

# The analytic fixed point function II

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**ABSTRACT.** Let  $\varphi$  be analytic in the unit disk  $\mathbb{D}$  and let  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi(0) \neq 0$ . Then  $w = z/\varphi(z)$  has an analytic inverse  $z = f(w)$  for  $w \in \mathbb{D}$ , the fixed point function. This paper studies the case that  $\varphi(1) = \varphi'(1) = 1$  with a growth condition for  $\varphi''(x)$  and determines the asymptotic behaviour of various combinations of the coefficients of  $\varphi$  connected with  $f$ . The results can be interpreted in various contexts of probability theory.

*Keywords and phrases.* Fixed point function, coefficients, Bürmann-Lagrange, asymptotics, equilibrium, first return, branching process.

*2000 Mathematics Subject Classification.* Primary: 30B10. Secondary: 60F99, 60J80.

**RESUMEN.** Sea  $\varphi$  analítica en el disco unitario  $\mathbb{D}$  y  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi(0) \neq 0$ . Entonces  $w = z/\varphi(z)$  tiene una inversa analítica  $z = f(w)$  para  $w \in \mathbb{D}$ , la función de punto fijo. Este artículo estudia el caso en que  $\varphi(1) = \varphi'(1) = 1$  con una condición de crecimiento para  $\varphi''(x)$  y determina el comportamiento asintótico de varias combinaciones de los coeficientes de  $\varphi$  conectados con  $f$ . Los resultados se pueden interpretar en varios contextos de la teoría de la probabilidad.

## 1. Introduction

Let the function  $\varphi$  be analytic in the unit disk  $\mathbb{D}$  and  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi(0) \neq 0$ . In [MePo05, Sec. 3] it was shown that there is a unique function  $f$  that maps  $\mathbb{D}$  conformally onto a starlike domain  $F$  in  $\mathbb{D}$  and satisfies  $f(0) = 0$ ,

$$w \varphi(f(w)) = f(w) \quad \text{for } w \in \mathbb{D}. \quad (1.1)$$

\*Supported by COLCIENCIAS.

†Supported by Deutsche Forschungsgemeinschaft (DFG).

Thus  $z = f(w)$  is the inverse function of  $w = z/\varphi(z)$ . We call  $f$  the *fixed point function* of  $\varphi$  because  $f(w)$  is the unique fixed point of  $w\varphi$  in  $\mathbb{D}$ .

The fixed point function  $f$  has a continuous and injective extension to  $\overline{\mathbb{D}}$ , see [MePo05, Th. 3.2]. Furthermore [MePo05, Th. 2.2] we have

$$\partial\mathbb{D} \cap \partial F = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1, |\varphi'(\zeta)| \leq 1\} \quad (1.2)$$

where  $\varphi(\zeta)$  and  $\varphi'(\zeta)$  are angular limits [Po92, Sect. 4.3]. It follows from (1.1) by differentiation that

$$w \frac{f'(w)}{f(w)} = \frac{1}{1 - w\varphi'(f(w))} = \frac{1}{1 - z\varphi'(z)/\varphi(z)} \quad (1.3)$$

for  $z = f(w)$ ,  $w \in \mathbb{D}$ .

We shall restrict ourselves to the case that  $\varphi(1) = 1$  and  $\varphi'(1) \leq 1$ ; since  $\varphi(\mathbb{D}) \subset \mathbb{D}$  the Julia-Wolff lemma [Po92, Prop. 4.13] shows that  $\varphi(1) = 1$  implies that the angular derivative  $\varphi'(1)$  is positive real or infinite. The case  $\varphi'(1) < 1$  will be considered only in the last section.

In Section 4 we study the condition

$$\varphi(x) = x + b(1-x)^\beta + o((1-x)^\beta) \quad \text{as } x \rightarrow 1- \quad (1.4)$$

where  $1 < \beta \leq 2$  and  $0 < b < \infty$ . Then  $\varphi''(1)$  is finite if and only if  $\beta = 2$ . Our main result is Theorem 4.3 about coefficients.

The results about the coefficients can be interpreted as results about probabilities. Let  $X$  denote a random variable with values in  $\mathbb{N}_0$  and the distribution  $a_k = \mathbb{P}(X = k)$  for  $k = 0, 1, \dots$ . Then

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k \quad (z \in \mathbb{D}) \quad (1.5)$$

is the generating function of  $X$  and satisfies  $\varphi(1) = 1$  and  $\varphi(\mathbb{D}) \subset \mathbb{D}$ . We assume that  $\varphi(0) = \mathbb{P}(X = 0) > 0$ .

Let  $S_n$  be the sum of  $n$  independent random variables all distributed like  $X$ . The Bürmann-Lagrange formula (Theorem 2.1) shows that the fixed point function  $f$  has a special affinity to probabilities of the form  $\mathbb{P}(S_n = n - k)$ .

The study of  $S_n$  is a classical chapter of probability theory, see e.g. the book of V.V. Petrov [Pe75]. Most of our results on probability are known, at least, in the case  $\beta = 2$  of finite variance.

## 2. The Bürmann-Lagrange formula

Let  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  be analytic with  $\varphi(0) \neq 0$  and let  $z = f(w)$  be the inverse function of  $w = z/\varphi(z)$ . We define  $a_{n,k}$  for  $n \in \mathbb{Z}$  and  $k \in \mathbb{N}_0$  by

$$\varphi(z)^n = \sum_{k=0}^{\infty} a_{n,k} z^k. \quad (2.1)$$

Now we present the Bürmann-Lagrange formula [PoSz25, p. 125] in a somewhat different form and also for functions  $\psi$  with a pole at 0.

The formulas still hold near  $w = 0$  if we only assume that  $\varphi$  is analytic near  $z = 0$  and  $\varphi(0) \neq 0$ .

**Theorem 2.1.** *Let  $m \geq 0$ ,  $0 < \rho \leq 1$  and*

$$\psi(z) = \sum_{k=-m}^{\infty} b_k z^k \text{ for } 0 < |z| < \rho. \quad (2.2)$$

*If  $0 < |w| < \rho$  then*

$$w f'(w) \psi(f(w)) = \sum_{n=-m+1}^{\infty} \left( \sum_{k=-m+1}^n b_{k-1} a_{n,n-k} \right) w^n, \quad (2.3)$$

$$\psi(f(w)) = b_0 - \sum_{k=1}^m b_{-k} a_k^* + \sum_{n=-m}^{\infty} \left( \sum_{k=-m}^n \frac{k}{n} b_k a_{n,n-k} \right) w^n \quad (2.4)$$

where  $n = 0$  is omitted in the last outer sum and where  $z \varphi'(z)/\varphi(z) = \sum a_k^* z^k$ .

*Proof.* Since  $|f(w)| \leq |w|$  by the Schwarz lemma and since  $f$  is univalent in  $\mathbb{D}$ , we have  $0 < |f(w)| < \rho$  for  $0 < |w| < \rho$  so that  $\psi \circ f$  is analytic in  $\{0 < |w| < \rho\}$ .

Let  $0 < r < \rho$  and  $C = \{|w| = r\}$ . Let  $n \in \mathbb{Z}$ . The coefficient of  $w^n$  of the function  $w f'(w) \psi(f(w))$  is

$$\frac{1}{2\pi i} \int_C \frac{\psi(f(w))}{w^n} f'(w) dw = \frac{1}{2\pi i} \int_{f(C)} \frac{\psi(z) \varphi(z)^n}{z^n} dz,$$

where we have substituted  $w = z/\varphi(z)$  with  $z = f(w)$ . This is the coefficient of  $z^{n-1}$  of the function

$$\psi(z) \varphi(z)^n = \sum_{k=-m+1}^{\infty} b_{k-1} z^{k-1} \sum_{j=0}^{\infty} a_{n,j} z^j$$

which is equal to the inner sum in (2.3).

Next we apply (2.3) to  $\psi'$ . We obtain

$$\frac{d}{dw} \psi(f(w)) = \sum_{n=-m}^{\infty} \left( \sum_{k=-m+1}^n k b_k a_{n,n-k} \right) w^{n-1}.$$

Integrating we obtain (2.4) except for a constant. The coefficient of  $w^0$  is

$$\frac{1}{2\pi i} \int_C \frac{\psi(f(w))}{w} dw = \frac{1}{2\pi i} \int_{f(C)} \frac{\psi(z)}{z} \left( 1 - z \frac{\varphi'(z)}{\varphi(z)} \right) dz$$

because of (1.3), which gives the value in (2.4). □

In particular we obtain

$$w f'(w) f(w)^{k-1} = \sum_{n=k}^{\infty} a_{n,n-k} w^n \quad \text{for } k \in \mathbb{Z}, \quad (2.5)$$

$$f(w)^k = \sum_{n=k}^{\infty} \frac{k}{n} a_{n,n-k} w^n \quad \text{for } k \in \mathbb{N}. \quad (2.6)$$

### 3. Some auxiliary estimates

A Stolz angle at 1 is an open triangle  $\Delta$  symmetric to  $\mathbb{R}$  that satisfies  $\overline{\Delta} \cap \partial\mathbb{D} = \{1\}$ . We say that a function has an angular limit at 1 if this limit exists for  $z \rightarrow 1$  in every Stolz angle  $\Delta$ .

**Proposition 3.1.** *Let  $g$  be analytic in  $\mathbb{D}$  and*

$$g(z) \sim b(1-z)^\beta \text{ as } z \rightarrow 1 \text{ angularly.} \quad (3.1)$$

where  $b \neq 0$  and  $\beta \neq 0$ . Then

$$g'(z) \sim -\beta b(1-z)^{\beta-1} \text{ as } z \rightarrow 1 \text{ angularly.} \quad (3.2)$$

*Proof.* By (3.1) the function  $(1-z)^{-\beta} g(z)$  has the angular limit  $b \neq \infty$  at 1. It follows [Po92, Prop. 4.8] that

$$(1-z)^{-\beta+1} g'(z) + \beta(1-z)^{-\beta} g(z) = (1-z) \frac{d}{dz} [(1-z)^{-\beta} g(z)]$$

has the angular limit 0 at 1. Hence (3.2) follows from (3.1).  $\square$

**Proposition 3.2.** *Let  $g$  be analytic in  $\mathbb{D}$  and*

$$(1-x)^\alpha g(x) \rightarrow 0 \text{ as } x \rightarrow 1-, \quad (3.3)$$

$$|1-z|^\alpha |g(z)| \leq c < \infty \text{ for } z \in \mathbb{D} \quad (3.4)$$

where  $1 < \alpha < \infty$ . Then

$$\int_{-\pi}^{\pi} |g(re^{it})| dt = o((1-r)^{1-\alpha}) \text{ as } r \rightarrow 1-. \quad (3.5)$$

*Proof.* We establish (3.5) for  $0 \leq t \leq \pi$ . The analytic function  $(1-z)^\alpha g(z)$  is bounded because of (3.4) and therefore has the angular limit 0 at 1 because of (3.3), see [Po92, Th. 4.3].

Given  $\varepsilon \in (0, 1)$  there exists  $r_0 \in (\frac{1}{2}, 1)$  such that

$$|1 - re^{it}|^\alpha |g(re^{it})| < \varepsilon \text{ for } r_0 < r < 1, |t| \leq \delta = (1-r)/\varepsilon.$$

For  $r_0 < r < 1$  we therefore have

$$\int_0^\delta |g(re^{it})| dt < \varepsilon \int_0^\delta \frac{|1 - re^{it}|^{2-\alpha}}{|1 - re^{it}|^2} dt. \quad (3.6)$$

If  $1 < \alpha \leq 2$  this is

$$\leq \varepsilon(1 + \varepsilon^{-2})^{(2-\alpha)/2} (1-r)^{2-\alpha} \int_0^\delta \frac{dt}{|1 - re^{it}|^2} < 4\pi\varepsilon^{\alpha-1}(1-r)^{1-\alpha}.$$

If  $2 \leq \alpha < \infty$  the last expression in (3.6) is

$$\leq \varepsilon(1-r)^{(2-\alpha)} \int_0^\delta \frac{dt}{|1 - re^{it}|^2} \leq 2\pi\varepsilon(1-r)^{1-\alpha}.$$

Since  $|1 - re^{it}| \geq 2rt/\pi$  we obtain from (3.4) that

$$\int_\delta^\pi |g(re^{it})| dt \leq \int_\delta^\infty \frac{c\pi^\alpha}{t^\alpha} dt = \frac{c\pi^\alpha \varepsilon^{\alpha-1}}{\alpha-1} (1-r)^{1-\alpha}$$

because  $\delta = (1-r)/\varepsilon$ . These estimates prove (3.5). ✓

It is well known that, for  $\alpha > 0$ ,

$$(-1)^n \binom{-\alpha}{n} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad (n \rightarrow \infty). \quad (3.7)$$

The following theorem is the key to the later results.

**Theorem 3.3.** *Let  $1 < \alpha < \infty$  and let*

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad (3.8)$$

*be analytic in  $\mathbb{D}$ . We suppose that*

$$(1-x)^{\alpha-1} h(x) \rightarrow a \in \mathbb{C} \text{ as } x \rightarrow 1, \quad (3.9)$$

$$\sup_{z \in \mathbb{D}} |1-z|^\alpha |h'(z)| < \infty. \quad (3.10)$$

*Then*

$$c_n \sim \frac{a}{\Gamma(\alpha-1)} n^{\alpha-2} \text{ as } n \rightarrow \infty. \quad (3.11)$$

*Proof.* It follows from (3.9) and (3.10) that

$$|h(x)| \leq \frac{c_0}{(1-x)^{\alpha-1}} \quad (0 \leq x \leq 1), \quad |h'(\zeta)| \leq \frac{c_1}{|1-\zeta|^\alpha} \quad (\zeta \in \mathbb{D}).$$

Let  $z \in \mathbb{D}$  and  $|1-z| < 1$ ; the case  $|1-z| \geq 1$  is simpler. Let  $C$  be the circular arc  $\{\zeta \in \mathbb{D} : |1-\zeta| = |1-z|\}$  and let  $x \in (0, 1)$  be the point where  $C$  intersects  $\mathbb{R}$ . Integrating over  $C$  we obtain

$$|h(z) - h(x)| \leq \int_x^z |h'(\zeta)| |d\zeta| \leq \frac{\pi}{2} |1-z| \frac{c_1}{|1-z|^\alpha},$$

and since  $1 - x = |1 - z|$  we conclude that

$$|1 - z|^{\alpha-1} |h(z)| \leq c_2 \text{ for } z \in \mathbb{D}. \quad (3.12)$$

It follows by (3.9) that  $(1 - z)^{\alpha-1} h(z)$  has the angular limit  $a$ ; see e.g. [Po92, Th. 4.3]. Therefore we conclude from Proposition 3.1 that  $(1 - x)^\alpha h'(x) \rightarrow (\alpha - 1)a$  as  $x \rightarrow 1 -$ . Hence we can apply Proposition 3.2 to the function

$$g(z) = z h'(z) - \frac{(\alpha - 1)a}{(1 - z)^\alpha} = \sum_{n=0}^{\infty} \left( n c_n - (\alpha - 1)a \binom{-\alpha}{n} (-1)^n \right) z^n; \quad (3.13)$$

the condition (3.4) is satisfied due to (3.10). We conclude from (3.5) with  $r = 1 - n^{-1}$  that the coefficients of  $g$  are  $o(n^{\alpha-1})$  so that (3.11) follows from (3.13) and (3.7).  $\square$

#### 4. A fractional derivative condition

In this section we consider the following condition and its consequences.

(A) The function  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is analytic and satisfies  $\varphi(0) \neq 0$  and

$$\varphi(x) - x \sim b(1 - x)^\beta \text{ as } x \rightarrow 1 - \quad (4.1)$$

where  $0 < b < \infty$  and  $1 < \beta \leq 2$ . Note that we only require radial and not unrestricted approach to  $z = 1$ .

**Proposition 4.1.** *If condition (A) holds then  $\varphi(1) = \varphi'(1) = 1$  as angular limits and, as  $z \rightarrow 1$  angularly,*

$$\varphi(z) - z \sim b(1 - z)^\beta, \quad (4.2)$$

$$1 - \varphi'(z) \sim \beta b(1 - z)^{\beta-1}, \quad (4.3)$$

$$\varphi''(z) \sim \beta(\beta - 1)b(1 - z)^{\beta-2}. \quad (4.4)$$

*Proof.* We see from (4.1) that  $(1 - \varphi(x))/(1 - x) \rightarrow 1$ . Hence  $\varphi$  has the angular derivative 1 at 1 so that  $\varphi'(1) = 1$  [Po92, Prop. 4.7] and it follows from the Julia-Wolff lemma [Po92, Th. 4.13] that

$$\frac{1 + \varphi(z)}{1 - \varphi(z)} = \frac{1 + z}{1 - z} + p(z) \quad (z \in \mathbb{D})$$

where  $\operatorname{Re} p(z) > 0$  and thus  $|\arg p(z)| < \frac{\pi}{2}$ . Hence

$$h(z) = \log \frac{\varphi(z) - z}{(1 - z)^\beta} = \log \frac{p(z)(1 - \varphi(z))}{2(1 - z)^{\beta-1}}$$

satisfies  $|\operatorname{Im} h(z)| < (\beta + 2)\pi/2$  and is therefore a Bloch function [Po92, Sect. 4.2]. Since  $h(x) \rightarrow \log b$  as  $x \rightarrow 1$  by (4.1), it follows that  $h$  has the angular limit at 1. This is the assertion (4.2), and we obtain (4.3) and (4.4) by applying Proposition 3.1 twice.  $\square$

Let  $f$  be again the fixed point function of  $\varphi$ , see (1.1).

**Theorem 4.2.** *Under the assumption (A) the domain  $F = f(\mathbb{D})$  has tangents of angles  $\pm \frac{\pi}{2\beta}$  at 1 and*

$$1 - f(w) \sim b^{-1/\beta}(1 - w)^{1/\beta}, \quad (4.5)$$

$$f'(w) \sim (\beta b)^{-1}(1 - f(w))^{1-\beta} \sim \beta^{-1} b^{-\frac{1}{\beta}}(1 - w)^{\frac{1}{\beta}-1}, \quad (4.6)$$

$$f''(w) \sim (\beta - 1) \beta^{-2} b^{-\frac{1}{\beta}}(1 - w)^{\frac{1}{\beta}-2} \quad (4.7)$$

as  $w \rightarrow 1$ ,  $w \in \mathbb{D}$ , thus for unrestricted approach.

*Proof.* (a) Let  $\Delta$  be a Stolz angle in 1 of opening  $\alpha > \pi/\beta$  and let  $\varepsilon > 0$ . If  $z = 1 - \rho e^{i\vartheta}$  with  $|\vartheta| < \frac{\pi}{2}$  then, by (4.2),

$$\begin{aligned} |\varphi(z)|^2 &= |z + b(1 - z)^\beta + o(\rho^\beta)|^2 \\ &= |z|^2 + 2b \operatorname{Re} [(1 - z)^\beta] + o(\rho^\beta) \end{aligned}$$

as  $\rho \rightarrow 0$  and thus

$$|\varphi(z)|^2 - |z|^2 = \rho^\beta (2b \cos(\beta\vartheta) + o(1)).$$

This is positive for  $\beta|\vartheta| < \frac{\pi}{2} - \varepsilon$  and negative for  $\beta|\vartheta| > \frac{\pi}{2} + \varepsilon$  for small  $\rho$ . Hence the domain  $F = \{z \in \mathbb{D} : |\varphi(z)| > |z|\}$  has tangents of angles  $\pm\pi/(2\beta)$  at 1. In particular,  $F$  lies within some Stolz angle near 1.

(b) We obtain from Proposition 4.1 that  $1 - z\varphi'(z)/\varphi(z) \sim \beta b(1 - z)^{\beta-1}$  as  $z \rightarrow 1$  angularly. Since  $f(\mathbb{D})$  lies in a Stolz angle by part (a), we conclude from (1.3) with  $z = f(w)$  that

$$f'(w) = \frac{1 + o(1)}{\beta b} (1 - f(w))^{1-\beta} \quad (4.8)$$

as  $w \rightarrow 1$ ,  $w \in \mathbb{D}$  and therefore

$$\begin{aligned} (1 - f(w))^\beta &= \beta \int_w^1 (1 - f(\omega))^{\beta-1} \frac{1 + o(1)}{\beta b} (1 - f(\omega))^{1-\beta} d\omega \\ &= (b^{-1} + o(1))(1 - w). \end{aligned}$$

Hence (4.5) holds, and (4.6) follows from (4.8).

By a short calculation we obtain from (1.3) that

$$f''(w) = \frac{w^2 f'(w)^3}{f(w)} \varphi''(f(w)) + 2 \frac{f'(w)^2}{f(w)} - 2 \frac{f'(w)}{w}. \quad (4.9)$$

Hence we see from (4.4) and (4.6) that

$$f''(w) \sim (\beta b)^{-3}(1 - f(w))^{3-3\beta} \beta(\beta - 1) b(1 - f(w))^{\beta-2}$$

which implies (4.7) in view of (4.5). □

Let  $a_{n,k}$  be the coefficients of  $\varphi(z)^n$ , see (2.1). We come to our main theorem.

**Theorem 4.3.** Suppose that condition (A) holds and that  $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$ . Let

$$\psi(z) = \frac{\chi(z)}{(1-z)^\gamma} = \sum_{k=0}^{\infty} b_k z^k, \quad \gamma \geq 0 \quad (4.10)$$

where  $\chi$  is analytic in  $\mathbb{D}$  and has a finite angular limit  $\chi(1) \neq 0$ . Then

$$\sum_{k=1}^n b_{k-1} a_{n,n-k} \sim \frac{\chi(1) b^{\frac{\gamma-1}{\beta}}}{\beta \Gamma(1 + (\gamma-1)/\beta)} n^{\frac{\gamma-1}{\beta}} \text{ as } n \rightarrow \infty. \quad (4.11)$$

*Proof.* (a) We apply Theorem 3.3 with  $\alpha = 2 + \frac{\gamma-1}{\beta} > 1$  and

$$h(w) = w\psi(f(w)) f'(w) = \sum_{n=0}^{\infty} c_n w^n. \quad (4.12)$$

We have  $\chi(f(w)) \rightarrow \chi(1)$  because  $f(1) = 1$  and  $F = f(\mathbb{D})$  lies in a Stolz angle by Theorem 4.2. Hence we obtain from (4.5), (4.6) and (4.10) that

$$(1-w)^{\alpha-1} h(w) \sim \chi(1) (1-w)^{1+\frac{\gamma-1}{\beta}} b^{\frac{\gamma}{\beta}} (1-w)^{-\frac{\gamma}{\beta}} \beta^{-1} b^{-\frac{1}{\beta}} (1-w)^{\frac{1}{\beta}-1}$$

which converges to  $\chi(1) \beta^{-1} b^{(\gamma-1)/\beta}$  as  $w \rightarrow 1$ . We shall verify (3.10) in part (b). Then it follows from (3.11) that

$$c_n \sim c n^{(\gamma-1)/\beta} \text{ as } n \rightarrow \infty \quad (4.13)$$

where  $c$  is the factor in (4.11), and (4.11) now is a consequence of (2.3) (with  $m = 0$ ) in the Bürmann-Lagrange formula.

(b) Since the angular limit  $\chi(1)$  exists, we have  $(1-z)\chi'(z) \rightarrow 0$  [Po92, Prop. 4.8] and thus, by (4.10),

$$\psi'(z) = \frac{\gamma\chi(z) + (1-z)\chi'(z)}{(1-z)^{\gamma+1}} = O\left(\frac{1}{|1-z|^{\gamma+1}}\right) \quad (4.14)$$

as  $z \rightarrow 1$ ,  $z \in F = f(\mathbb{D})$ . Hence we obtain from Theorem 4.2 that

$$\begin{aligned} h'(w) &= \psi(f(w))f'(w) + w\psi'(f(w))f'(w)^2 + w\psi(f(w))f''(w) \\ &= O(|1-f(w)|^{-\gamma-1+2-2\beta}) + O(|1-f(w)|^{-\gamma}|1-w|^{\frac{1}{\beta}-2}) \\ &= O(|1-w|^{(1-\gamma)/\beta-2}) = O(|1-w|^{-\alpha}) \end{aligned} \quad (4.15)$$

as  $w \rightarrow 1$ ,  $w \in \mathbb{D}$ . It follows that  $|1-w|^\alpha |h'(w)|$  is bounded for  $w \in \mathbb{D}$ ,  $|w-1| \leq \delta$  for some  $\delta > 0$ .

Furthermore  $f$  is continuous and injective in  $\overline{\mathbb{D}}$ . Since  $f(1) = 1$  it follows that  $|1-f(w)|$  is bounded away from 0 in  $U = \{w \in \mathbb{D} : |w-1| > \delta\}$ . By assumption we have  $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$  and it follows from [MePo05, Th. 2.2] that  $f$  is analytic in  $\overline{U}$ . Moreover  $\psi'(f(w))$  is bounded in  $U$ . Hence we see from (4.15) that  $|1-w|^\alpha |h'(w)|$  is bounded also in  $U$ .  $\square$



## 5. Applications to probability theory

Now we assume that  $\varphi$  has the form

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \geq 0 \quad (k = 0, 1, \dots) \quad (5.1)$$

and satisfies  $\varphi(0) \neq 0$ ,  $\varphi(1) = 1$  and  $\varphi'(1) = 1$ . Thus  $\varphi$  is the generating function of a random variable  $X$  with values in  $\mathbb{N}_0$  and expectation  $\mathbb{E}(X) = \varphi'(1) = 1$ . Let

$$S_n = X_1 + \dots + X_n \quad (n = 0, 1, \dots)$$

where the  $X_\nu$  are independent random variables with  $\mathbb{P}(X_\nu = k) = a_k$  for all  $\nu$  and  $k$ . Since the  $X_\nu$  are independent, the power  $\varphi(z)^n$  has the coefficients  $\mathbb{P}(S_n = k)$  and thus, by (2.1)

$$a_{n,k} = \mathbb{P}(S_n = k) \text{ for } n, k \in \mathbb{N}_0. \quad (5.2)$$

**Proposition 5.1.** *Let  $\varphi$  be given by (5.1) with  $\varphi(1) = \varphi'(1) = 1$  and suppose that*

$$\sum_{k=1}^m k^2 a_k \sim c m^{2-\beta} \quad (m \rightarrow \infty) \quad (5.3)$$

where  $1 < \beta \leq 2$  and  $0 < c < \infty$ . Then condition (A) of Section 4 is satisfied with

$$b = \frac{c \Gamma(3 - \beta)}{\beta(\beta - 1)}. \quad (5.4)$$

An explicit example is given [MePo05, Ex. 6.2] by

$$\varphi(z) = z + (2\beta)^{-1} (1 - z)^\beta + \frac{1}{4} (1 - z)^2.$$

*Proof.* The case  $\beta = 2$  is easy. Therefore we assume that  $1 < \beta < 2$ . It follows from (5.3) and (3.7) that

$$\sum_{k=1}^m k(k-1)a_k \sim c \Gamma(3 - \beta) (-1)^m \binom{\beta - 3}{m}$$

and therefore, as  $x \rightarrow 1$ ,

$$\frac{\varphi''(x)}{1-x} = \sum_{m=2}^{\infty} \left( \sum_{k=1}^m (k-1)k a_k \right) x^{m-2} \sim \frac{c \Gamma(3 - \beta)}{(1-x)^{3-\beta}}.$$

Now we multiply by  $1 - x$  and integrate twice using  $\varphi'(1) = 1$  and  $\varphi(1) = 1$ . We obtain (4.1) with  $b$  given by (5.4).  $\square$

The generating function  $\varphi$  is called *aperiodic* if there does not exist  $q > 1$  such that  $a_k = 0$  for  $k \not\equiv 0 \pmod{q}$ . If  $\varphi$  is aperiodic then  $|\varphi(z)| < 1$  for  $z \in \mathbb{D}$ ,  $z \neq 1$ , see e.g. [MePo05, Sect. 7]. Thus the condition  $\bar{f}(\mathbb{D}) \subset \mathbb{D} \cup \{1\}$  of Theorem 4.3 is satisfied. Hence we obtain from Theorem 4.3:

**Theorem 5.2.** *Let the generating function  $\varphi$  be aperiodic and let condition (A) of Section 4 be satisfied. Let  $\gamma \geq 0$  and*

$$\psi(z) = \frac{\chi(z)}{(1-z)^\gamma} = \sum_{k=0}^{\infty} b_k z^k \quad (5.5)$$

where  $\chi$  is analytic in  $\mathbb{D}$  and  $\chi(1) \neq 0$ . Then

$$\sum_{k=1}^n b_{k-1} \mathbb{P}(S_n = n-k) \sim \frac{\chi(1) b^{\frac{\gamma-1}{\beta}}}{\beta \Gamma(1 + (\gamma-1)/\beta)} n^{\frac{\gamma-1}{\beta}} \text{ as } n \rightarrow \infty. \quad (5.6)$$

If the variance  $\sigma^2$  of  $X$  is finite then we see from (4.4) that  $\beta = 2$  and  $b = \sigma^2/2$ . Hence (5.6) becomes

$$\sum_{k=1}^n b_{k-1} \mathbb{P}(S_n = n-k) \sim \frac{\chi(1) \sigma^{\gamma-1}}{2^{(\gamma+1)/2} \Gamma((1+\gamma)/2)} n^{\frac{\gamma-1}{2}}. \quad (5.7)$$

Now we give some specific applications where we always assume that condition (A) holds and that  $\varphi$  is aperiodic.

**5.1. The limit behaviour of  $S_n$ .** Let  $Z$  be any random variable with values in  $\mathbb{N}_0$  that is independent of the sums  $S_n$ . We apply Theorem 5.2 with  $\gamma = 0$  and  $\chi$  the generating function of  $Z$ . Then  $\chi(1) = 1$  and the sum in (4.11) becomes

$$\sum_{k=1}^n \mathbb{P}(Z = k-1) \mathbb{P}(S_n = n-k).$$

Since the  $Z$  and  $S_n$  are independent we obtain

$$\mathbb{P}(S_n + Z = n) \sim \frac{b^{-1/\beta}}{\beta \Gamma(1 - 1/\beta)} n^{-\frac{1}{\beta}} \text{ as } n \rightarrow \infty. \quad (5.8)$$

**5.2. Asymmetry near the equilibrium.** Since  $\mathbb{E}(S_n) = n$  the equilibrium is reached if  $S_n = n$ . Now we apply Theorem 5.2 with  $\gamma = 1$  and  $\chi(z) = 1$ . We obtain

$$\mathbb{P}(S_n < n) = \sum_{k=1}^n \mathbb{P}(S_n = n-k) \rightarrow \frac{1}{\beta} \text{ as } n \rightarrow \infty.$$

This value is  $> \frac{1}{2}$  if  $\beta < 2$ . This does not contradict the law of large numbers because this law only says that  $S_n/n \rightarrow 1$  but does not say anything about  $\mathbb{P}(S_n/n < 1)$ .

We introduce random variables  $T_n$  with values in  $\{1, \dots, n\}$  by  $T_n = n - S_n$  for  $S_n < n$  and  $\mathbb{P}(T_n = k) = \mathbb{P}(S_n = n-k \mid S_n < n)$ . It follows from Theorem 5.2 with  $\psi(z) = (1-z)^{-2}$  and  $\psi(z) = 2(1-z)^{-3}$  that

$$\mathbb{E}(T_n) \sim \frac{b^{1/\beta}}{\Gamma(1+1/\beta)} n^{\frac{1}{\beta}}, \quad \mathbb{V}(T_n) \sim \left( \frac{2b^{2/\beta}}{\Gamma(1+2/\beta)} - \frac{b^{2/\beta}}{\Gamma(1+1/\beta)^2} \right) n^{\frac{2}{\beta}}.$$

5.3. *The first return to equilibrium.* Now we introduce a random variable  $N$  with values in  $\mathbb{N}$  by

$$N = n \Leftrightarrow S_n = n, \quad S_\nu \neq \nu \quad (1 \leq \nu < n).$$

Thus  $N$  is when  $S_n$  reaches its equilibrium for the first time.

Now  $(S_n = n)$  is a recurrent event [Fe68, p. 311] and we see from (1.3) and from (2.5) with  $k = 0$  that

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n = n) w^n = \frac{1}{1 - w \varphi'(f(w))}.$$

Hence it follows [Fe68, p. 311] that

$$w \varphi'(f(w)) = \sum_{n=1}^{\infty} \mathbb{P}(N = n) w^n. \quad (5.9)$$

Since  $\varphi'(1) = 1$  and thus  $f(1) = 1$ , we see that the random variable  $N$  is not defective.

Now we argue as in the proof of Theorem 4.3 with  $\psi = \varphi''$ . It follows from (5.9) that

$$h(w) = w \varphi''(f(w)) f'(w) = \sum_{n=0}^{\infty} n \mathbb{P}(N = n+1) w^n; \quad (5.10)$$

this notation agrees with (4.12). We consider  $\alpha = 1 + \frac{1}{\beta} > 1$ . It follows from Proposition 4.1 and Theorem 4.2 that, as  $w \rightarrow 1-$ ,

$$\begin{aligned} (1-w)^{\alpha-1} h(w) &\sim (1-w)^{\frac{1}{\beta}} \beta(\beta-1) b(1-f(w))^{\beta-2} f'(w) \\ &\rightarrow (\beta-1) b^{\frac{1}{\beta}}. \end{aligned}$$

With some effort the condition (3.10) is verified as in part (b) of the proof of Theorem 4.3. Hence we obtain from (5.10) and Theorem 3.3 that

$$n \mathbb{P}(N = n+1) \sim (\beta-1) b^{\frac{1}{\beta}} \Gamma(1/\beta)^{-1} n^{\frac{1}{\beta}-1}$$

and therefore

$$\mathbb{P}(N = n) \sim \frac{(1-\beta) b^{1/\beta}}{\Gamma(1/\beta)} n^{\frac{1}{\beta}-2} \text{ as } n \rightarrow \infty.$$

If  $\sigma$  is finite then  $\beta = 2$  and we obtain  $\mathbb{P}(N = n) \sim \frac{\sigma}{\sqrt{2\pi}} n^{-3/2}$ .

5.4. *The total progeny in a branching process.* We consider a Galton-Watson branching process [Fe68] [AtNe72]. Let  $Z_k$  denote the number of individuals in the  $k$ -th generation where  $Z_0 = q$  is given; in general it is assumed that  $Z_0 = 1$ . These individuals reproduce independently and the number of children of each individual is distributed like  $X$ . Then

$$Y = \sum_{k=0}^{\infty} Z_k \leq \infty$$

is the total progeny, that is the total number of all individuals over all generations; see e.g. [Fe68, p. 298] [KaNa94]. It was shown in [MePo05, Sect. 6] that the fixed point function  $f$  of  $\varphi$  is the generating function of  $Y$  if  $Z_0 = 1$ . Since we now start with  $q$  individuals reproducing independently, we have

$$f(w)^q = \sum_{n=q}^{\infty} \mathbb{P}(Y = n) w^n.$$

Hence we obtain from (2.6) and (5.8) that

$$\mathbb{P}(Y = n) = \frac{q}{n} \mathbb{P}(S_n = n - q) \sim \frac{q b^{-1/\beta}}{\beta \Gamma(1 - 1/\beta)} n^{-\frac{1}{\beta} - 1}. \quad (5.11)$$

If  $\sigma < \infty$  we have  $\mathbb{P}(Y = n) \sim \frac{q}{\sqrt{2\pi}\sigma} n^{-3/2}$ .

## 6. The case $\varphi'(1) < 1$

Let  $\varphi$  again be analytic in  $\mathbb{D}$  and  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi(0) \neq 0$  and  $\varphi(1) = 1$ . Now we assume that

$$\varphi'(z) \rightarrow \mu < 1 \text{ as } z \rightarrow 1, \quad z \in \mathbb{D}. \quad (6.1)$$

The fixed point function  $f$  satisfies  $f(1) = 1$  and now

$$f'(w) \rightarrow \frac{1}{1 - \mu}, \quad \frac{1 - f(w)}{1 - w} \rightarrow \frac{1}{1 - \mu} \text{ as } w \rightarrow 1, \quad w \in \mathbb{D} \quad (6.2)$$

by (1.3). Hence  $f(\mathbb{D})$  is tangential to  $\partial\mathbb{D}$  at 1 [Po92, p. 80]. This is the reason why we have to allow unrestricted approach in (6.1). The situation is more complicated than for  $\varphi'(1) = 1$  and we only prove one result.

**Theorem 6.1.** *Suppose that  $1 < \beta < 2$ ,  $c \in \mathbb{C}$ ,  $c \neq 0$  and*

$$\varphi''(z) \sim c(1 - z)^{\beta-2}, \quad \varphi'''(z) = O(|1 - z|^{\beta-3}) \quad (6.3)$$

*as  $z \rightarrow 1$ ,  $z \in \mathbb{D}$ . If  $\overline{f(\mathbb{D})} \subset \mathbb{D} \cup \{1\}$  then, for every  $k \in \mathbb{Z}$ ,*

$$a_{n,n-k} \sim \frac{c(1 - \mu)^{-\beta-1}}{\Gamma(2 - \beta)} n^{-\beta} \text{ as } n \rightarrow \infty. \quad (6.4)$$

*Proof.* We have  $\alpha = 3 - \beta > 1$  because  $\beta < 2$ . We apply Theorem 3.3 to

$$h(w) = \frac{d}{dw} (f'(w) f(w)^{k-1}) = \sum_{n=k}^{\infty} (n-1) a_{n,n-k} w^{n-2}; \quad (6.5)$$

see (2.5). We restrict ourselves to the case  $k = 1$  to simplify some technical details.

It follows from (4.9), (6.2) and (6.3) that

$$h(w) = f''(w) \sim c(1 - \mu)^{-\beta-1} (1 - w)^{\beta-2} \text{ as } w \rightarrow 1, \quad w \in \mathbb{D}. \quad (6.6)$$

Now we differentiate (4.9) and obtain from (6.3) that

$$\begin{aligned} h'(w) &= f'''(w) = O(f''(w) \varphi''(f(w))) + O(\varphi'''(f(w))) \\ &= O(|1-w|^{2\beta-4}) + O(|1-w|^{\beta-3}) = O(|1-w|^{-\alpha}) \end{aligned}$$

because  $\beta > 1$ . As in part (b) of the proof of Theorem 4.3, we see that (3.10) holds. Hence it follows from (3.11), (6.5) and (6.6) that

$$(n-1)a_{n,n-1} \sim \frac{c(1-\mu)^{-\beta-1}}{\Gamma(2-\beta)} n^{1-\beta} \text{ as } n \rightarrow \infty$$

which implies (6.4) for  $k = 1$ . ✓

Now let  $\varphi$  be an aperiodic probability generating function with  $\mathbb{E}(X) < 1$  that satisfies (6.3) with  $1 < \beta < 2$ . Then it follows from (6.4) that

$$\mathbb{P}(S_n = n - q) \sim c_1 n^{-\beta} \quad (n \rightarrow \infty), \quad c_1 \neq 0 \quad (6.7)$$

and we obtain from (5.11) that the total progeny  $Y$  in a branching process satisfies  $\mathbb{P}(Y = n) \sim c_2 n^{-\beta-1}$ ,  $c_2 \neq 0$ .

The relation is not always (or never ?) true if  $\beta = 2$ , that is for finite variance. Consider for instance the case that  $\varphi$  is analytic in  $\{|z| < R\}$  with  $R > 1$ . Then large deviation theory shows that

$$\mathbb{P}(S_n = n - 1) = O(\rho^n) \quad (n \rightarrow \infty) \text{ for some } \rho < 1$$

which is very much smaller than (6.7). See e.g. [Gä77] and see [MePo05, Sect. 7] for details.

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(Recibido en febrero de 2006. Aceptado en abril de 2006)

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