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Existence of global entropy solutions to a non-strictly hyperbolic system with a source

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Abstract. In this paper we use the theory of compensated compactness coupled with some basic ideas of the Kinetic formulation to establish an existence theorem for global entropy solutions to the non-strictly hyperbolic system with a source.

$$\begin{cases} \rho_t + (\rho u)_x &= U(\rho, u, x, t) \\ u_t + (\frac{u^2}{2} + P(\rho))_x &= V(\rho, u, x, t) \end{cases}$$

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Resumen. En este artículo usamos la teoría de la compacidad compensada asociada con algunas ideas básicas de formulación Kinetica para establecer un teorema de existencia para soluciones de entropía global del sistema no estrictamente hiperbólico con fuente

$$\begin{cases} \rho_t + (\rho u)_x &= U(\rho, u, x, t) \\ u_t + (\frac{u^2}{2} + P(\rho))_x &= V(\rho, u, x, t) \end{cases}$$

1. Introduction

In this paper, we are concerned with the following Cauchy problem (1.1), (1.2) for the nonlinear, inhomogeneous, non-strictly hyperbolic system

$$\begin{cases} \rho_t + (\rho u)_x &= U(\rho, u, x, t) \\ u_t + \left(\frac{u^2}{2} + P(\rho)\right)_x &= V(\rho, u, x, t) \end{cases} \tag{1.1}$$

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)) \tag{1.2}$$

or,

$$\begin{cases} v_t + f(v)_x & = & H(v,x,t) \\ v\mid_{t=0} & = & v_0(x,t) \end{cases} \tag{1.3}$$

where $f(v) = \left(\rho u, \frac{u^2}{2} + P(\rho)\right)^T$, $H(v,x,t) = (U(\rho,u,x,t), V(\rho,u,x,t))^T$, $v = (\rho, u)^T$, $u_0(x)$ and $\rho_0(x) \geq 0 (\not\equiv 0)$ are bounded measurable functions. For polytropic gas, $P(\rho) = \frac{\theta}{2}\rho^{r-1}$, $\theta = \frac{r-1}{2}$ and $r > 3$ is a constant.

System (1.1) is a model of gas dynamics of nonconservative form with a source. For instance, if $H(v,x,t) = (0, \alpha(x,t))^T$, $\alpha(x,t)$ represents body force, usually gravity acting on all the fluid in any volume, when

$$H(v,x,t) = \left(-\frac{a'(x)}{a(x)}\rho u, 0\right),$$

the Cauchy problem models transonic nozzle flow through a variable-area duct.

An essential feature of the system is a non-strictly hyperbolicity, that is, a pair of wave speed coalesce on the vacuum $\rho = 0$.

The homogeneous system corresponding to system (1.1) is

$$\begin{cases} \rho_t + (\rho u)_x & = & 0 \\ u_t + (\frac{u^2}{2} + P(\rho))_x & = & 0 \end{cases} \tag{1.4}$$

System (1.4) was first derived by S.Earnshaw [4] in 1858 for isentropic flow, where ρ denotes the density, u the velocity and $P(\rho)$ the pressure of fluid. As to the study of the existence of global weak solutions for the Cauchy problem (1.4), (1.2), we can see [3, 10, 13]. Diperna [3] is the first one to study the Cauchy problem for the case of $1 < r < 3$ by using the Glimm's scheme method [5]. However,for the case $r > 3$, the strict hyperbolicity of system (1.4) fails since ρ could be zero at a finite time. In order to use the theory of compensated compactness, Lu[10] added a small perturbation δ to the nonlinear function $P(\rho)$ so that system (1.4) has a strictly convex entropy for any fixed $\delta \geq 0$ and hence both strong and weak entropy-entropy flux pairs of the perturbation system of (1.4) satisfy the H^{-1} compactness condition. Therefore the existence of entropy solutions is obtained for this perturbation system. Later in [13], Lu constructed three groups strong-weak entropy combination, and solved this problem completely.

The results concerned of the existence of global weak solution for the general inhomogeneous hyperbolic system comparatively less, which have been found in the works [1, 2, 6, 9]. In [9], T.P. Liu first studied existence and qualitative behavior of solutions for near constant data to resonant systems of this type by using Glimm's random choice method [5]. Chen and Glimm [1] introduced a Godunov shock capturing scheme to obtain L^∞ estimates and compensated compactness of corresponding approximate solutions to the compressible euler equations with geometrical structure. Their method incorporates natural building blocks from Riemann solutions and the existence theory of global weak entropy solutions for measurable initial data in L^∞; Klingenberg and Lu's method in [6] is vanishing viscosity together with compensated compactness.

In this paper, we use the theory of compensated compactness coupled with some basic ideas of the Kinetic formulation from [7, 8, 13] to establish an existence theorem for global entropy solutions to a more general inhomogeneous, non-strictly hyperbolic system (1.1),(1.2). The main results are as follows:

We assume that the functions U and V satisfy the following conditions:

A1 Both U and V are continuous functions, and

$$V\,|_{\rho=0 \text{ or } u=0} = 0 \tag{1.5}$$

A2 There exists a continuous function $F(\omega, z)$ and constants $h_0 > 0$, such that

$$X(\omega, z, x, t) \le F(\omega, z) \quad Y(\omega, z, x, t) \ge -F(\omega, z), \tag{1.6}$$

for $\omega - z \ge 0, 0 \le t \le h_0$, where

$$\begin{cases} X(\omega, z, x, t) &= \theta\rho^{\theta-1}U(\rho, u, x, t) + V(\rho, z, x, t)\big|_{\rho=\left(\frac{\omega-z}{2}\right)^{\frac{1}{\theta}},\, u=\frac{\omega+z}{2}} \\ Y(\omega, z, x, t) &= -\theta\rho^{\theta-1}U(\rho, u, x, t) + V(\rho, z, x, t)\big|_{\rho=\left(\frac{\omega-z}{2}\right)^{\frac{1}{\theta}},\, u=\frac{\omega+z}{2}} \end{cases}$$

$$\omega F(\omega, z) \le \Phi(r)r + c, \quad zF(\omega, z) \le \Phi(r)r + c, \tag{1.7}$$

where c is a positive constant, $r = \sqrt{\omega^2 + z^2}$ and $\Phi(r)$ is a nondecreasing positive function of $r \ge 0$ satisfying the condition $\int\limits_0^\infty \frac{d\tau}{\Phi(\tau)} = \infty$

A3

$$|H(v_2, x, t) - H(v_1, x, t)| \le C_K |v_2 - v_1|^\sigma, \quad 0 < \sigma \le 1 \tag{1.8}$$

Remark 1.1. *For $(U, V) = (\alpha(x,t)\rho, \alpha(x,t)u)$, $(0, \alpha(x,t))$ and $(0, \alpha(x,t)u \ln(|u|+1))$, where $|\alpha(x,t)| \le \alpha_0 < \infty$, it is easy to check that they satisfy the condition (A1-A3).*

Theorem 1.1. *Assume that the conditions (A1-A3) hold and the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \ge 0$, then the Cauchy problem (1.1)-(1.2) has a global bounded entropy solution.*

Remark 1.2. *A pair of functions $(\rho(x,t), u(x,t))$ is called an entropy weak solution of the Cauchy problem (1.1)-(1.2) if*

$$\begin{cases} \int_0^\infty \int_{-\infty}^{+\infty} \left(\rho\phi(x,t)_t + (\rho u)\phi(x,t)_x + U(\rho, u, x, t)\phi\right) dxdt & \\ \qquad + \int_{-\infty}^{+\infty} \rho_0(x)\phi(x, 0)\, dxdt &= 0 \\ \int_0^\infty \int_{-\infty}^{+\infty} \left(u\phi(x,t)_t + \left(\frac{u^2}{2} + P(\rho)\right)\phi(x,t)_x + U(\rho, u, x, t)\phi\right) dxdt & \\ \qquad + \int_{-\infty}^{+\infty} u_0(x)\phi(x, 0)\, dxdt &= 0 \end{cases}$$

for any test function $\phi(x,t) \in C_0^1(R \times R^+)$ and

$$\eta\left(\rho(x,t), u(x,t)\right)_t + q\left(\rho(x,t), u(x,t)\right)_x \le 0 \tag{1.9}$$

in the sense of distributions for any convex entropy $\eta(\rho, u)$ of system (1.1).

The rest of this paper is organized as follows: In Section 2, we give a priori-L^∞ estimate for the approximate solutions of the Cauchy problem (1.1),(1.2). In Sections 3 and 4, we use this estimate coupled with some basic ideas of Kinetic formulation in [7, 8, 13] to prove the main theorem.

2. L^∞ estimates of viscosity solutions

To prove theorem, we first consider the following perturbation system

$$\begin{cases} \rho_t + ((\rho-\delta)u)_x = U(\rho,u,x,t) \\ u_t + \left(\frac{u^2}{2} + \int_\delta^\rho \theta^2 (t-\delta) t^{r-3} dt\right)_x = V(\rho,u,x,t) \end{cases} \tag{2.1}$$

where $\delta > 0$ is the perturbation constant.

By simple calculations,two eigenvalues of system (2.1) are

$$\lambda_1 = u - \theta\rho^{\theta-1}(\rho-\delta), \quad \lambda_2 = u + \theta\rho^{\theta-1}(\rho-\delta), \tag{2.2}$$

and the two corresponding Riemann invariants are the same as system (1.1)

$$z = u - \rho^\theta, \quad \omega = u + \rho^\theta \tag{2.3}$$

Adding viscosity terms to the right-hand side of the (2.1) yields the following parabolic system:

$$\begin{cases} \rho_t + ((\rho-\delta)u)_x = U(\rho,u,x,t) + \varepsilon\rho_{xx} \\ u_t + \left(\frac{u^2}{2} + \int_\delta^\rho \theta^2 (t-\delta) t^{r-3} dt\right)_x = V(\rho,u,x,t) + \varepsilon u_{xx} \end{cases} \tag{2.4}$$

with the initial data

$$(\rho^\varepsilon(x,0), u^\varepsilon(x,0)) = (\rho_0^\varepsilon, u_0^\varepsilon), \tag{2.5}$$

where

$$\rho_0^\varepsilon(x) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} H\left(\frac{x-y}{\varepsilon}\right)(\rho_0(y)+\delta)dy \tag{2.6}$$

$$u_0^\varepsilon(x) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} H\left(\frac{x-y}{\varepsilon}\right)u_0(y)dy$$

and $H(x)$ is a mollifier.

Therefore, by virtue of the condition in theorem, we have:

$$(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \in C_0^\infty \times C_0^\infty \tag{2.7}$$

$$\|\rho_0^\varepsilon(x)\|_\infty + \|u_0^\varepsilon(x)\|_\infty \le \|\rho_0(x)\|_\infty + \|u_0(x)\|_\infty + \delta \tag{2.8}$$

and

$$(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \longrightarrow (\rho_0(x), u_0(x)) \quad \text{a. e. on } R.$$

We first give the L^∞ estimate of the viscosity solution for perturbation system (2.1).

Lemma 2.1. *Assume that the conditions in Theorem 1.1 are satisfied and the solutions $\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)$ of the Cauchy problem (2.4), (2.5) exist in $R\times[0,+\infty)$, then $\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)$ satisfy the following estimates:*

$$\begin{aligned} \omega\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)&\le M(T), \\ z\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)&\ge -M(T), \end{aligned} \tag{2.9}$$

where $M(T)$ is a constant independent of ε,δ for arbitrary fixed $T>0$.

For simplicity, in the following we still take (ρ,u) for $\left(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta}\right)$.

Proof. We multiply (2.4) by (ω_ρ,ω_u) and (z_ρ,z_u) respectively, and we obtain:

$$\begin{aligned} \omega_t+\lambda_2\omega_x&=\varepsilon\omega_{xx}-\varepsilon\theta(\theta-1)\rho^{\theta-2}\rho_x^2+\theta\rho^{\theta-1}U(\rho,u,x,t)+V(\rho,u,x,t)\\ &\le\varepsilon\omega_{xx}+X(\omega,z,x,t)\\ &\le\varepsilon\omega_{xx}+F(\omega,z); \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} z_t+\lambda_1 z_x&=\varepsilon z_{xx}+\varepsilon\theta(\theta-1)\rho^{\theta-2}\rho_x^2-\theta\rho^{\theta-1}U(\rho,u,x,t)+V(\rho,u,x,t)\\ &\ge\varepsilon z_{xx}+Y(\omega,z,x,t)\\ &\ge\varepsilon z_{xx}-F(\omega,z). \end{aligned} \tag{2.11}$$

For the inequality

$$\omega_t+\lambda_2\omega_x\le\varepsilon\omega_{xx}+F(\omega,z), \tag{2.12}$$

we make the transformation $\omega=\phi(v)$, where the function ϕ satisfies the equation $\int_c^{\phi(\xi)}\frac{d\tau}{\Phi(\sqrt{2\tau})}=\ln\xi$, then we have

$$v_t+\lambda_2 v_x\le\varepsilon\left[\frac{\phi''(v)}{\phi'(v)}(v_x)^2+v_{xx}\right]+\frac{F(\omega,z)}{\phi'(v)}. \tag{2.13}$$

Also let $v=\hat{v}e^{\lambda t}$, $\lambda>0$, we have the inequality

$$\hat{v}_t+\lambda_2\hat{v}_x-\varepsilon\hat{v}_{xx}\le\varepsilon\frac{\phi''(v)}{\phi'(v)}(\hat{v}_x)^2e^{\lambda t}-\lambda\hat{v}+\frac{F(\omega,z)}{\phi'(v)}e^{-\lambda t}. \tag{2.14}$$

If $\hat{v}$ takes its greatest value at some interior point (x_0,t_0), suppose that $\hat{v}(x_0,t_0)\ge e^{-\lambda t_0}$ (in fact, if $\hat{v}<e^{-\lambda t_0}$, then $v=\hat{v}e^{\lambda t}<e^{\lambda(t-t_0)}\le 1$. By virtue of the continuity and monotonicity of ϕ, we can get the boundedness of ω). Then on the basis of (2.14), we have at this point $\hat{v}_t\ge 0$, $\hat{v}_x=0$, $\hat{v}_{xx}<0$, hence

$$\lambda\hat{v}\mid_{(x_0,t_0)}\le\frac{F(\omega,z)}{\phi'(v)}e^{-\lambda t}\mid_{(x_0,t_0)} \tag{2.15}$$

$$\lambda\hat{v}\phi'(v)\mid_{(x_0,t_0)}\le F(\omega,z)e^{-\lambda t}\mid_{(x_0,t_0)} \tag{2.16}$$

Since by assumption: $\hat{v}(x_0,t_0)\ge e^{-\lambda t_0}$, we have $v(x_0,t_0)\ge 1$ and hence $\omega(x_0,t_0)\ge 0$.

Multiplying (2.16) by $\omega\left(x_0, t_0\right)$, we obtain:

$$\omega \lambda v \phi'(v) - \Phi(r) r - c \,|_{(x_0,t_0)} \leq 0. \tag{2.17}$$

Since $\phi'(v) \frac{1}{\Phi\left(\sqrt{2}\phi(v)\right)} = \frac{1}{v}$ we have,

$$\omega \Phi(0)\left(\lambda - \sqrt{2}\right) \leq C \tag{2.18}$$

namely for $\lambda > \sqrt{2}$,

$$\omega(x_0, t_0) \leq \frac{C e^{\lambda t_0}}{\Phi(0)\left(\lambda - \sqrt{2} e^{\lambda t}\right)} \tag{2.19}$$

Also by virtue of the condition (2.7), (2.8) and the Theorem 2.1 in [11], we have

$$\omega(x, 0) \leq M, \lim_{|x| \to \infty} \omega(x, t) = 0. \tag{2.20}$$

Hence there exists a $R > 0$, such that if $|x| \geq R$, for arbitrary T and $t \in [0, T]$, we have $\omega(x, t) \leq M$.

According to all of the above, we obtain the estimate $\omega\left(\rho(x,t), u(x,t)\right) \leq M(T)$ for arbitrary $(x,t) \in (-\infty, +\infty) \times [0, T]$. Similarly we can get the estimates $z\left(\rho(x,t), u(x,t)\right) \geq -M(T)$. This completes the proof of Lemma 2.1. ☑

From lemma 2.1, we can obtain the following lemma directly.

Lemma 2.2. *If the conditions in Theorem 1.1 are satisfied, the solutions of the cauchy problem (2.4), (2.5) have a prior-L^∞ estimate for arbitrary $T > 0$ and $t \in [0, T]$,*

$$\delta \leq \rho^{\varepsilon,\delta} \leq M(T), \quad \left|u^{\varepsilon,\delta}(x,t)\right| \leq M(T) \tag{2.21}$$

where $M(T)$ is a positive constant depending only on the initial data and fixed T.

Notice that the system (1.1) has a strictly convex entropy

$$\eta^* = \frac{1}{2} u^2 + \frac{r-1}{4(r-1)} \rho^{r-1}. \tag{2.22}$$

Consequently we have the following lemma:

Lemma 2.3. *If the conditions in Theorem 1.1 are satisfied, then for arbitrary fixed $\varepsilon > 0$, $\sqrt{\varepsilon}\theta\left(\rho^{\varepsilon,\delta}\right)^{\frac{r-3}{2}} \rho_x^{\varepsilon,\delta}$ and $\sqrt{\varepsilon} u_x^{\varepsilon,\delta}$ are uniformly bounded in $L^2_{loc}(R \times R^+)$ in the sense of distribution.*

3. Entropy Waves

This section is concerned with entropy wave for the system (1.1).

One family of weak entropies of system (1.1) (also system (2.1)) is given by

$$\eta_0(\rho,u)=\int_R g(\xi)G_0(\rho,\xi-u)\,d\xi \tag{3.1}$$

and the weak entropy flux q_0 of system (1.1) associated with η_0 is

$$q_0(\rho,u)=\int_R g(\xi)\,[\theta\xi+(1-\theta)u]\,G_0(\rho,\xi-u)d\xi \tag{3.2}$$

two families of strong entropies of system (1.1) are given as follows (cf.[7, 8, 13])

$$\eta_\pm(\rho,u)=\int_R g(\xi)G_\pm(\rho,\xi-u)d\xi \tag{3.3}$$

and the strong entropy fluxes $q_\pm$ of system (1.1) associated with $\eta_\pm$ are

$$q_\pm(\rho,u)=\int_R g(\xi)\,[\theta\xi+(1-\theta)u]\,G_\pm(\rho,\xi-u)d\xi \tag{3.4}$$

where $g(\xi)$ is a smooth function with a compact support set in $(-\infty,+\infty)$ and the fundamental solutions

$$\left\{\begin{array}{lcl} G_0(\rho,\xi-u) & = & [(\omega-\xi)(\xi-z)]_+^\lambda \\ G_+(\rho,\xi-u) & = & (\xi-z)^\lambda(\xi-\omega)_+^\lambda \\ G_-(\rho,\xi-u) & = & (\omega-\xi)^\lambda(z-\xi)_+^\lambda \end{array}\right. \tag{3.5}$$

and $\lambda=\frac{3-r}{2(r-1)}>-\frac{1}{2}$. Here we use the notation $x_+=\max(0,x)$.

Lemma 3.1. *For the viscosity solutions $\left(\rho^{\delta,\varepsilon}(x,t),u^{\varepsilon,\delta}(x,t)\right)$ of the Cauchy problem (2.4) and (2.5), if the entropy $\eta(\rho,u)$ of system (1.1) satisfies that*

$$\eta_\rho(0,u)=0,\ \frac{\partial^i\eta(\rho,u)}{\partial u^i},\quad i=0,1,2,3 \tag{3.6}$$

are bounded in $0\le\rho\le M,|u|\le M$, then

$$\eta\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)_t+q\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)_x \tag{3.7}$$

is compact in $H^{-1}_{loc}(R\times R^+)$ as ε and δ tends to zero, where q is the entropy flux of system (1.1) associated with η.

Lemma 3.2. *For the viscosity solutions $\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)$ of the Cauchy problem (2.4) and (2.5)*

$$\eta_j\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)_t+q_j\left(\rho^{\varepsilon,\delta}(x,t),u^{\varepsilon,\delta}(x,t)\right)_x,\quad j=1,2,3 \tag{3.8}$$

are compact in $H^{-1}_{loc}(R\times R^+)$ as ε and δ tends to zero, where

$$C=-\frac{2\lambda\theta\int_0^\infty(s+2)^{\lambda-1}s^\lambda ds}{\int_{-1}^1(1-s^2)^\lambda ds}>0 \tag{3.9}$$

$$\eta_1 = \eta_+ + C\eta_0, \quad \eta_2 = \eta_- + C\eta_0, \quad \eta_3 = \eta_+ - \eta_-, \tag{3.10}$$

$\eta_\pm, \eta_0$ being given by (3.1),(3.3), and q_j are corresponding entropy fluxes of η_j.

Proof. See [13]. ☑

4. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1 by using the estimate in Section 2 together with some ideas of Kinetic Formulation in [7, 8, 13].

Consider a compactly supported probability measure ν on R^2. By using Lemma 3.2,we have the following measure equation:

$$< \nu, \eta_i >< \nu, q_j > - < \nu, \eta_j >< \nu, q_i > = < \nu, \eta_i q_j - \eta_j q_i > \tag{4.1}$$
$$i, j = 1, 2, 3$$

By virtue of the arbitrariness of the function g and h, we have

$$\begin{aligned} & \overline{G_i(\xi_1)}\ \overline{[\theta\xi_2 + (1-\theta)u]G_j(\xi_2)} - \overline{G_j(\xi_2)}\ \overline{[\theta\xi_1 + (1-\theta)u]G_i(\xi_1)} \\ = & \overline{G_i(\xi_1)[\theta\xi_2 + (1-\theta)u]G_j(\xi_2)} - \overline{G_j(\xi_2)[\theta\xi_1 + (1-\theta)u]G_j(\xi_1)} \\ = & \theta(\xi_2 - \xi_1)\overline{G_i(\xi_1)G_j(\xi_2)} \end{aligned} \tag{4.2}$$

where G_i are fundamental solutions corresponding to the entropies η_i and $\overline{G(\xi)} = \int G(\rho, u - \xi) d\nu_{x,t}(\rho, u)$ indicate the usual integration with respect to the Young measure.

In what follows, we shall prove that the positive measures $\nu_{x,t}$ must be Dirac measures by using compensated compactness theory. Now we discuss it from two respects. Let

$$\xi_+ = \inf_{(\rho,u)\in supp\nu_{x,t}} \omega(\rho, u), \quad \xi_- = \sup_{(\rho,u)\in supp\nu_{x,t}} z(\rho, u) \tag{4.3}$$

Proof. **Case 1: $\xi_- \le \xi_+$.**
If $\xi_- \le \xi_+$, we choose $G_i = G_j = G_3$ and $\xi_1, \xi_2 \in (\xi_+, \infty)$. Since $G_-(\xi_1) = G_-(\xi_2) = 0$, we may rewrite (4.2) as

$$\frac{\theta}{1-\theta}\left[\frac{\overline{G_+(\xi_1)G_+(\xi_2)}}{\overline{G_+(\xi_1)}\ \overline{G_+(\xi_2)}} - 1\right] = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}} - \frac{\overline{uG_+(\xi_1)}}{\overline{G_+(\xi_1)}}\right]. \tag{4.4}$$

Similarly

$$\frac{\theta}{1-\theta}\left[\frac{\overline{G_-(\xi_1)G_-(\xi_2)}}{\overline{G_-(\xi_1)}\ \overline{G_-(\xi_2)}} - 1\right] = \frac{1}{\xi_2 - \xi_1}\left[\frac{\overline{uG_-(\xi_2)}}{\overline{G_-(\xi_2)}} - \frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}}\right] \tag{4.5}$$

for $\xi_1, \xi_2 \in (-\infty, \xi_-)$.

As done in [8, 13], we can obtain $\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}$ and $\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}$ are both non-increasing.

However,

$$\begin{aligned}\overline{u} &= \lim_{\xi\to\infty}\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} \le \lim_{\xi\to\xi_+}\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} \le \frac{\xi_+ + \xi_-}{2} \\ &\le \lim_{\xi\to\xi_-}\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} \le \lim_{\xi\to-\infty}\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \overline{u} \end{aligned} \tag{4.6}$$

then $\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}}$ and $\frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}}$ are both constant on $\xi_2 \in (\xi_+, \infty)$, $\xi_1 \in (-\infty, \xi_-)$.

Let $I_\alpha(\xi)$ be a nonnegative smooth function with compact set in $(-\frac{1}{\alpha}, \frac{1}{\alpha})$ and $I_\alpha(\xi) \to 1$ as $\alpha \to 0^+$, $\psi_\alpha(\xi) \ge 0$ be a unit mass mollifier, denote $G_\alpha^\pm = (G^\pm I_\alpha) * \psi_\alpha$, using (4.4), (4.5) again, we have

$$\overline{G_\alpha^\pm(\xi_1)G_\alpha^\pm(\xi)} = \overline{G_\alpha^\pm(\xi_1)}\ \overline{G_\alpha^\pm(\xi)}. \tag{4.7}$$

Letting $\xi_1 \to \xi$ in (4.7), we get

$$\overline{(G_\alpha^\pm(\xi))^2} = \left(\overline{G_\alpha^\pm(\xi)}\right)^2, \tag{4.8}$$

which implies that

$$\overline{\left(G_\alpha^\pm(\xi) - \overline{G_\alpha^\pm(\xi)}\right)^2} = 0 \tag{4.9}$$

on $(\omega, z) \in \operatorname{supp} \nu_{x,t}$ and hence $G_\alpha^\pm(\omega, z, \xi) - \overline{G_\alpha^\pm(\xi)} = 0$ on the support of $\nu_{x,t}$, and by letting $\alpha \to 0$, so does $G_\pm(\omega, z, \xi) = \overline{G_\pm(\xi)}$. This shows that $\nu_{x,t}$ is a Dirac mass.

Case 2: $\xi_- > \xi_+$.

If $\xi_- > \xi_+$ similarly we have that

$$\frac{\overline{uG_+(\xi_2)}}{\overline{G_+(\xi_2)}}, \frac{\overline{uG_-(\xi_1)}}{\overline{G_-(\xi_1)}} \tag{4.10}$$

are both non-increasing for $\xi_2 \in (\xi_-, \infty)$, $\xi_1 \in (-\infty, \xi_+)$.

However since the following estimates from (4.6),

$$\lim_{\xi\to\infty}\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \overline{u}, \quad \lim_{\xi\to-\infty}\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \overline{u}, \tag{4.11}$$

we have the following inequality

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} \ge \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{4.12}$$

Now we choose $G_i = G_1$, $G_j = G_2$ and $\xi_1 = \xi_2 = \xi$ in (4.2) to obtain

$$\begin{aligned} &\overline{uG_+(\xi)G_-(\xi)} + C\overline{uG_0(\xi)}\ \overline{G_-(\xi)} + C\overline{uG_+(\xi)}\ \overline{G_0(\xi)} \\ =\ &\overline{uG_-(\xi)}\ \overline{G_+(\xi)} + C\overline{uG_0(\xi)}\ \overline{G_+(\xi)} + C\overline{uG_-(\xi)}\ \overline{G_0(\xi)} \end{aligned} \tag{4.13}$$

where C is given by (3.9).

Let

$$\omega_+ = \sup_{(\rho,u)\in supp\nu_{x,t}} \omega(\rho,u), \quad z_- = \inf_{(\rho,u)\in supp\nu_{x,t}} z(\rho,u). \tag{4.14}$$

If we choose $\xi \in (\xi_-, \omega_+)$, then $\overline{uG_-(\xi)} = \overline{G_-(\xi)} = 0$ and hence by (4.13)

$$\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}} \tag{4.15}$$

for $\xi \in (\xi_-, \omega_+)$. In particular,

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} = \frac{\overline{uG_0(\xi_-)}}{\overline{G_0(\xi_-)}} \tag{4.16}$$

Similarly, if choosing $\xi \in (z_-, \xi_+)$, we have

$$\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} = \frac{\overline{uG_0(\xi)}}{\overline{G_0(\xi)}} \tag{4.17}$$

for $\xi \in (z_-, \xi_+)$. In particular,

$$\frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}} = \frac{\overline{uG_0(\xi_+)}}{\overline{G_0(\xi_+)}}. \tag{4.18}$$

using (4.13), we have

$$\begin{aligned} &\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}} + C\frac{\overline{uG_0(\xi)}}{\overline{G_+(\xi)}} + C\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}\frac{\overline{G_0(\xi)}}{\overline{G_-(\xi)}} \\ = \;&\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}} + C\frac{\overline{uG_0(\xi)}}{\overline{G_-(\xi)}} + C\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}\frac{\overline{G_0(\xi)}}{\overline{G_+(\xi)}} \end{aligned} \tag{4.19}$$

Below we use ξ^{+0} to indicate the right limit and ξ^{-0} the left limit at ξ. Letting $\xi \to \xi_-$ in (4.19) and using (4.16), we have

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}\left(1 + C\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}}\right) = \frac{\overline{uG_-\left(\xi_-^{-0}\right)}}{\overline{G_-\left(\xi_-^{-0}\right)}}\left(1 + C\frac{\overline{G_0(\xi_-)}}{\overline{G_+(\xi_-)}}\right), \tag{4.20}$$

which implies

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} = \frac{\overline{uG_-\left(\xi_-^{-0}\right)}}{\overline{G_-\left(\xi_-^{-0}\right)}}. \tag{4.21}$$

Similarly, letting $\xi \to \xi_+$ in (4.19) and using (4.18), we have

$$\frac{\overline{uG_+\left(\xi_+^{+0}\right)}}{\overline{G_+\left(\xi_+^{+0}\right)}} = \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{4.22}$$

Let $G_i = G_j = G_1$ in (4.2). By the same treatment in Case 1, we have that $\frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}}$ is non-increasing for $\xi \in (\xi_+, \xi_-)$. However

$$\lim_{\xi \to \xi_-} \frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}} = \frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}}, \tag{4.23}$$

where (4.16) is used in the last equality, and

$$\lim_{\xi \to \xi_+} \frac{\overline{uG_1(\xi)}}{\overline{G_1(\xi)}} = \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}, \tag{4.24}$$

where (4.18) and (4.22) are used in the last equality. Thus

$$\frac{\overline{uG_+(\xi_-)}}{\overline{G_+(\xi_-)}} \leq \frac{\overline{uG_-(\xi_+)}}{\overline{G_-(\xi_+)}}. \tag{4.25}$$

(4.25) and (4.12) imply that $\frac{\overline{uG_+(\xi)}}{\overline{G_+(\xi)}}$ is a constant for $\xi \in (\xi_+, \infty)$ and $\frac{\overline{uG_-(\xi)}}{\overline{G_-(\xi)}}$ is a constant for $\xi \in (-\infty, \xi_-)$. Hence Young measure ν is also a Dirac mass from the proof in Case 1. This is contrary to the assumption $\xi_+ < \xi_-$ since $\omega \geq z$. Therefore only Case 1, i. e., $\xi_+ \geq \xi_-$ is permitted, and ν is a Dirac mass. So we end the proof of the Theorem 1.1. ☑

References

[1] G. Q. CHEN & J. GLIMM, Global solutions to the compressible Euler equations with geometric structure, *Commun. Math. Phys.* **180** (1996), 153–193.

[2] X.DING, G-Q. CHEN & P. LUO, Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics, *Commn. Math. Phys* **12** (1989), 63–84.

[3] R. J. DIPERNA, Global solutions to a class of nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **26** (1973), 1–28.

[4] S. EARNSHAW,On the mathematical theory of sound, *Philos.Trans.* **150** (1858), 1150–1154.

[5] J. GLIMM, Solutions in the large for nonliear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 95–105.

[6] C. KLINGENBERG & Y.-G. LU, Existence of solutions to hyperbolic conservation Laws with a source, *Commun. Math. Phys.* **187** (1997), 327–340.

[7] P. L. LIONS, B. PERTHAME & P. E. SOUGANIDIS, Existence and stabibility of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure Appl. Math.* **49** (1996), 599–638.

[8] P.L. LIONS, B. PERTHAME & E. TADMOR, Kinetic formulation of the isentropic gas dynamics and *p*-system, *Comm. Pure Appl. Math.* **163** (1994), 415–431.

[9] T. P. LIU, Quasilinear hyperbolic systems, *Commn. Math. Phys.* **68** (1979), 141–172.

[10] Y. -G. LU, Convergence of the viscosity method for nonstrictly hyperbolic conservation laws, *Commun. Math. Phys.* **150** (1992), 59–64.

[11] Y. G. Lu, Existence of generalized solutions for some coupled system of nonlinear hyperbolic equations, *J. Sys. Sci.& Math. Scis.* **16** (1996) (in Chineses), 125–135.
[12] Y. -G. Lu, *Hyperbolic Conservation Laws and the Compensated Compactness Method*, Chapman and Hall, CRC Press, New York, 2002.
[13] Y. -G. Lu, Existence of global entropy solutions to a nonstrictly hyperbolic system, *Arch. Rat. Mech. Anal.* **178** (2005), 287–299.

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