I)^m x = 0$ and $(T^2 + aT + b)^n y = 0$. We use induction on $m + n$ to show that $\langle x, y \rangle = 0$. For $m + n = 1$ then either $x = 0$ or $y = 0$, and, hence, $\langle x, y \rangle = 0$. For $m + n > 1$, let $x_1 = (T - \alpha I)x$, $x_2 = (T - \alpha I)^2 x$ and $y_1 = (T^2 + aT + b) y$. Then, using the induction hypothesis

  (i) $\langle x_1, y \rangle = 0$; This implies that $\langle Tx, y \rangle = \alpha \langle x, y \rangle$
  (ii) $\langle x_2, y \rangle = 0$; This along with (i) implies that $\langle T^2 x, y \rangle = \alpha^2 \langle x, y \rangle$
  (iii) Finally, $\langle x, y_1 \rangle = 0$. Then using the symmetry of T along with the results of (i) and (ii) we have

$$(\alpha^2 + a\alpha + b) \langle x, y \rangle = 0.$$

For α real, $\alpha^2 + a\alpha + b \neq 0$. Thus $\langle x, y \rangle = 0$.

Complexification. To continue the rest of our construction, let $E = V + iV$ be the complexification of V. Then, E is a pseudo-unitary space with the inner product defined by

$$\langle x_1 + iy_1, x_2 + iy_2 \rangle = (\langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle) + (\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle) i.$$

The transformation T extends to a self-adjoint, linear transformation, which we again denote by T, via

$$T(x + iy) = Tx + iTy.$$

Note that considering T either as a linear transformation on V or as a linear transformation on the complexification E, it has the same eigenvalues. Next we prove that null spaces corresponding to distinct non-conjugate complex eigenvalues of T are orthogonal.

Lemma 4.3. *Let $\lambda = a + ib$ and $\gamma = c + id$ be distinct, and non-conjugate, complex eigenvalues of T. Let*

$$V_1 = \left\{ x \mid \left[(T - a)^2 + b^2 \right]^m x = 0 \right\},$$

and

$$V_2 = \left\{ x' \mid \left[(T - c)^2 + d^2 \right]^n x' = 0 \right\},$$

be subspaces of V. Then V_1 and V_2 are orthogonal.

Proof. Let $E_1 \subseteq E$ be the sum of the eigenspaces corresponding to λ and $\overline{\lambda}$. Similarly, let $E_2 \subseteq E$ be the sum of the eigenspaces corresponding to γ and $\overline{\gamma}$. We have $v = x + iy \in E_1$ if and only if $x, y \in V_1$ and $w = x' + iy' \in E_2$ if and only if $x', y' \in V_2$. Since $\overline{v} \in E_1$ as well, we have $\langle v, w \rangle = 0$ and $\langle \overline{v}, w \rangle = 0$. This implies that $\langle x, y \rangle = 0, \langle x', y \rangle = 0, \langle x, y' \rangle = 0$, and $\langle x', y' \rangle = 0$, and thus proves the lemma. ☑

The nondegeneracy of the inner product implies that complex eigenvalues of T come in conjugate pairs, and therefore, we can state the following:

Proposition 4.1. *Let T be a self-adjoint, linear transformation on a pseudo-euclidean space V. Then V can be decomposed into the direct sum of pairwise orthogonal, invariant subspaces, $V = V_1 + V_2 + \cdots + V_s$, where the restriction of T to each V_i has either a single real eigenvalue α or a pair of complex conjugate eigenvalues λ and $\overline{\lambda}$.*

Thus, we reduce our construction of an orthonormal basis to two cases.

Case I. T has a single real eigenvalue.

Similar to the case of a pseudo-unitary space we construct a special Jordan basis as follows:

Lemma 4.4. *Let T be a self-adjoint, linear transformation on a pseudo-euclidean space V with a single real eigenvalue α. Then $V = V_1 + V_2 + \cdots + V_r$ is the direct sum of orthogonal, invariant subspaces such that the restriction of*

T to each subspace produces a Jordan block. The Jordan basis, $\{x_1, \cdots, x_k\}$, for a subspace V_p, is chosen such that $\langle x_i, x_j \rangle = \langle x_1, x_k \rangle = \epsilon$ if $i + j = k + 1$, where again $\epsilon = \pm 1$, is called the sign characteristic *of the inner product on E_p and $\langle x_i, x_j \rangle = 0$ otherwise.*

Proof. Choose k such that $(T - \alpha I)^{k-1} \neq 0$ and $(T - \alpha I)^k = 0$. Define $B(x, y) = \langle (T - \alpha I)^{k-1} x, y \rangle$. Since T is self-adjoint, (and the inner product is non-degenerative) B is a symmetric bilinear form on V which is not identically zero. Thus there is an $x \in V$ such that $B(x, x) = r \neq 0$. By replacing x with $\frac{x}{\sqrt{|r|}}$ if necessary we can assume that $B(x, x) = \epsilon$. Let $x_1 = x, x_p = (T - \alpha I) x_{p-1}$ for $2 \le p \le k$. Then $\langle x_i, x_j \rangle = 0$ if $i + j > k + 1$ and $\langle x_i, x_j \rangle = \langle x_1, x_k \rangle = B(x, x) = \pm 1$ if $i + j = k + 1$. Replace x_1 with a suitable linear combination of $x_1, x_2, \cdots, x_k$ (as is necessary) to have $\langle x_i, x_j \rangle = 0$ if $i + j \neq k + 1$ and $\langle x_i, x_j \rangle = \langle x_1, x_k \rangle = \epsilon$ for $i + j = k + 1$. Let $V_1 = \text{span}\{x_1, \cdots, x_k\}$. V_1 is invariant under T, and $\{x_1, \cdots, x_k\}$ is a Jordan basis. The restriction of T to this subspace is a Jordan block. Decompose $V = V_1 + V_1^\perp$ and repeat the same construction on $V_1^\perp$.   ☑

Proposition 4.2. *Let V_p be an invariant subspace of dimension k with a single real eigenvalue α as in Lemma 4.4. Then, there is an orthonormal basis β of V_p such that the matrix of T with respect to β is:*

a) *for $k = 2l$ even*

$$\begin{bmatrix} (A_\epsilon)^r(\alpha) & (_\epsilon \widehat{D})^r \\ (_\epsilon \widehat{D}^r)^* & (-^\epsilon A^r)(\alpha) \end{bmatrix},$$

(Each square block is of dimension l.) where again $\epsilon = \pm 1$.

b) *for $k = 2l + 1$ odd and the sign characteristic is positive*

$$\begin{bmatrix} B^r(\alpha) & \widehat{N}^r \\ (\widehat{N}^r)^* & A^r(\alpha) \end{bmatrix}.$$

($A^r(\alpha)$ and $B^r(\alpha)$ are square matrices of dimensions l and $l + 1$ respectively).

c) *for $k = 2l + 1$ odd and the sign characteristic is negative*

$$\begin{bmatrix} A^r(\alpha) & \widehat{E}^r \\ (\widehat{E}^r)^* & M^r(\alpha) \end{bmatrix}.$$

($A^r(\alpha)$ and $M^r(\alpha)$ are square matrices of dimension l and $l + 1$ respectively).

Proof.

(1) If $k = 2l$ is even, and the sign characteristic is positive, then let

$$a_1 = \frac{1}{\sqrt{2}}(x_1 + x_{2l}), \quad a_2 = \frac{1}{\sqrt{2}}(x_2 + x_{2l-1}), \cdots, \quad a_l = \frac{1}{\sqrt{2}}(x_l + x_{l+1}),$$

$$a_{2l} = \frac{1}{\sqrt{2}}(x_1 - x_{2l}), \quad a_{2l-1} = \frac{1}{\sqrt{2}}(x_2 - x_{k-1}), \cdots, a_{l+1} = \frac{1}{\sqrt{2}}(x_l - x_{l+1}).$$

The set $\{a_i\}, 1 \leq i \leq 2l$, form the orthonormal basis. If the sign characteristic is negative then replace a_i by a_{2l+1-i} and viceversa for $1 \leq i \leq l$.

(2) If $k = 2l + 1$ is odd and the sign characteristic is positive, let

$$a_1 = \frac{1}{\sqrt{2}}(x_1 + x_{2l+1}), \quad a_2 = \frac{1}{\sqrt{2}}(x_2 + x_{2l}), \cdots, a_l = \frac{1}{\sqrt{2}}(x_l + x_{l+2}),$$

$$a_{l+1} = x_{l+1},$$

$$a_{2l+1} = \frac{1}{\sqrt{2}}(x_1 - x_{2l+1}), \quad a_{2l} = \frac{1}{\sqrt{2}}(x_2 - x_{2l}), \cdots, a_{l+2} = \frac{1}{\sqrt{2}}(x_l - x_{l+2}).$$

If the sign characteristic is negative then replace a_i by a_{2l+2-i} and viceversa for $1 \leq i \leq l$.

☑

Case II. T has a single pair of complex conjugate eigenvalues.

The following derivation of the *real Jordan form*, (see [7]), will help us to construct the orthonormal basis in Proposition 4.3.

Lemma 4.5. *Let T be a self-adjoint, linear transformation on a pseudo-euclidean space V with $\lambda = a + bi$ and $\overline{\lambda} = a - bi$ the only eigenvalues. Then, $V = V_1 + \cdots + V_r$ is the direct sum of orthogonal, invariant subspaces, and the matrix of T when restricted to each subspace produces a "generalized real Jordan block" of the form*

$$\begin{bmatrix} a & b & & & & & & & \\ -b & a & & & & & & & \\ 1 & 0 & a & b & & & & & \\ 0 & 1 & -b & a & & & & & \\ & & 1 & 0 & a & b & & & \\ & & 0 & 1 & -b & a & & & \\ & & & & \ddots & & \ddots & & \\ & & & & & & 1 & 0 & a & b \\ & & & & & & 0 & 1 & -b & a \end{bmatrix}$$

Proof. Let E be the complexification of V. Choose k such that $(T - \lambda I)^k = 0$ and $(T - \lambda I)^{k-1} \neq 0$. Consider $B(v, w) = \langle (T - \lambda I)^{k-1}v, \overline{w} \rangle$. Since T is self-adjoint, B is Hermitian, and because the inner product is non-degenerate it is not identically zero, and so we can choose v' such that $B\left(v', v'\right) = \zeta \neq 0$. Let

$v = \frac{v'}{\sqrt{\zeta}}$. Then

$$B(v,v) = B\left(\frac{v'}{\sqrt{\zeta}}, \frac{v'}{\sqrt{\zeta}}\right) = \left\langle (A - \lambda I)^{k-1} \frac{v'}{\sqrt{\zeta}}, \frac{\overline{v'}}{\sqrt{\overline{\zeta}}} \right\rangle$$

$$= \frac{1}{\zeta}\left\langle (A - \lambda I)^{k-1} v', \overline{v'} \right\rangle = \frac{\zeta}{\zeta} = 1.$$

Thus, in the proof of Lemma 3.2 we can choose $w = \overline{v}$ and decompose E into the direct sum of orthogonal, invariant subspaces, i.e., $E = E_1 + E_2 + \cdots + E_r$, where the restriction of T to each subspace produces a pair of Jordan blocks. For example, the Jordan basis chosen for E_p is

$$v_1 = v, \quad v_i = (T - \lambda I)v_{i-1},$$

$$w_1 = \overline{v}, \quad w_i = (T - \overline{\lambda})w_{i-1} = \overline{v}_i, \quad 2 \le i \le k,$$

and

$$\langle v_i, v_j \rangle = \langle \overline{v}_i, \overline{v}_j \rangle = 0 \quad \text{for all } i, j,$$

$$\langle v_i, \overline{v}_j \rangle = 1 \quad \text{for } i + j = k + 1,$$

$$\langle v_i, \overline{v}_j \rangle = 0 \quad \text{for } i + j \ne k + 1.$$

Now let $v_j^r = \frac{1}{2}(v_j + \overline{v}_j)$ and $v_j^c = \frac{1}{2i}(v_j - \overline{v}_j)$. Then

$$\beta = \{v_1^r, v_1^c, v_2^r, v_2^c, \cdots, v_k^r, v_k^c\}$$

form a real Jordan basis for V_1, a subspace of V, and the matrix of T with respect to β is as claimed. ☑

Proposition 4.3. *Let V_p be an invariant subspace with a single pair of complex eigenvalues, $\lambda = a + bi$ and $\overline{\lambda} = a - bi$ with $b > 0$ as in Lemma 4.5, and assume it is of dimension $2k$. Then there is an orthonormal basis β of V_p such that the matrix of T with respect to β is*

a) *for $k = 2l$ even*

$$\begin{bmatrix} W^c(a) & Y^c(b) \\ (Y^c(b))^* & W^c(a) \end{bmatrix},$$

where

$$W^c(a) = \begin{bmatrix} (A_1)^c(a) & \\ & (A_{-1})^c(a) \end{bmatrix} \quad Y^c(b) = \begin{bmatrix} bI_l & \left(-\widehat{A}_1\right)^c \\ \left(\left(-\widehat{A}_1\right)^c\right)^* & -bI_l \end{bmatrix}.$$

b) *for $k = 2l + 1$ odd*

$$\begin{bmatrix} P^c(a) & Q^c(b) \\ (Q^c(b))^* & P^c(a) \end{bmatrix},$$

where

$$P^c(a) = \begin{bmatrix} B^c(a) & \\ & A^c(a) \end{bmatrix} \quad Q^c(b) = \begin{bmatrix} bI_{l+1} & -\widehat{C}^c \\ \left(-\widehat{C}^c\right)^* & -bI_l \end{bmatrix}.$$

Proof. Notice that $\langle v_i, \overline{v}_j \rangle = 1$ for $i + j = k + 1$. This implies that $\langle v_i^r, v_j^r \rangle - \langle v_i^c, v_j^c \rangle = 1$. Similarly, since $\langle v_i, v_j \rangle = 0$, we get $\langle v_i^r, v_j^r \rangle + \langle v_i^c, v_j^c \rangle = 0$. Thus

$$\langle v_i^r, v_j^r \rangle = \frac{1}{2} \quad \text{and} \quad \langle v_i^c, v_j^c \rangle = -\frac{1}{2} \quad \text{for} \quad i + j = k + 1.$$

Again, from $\langle v_i, v_j \rangle = 0$, we have

$$\langle v_i^c, v_j^r \rangle = \langle v_i^r, v_j^c \rangle = 0 \quad \text{for all} \quad i, j,$$

and

$$\langle v_i^r, v_j^r \rangle = \langle v_i^c, v_j^c \rangle = 0 \quad \text{if} \quad i + j \neq k + 1.$$

We construct an orthonormal basis, β, as follows:

Case A: $k = 2l$ is even, i.e., $\dim(V) = 4l$. Let

$$x_1 = v_1^r + v_{2l}^r, \quad x_2 = v_2^r + v_{2l-1}^r, \cdots, x_l = v_l^r + v_{l+1}^r,$$
$$x_{l+1} = -v_1^c + v_{2l}^c, \quad x_{l+2} = -v_2^c + v_{2l-1}^c, \cdots, x_{2l} = -v_l^c + v_{l+1}^c,$$
$$x_{2l+1} = v_1^c + v_{2l}^c, \quad x_{2l+2} = v_2^c + v_{2l-1}^c, \cdots, x_{3l} = v_l^c + v_{l+1}^c,$$
$$x_{3l+1} = -v_1^r + v_{2l}^r, \quad x_{3l+2} = -v_2^r + v_{2l-1}^r, \cdots, x_{4l} = -v_l^r + v_{l+1}^r.$$

Case B: $k = 2l + 1$ odd, i.e., $dim(V) = 4l + 2$. Then the orthonormal basis is

$$x_1 = v_1^r + v_{2l+1}^r, \quad x_2 = v_2^r + v_{2l}^r, \cdots, x_l = v_l^r + v_{l+2}^r,$$
$$x_{l+1} = \sqrt{2} v_{l+1}^r,$$
$$x_{l+2} = -v_1^c + v_{2l+1}^c, \cdots, x_{2l+1} = -v_l^c + v_{l+2}^c,$$
$$x_{2l+2} = v_1^c + v_{2l+1}^c, \cdots, x_{3l+1} = v_l^c + v_{l+2}^c,$$
$$x_{3l+2} = \sqrt{2} v_{l+1}^c,$$
$$x_{3l+3} = -v_1^r + v_{2l+1}^r, \cdots, x_{4l+2} = -v_l^r + v_{l+2}^r.$$

☑

Finally, when T has several eigenvalues, as in the complex case above, we decompose V into the direct sum of pairwise, orthogonal subspaces such that each subspace is invariant under the transformation T and the restriction of T to each subspace is given by a Jordan block. We use Propositions 4.2 and 4.3 accordingly to construct the orthonormal basis on each subspace and form an orthonormal basis for V by listing all the basis elements of norm -1 after listing the basis elements of norm $+1$.

Theorem 4.4. *Let V be a pseudo-euclidean space, and let T be a self-adjoint linear transformation on V with real eigenvalues $\alpha_1, \cdots, \alpha_p$ and complex eigenvalues $\lambda_1, \cdots, \lambda_s$ with $\lambda_j = a_j + b_j i$ and $b_j > 0$. Then, there is an orthonormal basis β for E such that the transformation is represented by the following matrix with respect to β:*

$$\begin{bmatrix} e^r & & h^r & \\ & a^c & & (\widehat{a})^c \\ (h^r)^* & & g^r & \\ & (\widehat{a})^c & & a^c \end{bmatrix}.$$

Blocks with the superscript c are associated with the complex eigenvalues, and blocks with the superscript r are obtained from the real eigenvalues.

Furthermore, using the notation of Propositions 4.2 and 4.3 each superscripted lower case letter is a block diagonal matrix whose structure is given by

(1) $e^r = (E^r(\alpha_1), \cdots, E^r(\alpha_p))$ *where:*

$$E^r(\alpha_j) = \begin{cases} (A_\epsilon)^r(\alpha_j) & \text{if } k \text{ is even;} \\ B^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ A^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(2) $h^r = (H^r(\alpha_1), \cdots, H^r(\alpha_p))$ *where:*

$$H^r(\alpha_j) = \begin{cases} \left(_\epsilon \widehat{D}\right)^r & \text{if } k \text{ is even;} \\ \widehat{N}^r & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ \widehat{E}^r & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(3) $g^r = (G^r(\alpha_1), \cdots, G^r(\alpha_p))$ *where:*

$$G^r(\alpha_j) = \begin{cases} {}^{-\epsilon}(A^r)(\alpha_j) & \text{if } k \text{ is even;} \\ A^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is positive;} \\ M^r(\alpha_j) & \text{if } k \text{ is odd and the sign characteristic is negative.} \end{cases}$$

(4) $a^c = (U^c(a_1), \cdots, U^c(a_s))$ *where:*

$$U^c(a_j) = \begin{cases} W^c(a_j) & \text{if } k \text{ is even;} \\ P^c(a_j) & \text{if } k \text{ is odd.} \end{cases}$$

(5) $(\widehat{a})^c = (V^c(b_1), \ldots, V^c(b_s))$ *where:*

$$V^c(b_j) = \begin{cases} Y^c(b_j) & \text{if } k \text{ is even;} \\ Q^c(b_j) & \text{if } k \text{ is odd.} \end{cases}$$

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