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Spectral properties of compressible stratified flows

Propiedades espectrales de los flujos estratificados comprimibles

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ABSTRACT. For bounded and unbounded domains in R^3, we establish the localization and the structure of the spectrum of normal vibrations described by systems of partial differential equations modelling small displacements of compressible stratified fluid in the homogeneous gravity field. We also compare the spectral properties of gravitational and rotational operators. Our main result is the construction of Weyl sequence for the essential spectrum, which is an explicit form of non-uniqueness of the solutions.

Key words and phrases. Partial differential equations, essential spectrum, Sobolev spaces, stratified fluid, internal waves

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RESUMEN. Para los dominios acotados y no-acotados en R^3, estudiamos la localización y la estructura del espectro de las vibraciones normales que se describen mediante sistemas de ecuaciones en derivadas parciales que modelan los movimientos pequeños de un líquido estratificado comprensible en el campo gravitacional homogéneo. También comparamos las propiedades espectrales de los operadores rotacionales y gravitacionales. Nuestro resultado principal es la construcción de la sucesión de Weyl para el espectro esencial, la cual representa explícitamente la no-unicidad de las soluciones.

Palabras y frases clave. Ecuaciones diferenciales parciales, espectro esencial, espacios de Sobolev, líquido estratificado, ondas internas.

1. Introduction

The objective of this paper is to study the structure and the localization of the spectrum of partial differential operators which arise in the description of small motions of an exponentially stratified compressible fluid in the gravity field.

We consider a system of equations in the form

$$\left\{\begin{array}{rcl} \rho_* \frac{\partial u_1}{\partial t} + \frac{\partial p}{\partial x_1} & = & 0 \\ \rho_* \frac{\partial u_2}{\partial t} + \frac{\partial p}{\partial x_2} & = & 0 \\ \rho_* \frac{\partial u_3}{\partial t} + g\rho + \frac{\partial p}{\partial x_3} & = & 0 \\ \frac{\partial \rho}{\partial t} - \frac{N^2 \rho_*}{g} u_3 & = & 0 \\ \frac{\partial p}{\partial t} + \rho_* \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) & = & 0 \end{array}\right. , \tag{1.1}$$

in the domain $\{x \in \Omega \subset R^3\}$, $t \geqslant 0$, where $\overrightarrow{u}(x,t)$ is a velocity field with components u_1, u_2, u_3, the function $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamycal density and ρ_*, g, N are positive constants. The equations (1.1) are deduced in [3] under the assumption that the function of stationary distribution of density is performed by the function $\rho_* e^{-Nx_3}$.

The system (1.1) was studied from different angles, some of the results may be found in [10], [12], [18], [9], [11], [7]. Particularly, the smoothness of the solution of stratified system for the case of the intrusion was studied in [18]. The isolated case of uniqueness of solutions for stratified fluid in a class of increasing functions was considered in [9]. The case of essential spectrum for ideal (non-compressible) fluid was considered in [10], [12]. The general smoothness of solutions was considered in [11]. The essential spectrum for rotational (non-stratified) ideal and compressible flows was considered in [17], and mathematical properties for different problems concerning rotational fluids were considered in [20] and [16].

Without loss of generality, we may assume $g = 1$ and $\rho_* = 1$ in (1.1), which can be achieved by introducing new unknown functions and renaming them as follows:

$$\overrightarrow{u} := \rho_* \overrightarrow{u}, \ \rho := g\rho.$$

Thus, we obtain the system

$$\left\{\begin{array}{r} \frac{\partial u_1}{\partial t} + \frac{\partial p}{\partial x_1} = 0 \\ \frac{\partial u_2}{\partial t} + \frac{\partial p}{\partial x_2} = 0 \\ \frac{\partial u_3}{\partial t} + \rho + \frac{\partial p}{\partial x_3} = 0 \\ \frac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \frac{\partial p}{\partial t} + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \end{array}\right. . \tag{1.2}$$

Let us observe certain mathematical similarity of the incompressible case of the system (1.2) and the system which describes rotational motions of ideal fluid over the vertical axis ($\overrightarrow{\omega} = (0, 0, \omega)$):

$$\begin{cases} \frac{\partial^2 \overrightarrow{v}}{\partial t^2} + \overrightarrow{\omega} \times \overrightarrow{v} + \nabla p &= 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0 \end{cases} .$$

Particularly, we would like to compare the scalar form of the two systems

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2}\right) + N^2\left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2}\right) &= 0, \\ \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2}\right) + \omega^2 \frac{\partial^2 \Phi}{\partial x_3^2} &= 0, \end{aligned}$$

and their corresponding singular solutions ([12]):

$$\mathcal{E}(x,t) = \frac{1}{4\pi\,|x_3|} \int\limits_0^{\frac{Nt|x_3|}{|x|}} J_0(\alpha)\, d\alpha \ ,$$

$$\mathcal{E}(x,t) = \frac{1}{4\pi\,|\overline{x}|\,|x|} \int\limits_0^{\frac{\omega t|\overline{x}|}{|x|}} J_0(\alpha)\, d\alpha \ .$$

This mathematical analogy between gravitational and rotational waves, may lead to the corresponding analogy in spectral properties.

In [17] we proved that the essential spectrum of normal vibrations generated by rotational inner waves for compressible fluids, is the interval of the real axis $[-\omega, \omega]$ for bounded domains, and it was the whole axis R^1 for the case $\Omega = R^3$. Thus, it seems appropriate to express the conjecture that the operators generated by (1.2) should possess spectral properties, analogous to the rotational system. Here we prove that this conjecture is true.

2. Spectral problem formulation

Let Ω be a bounded domain in R^3 and let us consider the boundary condition $\overrightarrow{u} \cdot \overrightarrow{n}|_{\partial\Omega} = 0$ for the system (1.2). We consider the following problem of normal vibrations

$$\begin{aligned} \overrightarrow{u}(x,t) &= \overrightarrow{v}(x)\, e^{-\lambda t} \\ \rho(x,t) &= N v_4(x) e^{-\lambda t} \\ p(x,t) &= v_5(x) e^{-\lambda t} \qquad , \lambda \in C. \end{aligned} \tag{2.1}$$

We denote $\widetilde{v} = (v_1, v_2, v_3, v_4, v_5)$ and write the system (1.2) in the matrix form

$$L\widetilde{v} = 0\,, \tag{2.2}$$

where

$$L = M - \lambda I$$

and

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}.$$

Now, let us define the main symbol $L^0(\xi)$ of the operator L. According to [8], [1], we can choose the numbers $s_i = t_j = 0$ for $i, j = 1, 2, 3, 4$ and $s_5 = t_5 = 1$, such that the elements l_{ij} in the matrix L will have the differential order not greater than $s_i + t_j$. In this way, the main symbol takes the following form:

$$L^0(\xi) = \begin{pmatrix} -\lambda & 0 & 0 & 0 & \xi_1 \\ 0 & -\lambda & 0 & 0 & \xi_2 \\ 0 & 0 & -\lambda & N & \xi_3 \\ 0 & 0 & -N & -\lambda & 0 \\ \xi_1 & \xi_2 & \xi_3 & 0 & 0 \end{pmatrix},$$

and thus

$$\det L^0(\xi) = \lambda\left(\lambda^2 |\xi|^2 + N^2 |\xi'|^2\right), \tag{2.3}$$

where $|\xi'|^2 = \xi_1^2 + \xi_2^2$.

We can see that if $\lambda \notin [-iN, iN]$, then for every $\xi \neq 0$ we have $\det L^0(\xi) \neq 0$ and, consequently, the operator L is elliptic in sense of Douglis-Nirenberg.

Our aim is to investigate the spectrum of the operator M. Let us define the domain of the operator M as follows:

$$D(M) = \left\{ \begin{array}{c} \vec{u} \in L_2(\Omega) \,|\, \exists f \in L_2(\Omega): \\ (\vec{u}, \nabla\varphi) = (f, \varphi)\, \forall\varphi \in W_2^1(\Omega) \end{array} \right\} \times W_2^1(\Omega) \times W_2^1(\Omega),$$

where $(\cdot,\cdot)$ is an inner product in $L_2(\Omega)$ and $W_2^1(\Omega)$ is a Sobolev space with the norm

$$\|f\|_{W_2^1(\Omega)} = \left(\int_\Omega \left[|\nabla f|^2 + f^2 \right] dx \right)^{\frac{1}{2}}.$$

First, we will show that M is skew-selfadjoint and thus its spectrum will belong to the imaginary axis. Then, we will find its structure and localization either for bounded domains $\Omega \subset R^3$, or for the whole space R^3.

From the physical point of view, the separation of variables (2.1) serves as a tool to establish the possibility to represent every non-stationary process described by (1.2) as a linear superposition of the normal vibrations. The knowledge of the spectrum of normal vibrations may be very useful for studying the stability of the flows. Also, the spectrum of operator M is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator M.

3. Spectral problem solution for compressible fluid

Lemma 3.0.1. *The operator M is skew-selfadjoint.*

Proof. We observe first that, for compressible fluid, the Lemma cannot be proved by using the projection of $L_2(\Omega)$ onto the space of the solenoidal fields, as it was done in [10], [12]. Here we will use directly the definition of an adjoint operator.

Since M can be represented as $M = M_0 + B$, where B is an anti-symmetric bounded operator

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & -N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then it is sufficient to verify the skew-selfadjointness for the operator M_0 with the domain $D(M_0) = D(M)$.

Let $\widetilde{u}, \widetilde{v} \in D(M_0)$. Integrating by parts, we obtain

$$(M_0\widetilde{u}, \widetilde{v}) = \left(\nabla u_5, \overrightarrow{v}\right) + \left(div\,\overrightarrow{u}, v_5\right) = -\left(div\,\overrightarrow{v}, u_5\right) - \left(\overrightarrow{u}, \nabla v_5\right) = -\left(\widetilde{u}, M_0\widetilde{v}\right).$$

Now, we shall prove the equality

$$D(M_0^*) = D(M_0).$$

First, we verify that $D(M_0^*) \subset D(M_0)$. Since the operator M_0 is not acting on the fourth component of the vector $\widetilde{u}$, then, without loss of generality, we may consider $u_4 = v_4 = f_4 = 0$.

Let $\widetilde{v} \in D(M_0^*)$. It means that $\widetilde{v} \in L_2(\Omega)$ and that there exists $\widetilde{f} \in L_2(\Omega)$ such that

$$(M_0\widetilde{u}, \widetilde{v}) = \left(\widetilde{u}, \widetilde{f}\right) \quad \text{for all } \widetilde{u} \in D(M_0) \; :$$

$$(M_0\widetilde{u}, \widetilde{v}) = (\nabla u_5, \overrightarrow{v}) + (div\,\overrightarrow{u}, v_5) = \left(\overrightarrow{u}, \overrightarrow{f}\right) + (u_5, f_5).$$

Let $\widetilde{u} = (0, 0, 0, 0, u_5)$, $u_5 \in W_2^1(\Omega)$. For such $\widetilde{u}$ we have

$$(\nabla u_5, \overrightarrow{v}) = (u_5, f_5).$$

Now, take $\widetilde{u} = (u_1, u_2, u_3, 0, 0)$. For such $\widetilde{u}$ we obtain

$$(div\,\overrightarrow{u}, v_5) = \left(\overrightarrow{u}, \overrightarrow{f}\right).$$

It follows from the last relation that v_5 has a weak gradient from $L_2(\Omega)$ and $v_5 \in W_2^1(\Omega)$. Finally, $D(M_0^*) \subset D(M_0)$. The reciprocal inclusion can be proved analogously and thus the lemma is proved. ☑

We recall that the *essential* spectrum is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([14], [19]). We shall use the following criterion which is attributed to Weyl ([14], [19]) : a necessary and sufficient condition that a real finite value μ be a point of the essential spectrum of a self-adjoint operator A is that there exist a sequence of elements $x_n \in D(A)$ such that

$$\|x_n\| = 1, \qquad x_n \rightharpoonup 0, \qquad \|(A - \mu I)x_n\| \to 0. \tag{3.1}$$

Theorem 3.1. *The essential spectrum of the operator M is the interval of the imaginary axis $[-iN, iN]$.*

Proof. From Lemma 3.1 we know that the spectrum of the operator M belongs to the imaginary axis. Taking into account (2.3), we consider $\lambda_0 \in (-iN, iN) \setminus \{0\}$ and choose a vector $\xi \neq 0$ such that

$$\lambda_0^2\xi_3^2 + |\xi'|^2\left(\lambda_0^2 + N^2\right) = 0.$$

Therefore, there exists the vector $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ such that $L^0(\xi)\eta = 0$:

$$\begin{cases} -\lambda_0\eta_1 + \xi_1\eta_5 &= 0 \\ -\lambda_0\eta_2 + \xi_2\eta_5 &= 0 \\ -\lambda_0\eta_3 + N\eta_4 + \xi_3\eta_5 &= 0 \\ -N\eta_3 - \lambda_0\eta_4 &= 0 \\ \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 &= 0 \end{cases}. \tag{3.2}$$

Solving (3.2) with respect to η, we obtain one of possible solutions:

$$\begin{cases} \eta_1 &= \frac{\xi_1}{\lambda_0} \\ \eta_2 &= \frac{\xi_2}{\lambda_0} \\ \eta_3 &= \frac{\xi_3\lambda_0}{\lambda_0^2+N^2} \\ \eta_4 &= \frac{-\xi_3 N}{\lambda_0^2+N^2} \\ \eta_5 &= 1 \end{cases}.$$

We observe that $\eta_i \neq 0$, $i = 1, 2, 3, 4, 5$.

Now, let $C_0^\infty(\Omega)$ be a space of smooth functions with compact support in Ω and let us choose a function

$$\psi_0(x) \in C_0^\infty(\Omega), \int\limits_{|x|\leqslant 1} \psi_0^2(x)\,dx = 1.$$

We fix $x_0 \in \Omega$ and define

$$\psi_k(x) = k^{\frac{3}{2}}\psi_0\big(k(x - x_0)\big), \qquad k = 1, 2, \ldots$$

One can easily see that, starting from certain k,

$$\|\psi_k\|_{L_2(\Omega)} = 1, \quad \left\|\frac{\partial\psi_k}{\partial x_j}\right\|_{L_2(\Omega)} = C_j^1 k, \quad \left\|\frac{\partial^2\psi_k}{\partial x_j^2}\right\|_{L_2(\Omega)} = C_j^2 k^2,$$

where the constants $C_j^i \neq 0$ do not depend on k.

We define the Weyl sequence

$$\widetilde{u}^k = \left(u_1^k, u_2^k, u_3^k, u_4^k, q^k\right)$$

as follows:

$$\begin{cases} u_j^k(x) &= \eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k + \frac{i}{k^3\xi_j}\frac{\partial\psi_k}{\partial x_j}\right) \quad ,j=1,2,3 \\ u_4^k(x) &= \eta_4 e^{ik^3\langle x,\xi\rangle}\psi_k \\ q^k(x) &= -\frac{i}{k^3}\psi_k e^{ik^3\langle x,\xi\rangle} \\ \langle x,\xi\rangle &= x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \quad ,k=1,2,\dots \end{cases} \tag{3.3}$$

Now we have to show that the sequence (3.3) actually satisfies all the conditions (3.1).

For the functions (3.3), the weak convegrence to zero follows from the weak convergence to zero of the functions $e^{ik^3\langle x,\xi\rangle}$ and the estimates $\|\psi_k\|_{L_2(\Omega)} = 1$, $\left\|\frac{\partial\psi_k}{\partial x_j}\right\|_{L_2(\Omega)} = C_j^1 k$.

The condition $\|x_n\| = 1$, actually, is equivalent to the condition that the norms of the Weyl sequence are separated from zero, and, it is sufficient to prove that at least the norms of one of the coordinates of the field $\widetilde{u}^k$ are separated from zero. Let us consider the first coordinate

$$u_1^k(x) = \eta_1 e^{ik^3\langle x,\xi\rangle}\psi_k + \eta_1 e^{ik^3\langle x,\xi\rangle}\frac{i}{k^3\xi_1}\frac{\partial\psi_k}{\partial x_1} \tag{3.4}$$

For the second term in the sum (3.4) we have

$$\lim_{k\to\infty}\left\|\eta_1 e^{ik^3\langle x,\xi\rangle}\frac{i}{k^3\xi_1}\frac{\partial\psi_k}{\partial x_1}\right\|_{L_2} = \lim_{k\to\infty}\frac{|\eta_1|}{k^3|\xi_1|}\left\|\frac{\partial\psi_k}{\partial x_1}\right\|_{L_2} = 0.$$

However, for the first term we obtain

$$\left\|\eta_1 e^{ik^3\langle x,\xi\rangle}\psi_k\right\|_{L_2} = |\eta_1|\,\|\psi_k\|_{L_2} = |\eta_1| \neq 0.$$

Now, it remains to verify that $\left\|(M-\lambda_0 I)\,\widetilde{u}^k\right\|_{L_2} \to 0$. Let us denote

$$\widetilde{f}^k = (M-\lambda_0 I)\,\widetilde{u}^k.$$

For the first component f_1^k we have

$$\begin{aligned} f_1^k &= -\lambda_0 u_1^k + \frac{\partial q^k}{\partial x_1} = \\ &= (-\lambda_0\eta_1 + \xi_1)\, e^{ik^3\langle x,\xi\rangle}\psi_k - \frac{i}{k^3}\left(\frac{\lambda_0\eta_1}{\xi_1} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_1} = \\ &= -\frac{i}{k^3}\left(\frac{\lambda_0\eta_1}{\xi_1} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_1}, \end{aligned}$$

since $-\lambda_0\eta_1 + \xi_1 = 0$, which follows from the first equation of (3.2). Thus, we have

$$\|f_1^k\|_{L_2(\Omega)} \le \frac{Const}{k^3}\left\|\frac{\partial\psi_k}{\partial x_1}\right\|_{L_2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0.$$

Analogously,

$$\begin{aligned} f_2^k &= -\lambda_0 u_2^k + \frac{\partial q^k}{\partial x_2} = \\ &= (-\lambda_0\eta_2 + \xi_2)\, e^{ik^3\langle x,\xi\rangle}\psi_k - \frac{i}{k^3}\left(\frac{\lambda_0\eta_2}{\xi_2} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_2} = \\ &= -\frac{i}{k^3}\left(\frac{\lambda_0\eta_2}{\xi_2} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_2}, \end{aligned}$$

and

$$\|f_2^k\|_{L_2(\Omega)} \le \frac{Const}{k^3}\left\|\frac{\partial\psi_k}{\partial x_2}\right\|_{L_2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0.$$

In a similar way we have

$$\begin{aligned} f_3^k &= -\lambda_0 u_3^k + N u_4^k + \frac{\partial q^k}{\partial x_3} = \\ &= (-\lambda_0\eta_3 + N\eta_4 + \xi_3)\, e^{ik^3\langle x,\xi\rangle}\psi_k - \frac{i}{k^3}\left(\frac{\lambda_0\eta_3}{\xi_3} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_3} = \\ &= -\frac{i}{k^3}\left(\frac{\lambda_0\eta_3}{\xi_3} + 1\right) e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_3}, \end{aligned}$$

and thus

$$\|f_3^k\|_{L_2(\Omega)} \le \frac{Const}{k^3}\left\|\frac{\partial\psi_k}{\partial x_3}\right\|_{L_2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0.$$

For the fourth component we have the expression

$$\begin{aligned} f_4^k &= -N u_3^k - \lambda_0 q^k = \\ &= (-\lambda_0\eta_4 - N\eta_3)\, e^{ik^3\langle x,\xi\rangle}\psi_k - \frac{i}{k^3}\frac{N\eta_3}{\xi_3} e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_3} = \\ &= -\frac{i}{k^3}\frac{N\eta_3}{\xi_3} e^{ik^3\langle x,\xi\rangle}\frac{\partial\psi_k}{\partial x_3}, \end{aligned}$$

which is followed by the estimate

$$\|f_4^k\|_{L_2(\Omega)} \leq \frac{Const}{k^3}\left\|\frac{\partial\psi_k}{\partial x_3}\right\|_{L_2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0.$$

To evaluate f_5^k, we use the last equation of (3.2)

$$\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 = 0,$$

which leads to

$$\begin{aligned} f_5^k &= div\overrightarrow{u}^k - \lambda_0 q^k = \\ &= \frac{i}{k^3}e^{ik^3(x,\xi)}\left(\lambda_0\psi_k + \sum_{j=1}^{3}\frac{\eta_j}{\xi_j}\frac{\partial^2\psi_k}{\partial x_j^2}\right). \end{aligned}$$

Finally, we obtain the estimate

$$\|f_5^k\|_{L_2(\Omega)} \leq \frac{C_1}{k^3} + \frac{C_2}{k} \underset{k\to\infty}{\longrightarrow} 0. \tag{3.5}$$

We have verified that for $\lambda \in (-iN, iN)\setminus\{0\}$ the functions defined by (3.3) actually represent a Weyl sequence. Since the essential spectrum is a closed set, the points $\lambda = 0$, $\lambda = \pm iN$, belong to it.

It was proved in [13] that the essential spectrum of the operator M is equal to $Q\cup S$, where

$$Q = \{\lambda \in \mathbb{C} : (M-\lambda I) \text{ is not elliptic in sense of Douglis-Nirenberg}\}$$

and

$$S = \left\{ \begin{array}{ll} \lambda\in\mathbb{C}\setminus Q: & \text{the boundary conditions of the operator } (M-\lambda I) \\ & \text{do not satisfy Lopatinsky conditions} \end{array} \right\}.$$

We have seen that for $\lambda \notin [-iN, iN]$ the system $M - \lambda I$ is elliptic in sense of Douglis-Nirenberg. Let us prove that, for this case, the boundary condition $\overrightarrow{u}\cdot\overrightarrow{n}|_{\partial\Omega} = 0$ satisfy Lopatinsky conditions.

Here we remind that the Lopatinsky conditions (see [13]) consist of the linear independence of the rows of the matrix

$$G\left(x,\widetilde{\xi},\tau\right)\widehat{L}\left(\widetilde{\xi},\tau\right)$$

with respect to the module $M^+\left(\widetilde{\xi},\tau\right)$, for $\left|\widetilde{\xi}\right| \neq 0$.

Here $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, $\widehat{L^0}(\xi)$ is the matrix of the algebraic complements of the main symbol matrix $L^0(\xi)$, $G(x,\xi)$ is the symbol of the matrix $G(x,D)$ which defines the boundary conditions, $M^+\left(\widetilde{\xi},\tau\right) = \prod\left(\tau - \tau_j\left(\widetilde{\xi}\right)\right)$, $\tau_j\left(\widetilde{\xi}\right)$ are the roots of the equation $\det L\left(\widetilde{\xi},\tau\right) = 0$ with positive imaginary part.

Since $\lambda \notin [-iN, iN]$, then we can introduce the parameter $a \neq 0$ as

$$a^2 = \left(\frac{\lambda^2 + N^2}{\lambda^2} \right),$$

so that the equation $\det L(\xi) = 0$ takes the form

$$a^2 |\xi'|^2 + \xi_3^2 = 0. \tag{3.6}$$

In the upper half of the complex plane, the equation has only one root

$$\tau = ia |\xi'| .$$

Let us choose a local system of coordinates so that $\xi_1 = 1$, $\xi_2 = 0$.

Then, we have

$$M^+ \left(\widetilde{\xi}, \tau \right) = \tau - ia ,$$

$$L^0(\tau) = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & N & \tau \\ 0 & 0 & -N & -\lambda & 0 \\ 1 & 0 & \tau & 0 & 0 \end{pmatrix},$$

$$\widehat{L^0}(\tau) = \begin{pmatrix} -\lambda^2\tau^2 & 0 & \lambda^2\tau & -\lambda N\tau & \lambda(\lambda^2+N^2) \\ 0 & -\lambda^2(1+\tau^2) - N^2 & 0 & 0 & 0 \\ \lambda^2\tau & 0 & -\lambda^2 & \lambda N & \lambda^2\tau \\ \lambda N\tau & 0 & -N\lambda & -\lambda^2(1+\tau^2) & \lambda^2 N\tau \\ \lambda(\lambda^2+N^2) & 0 & \lambda^3\tau & -\lambda^2 N\tau & \lambda^2(\lambda^2+N^2) \end{pmatrix}.$$

If we write the boundary conditions in form

$$G(\overrightarrow{u}, p)|_{\partial\Omega} = 0$$

we obtain immediately that

$$G = (n_1, n_2, n_3, 0, 0) .$$

and G is a vector row. Since $\widehat{L^0}(\tau)$ is a matrix whose size is 5×5, then $G\widehat{L}(\tau)$ is a row with five components. In other terms, the Lopatinsky condition is satisfied, which completes the proof. ☑

Now, let us consider the system (2.2) in $\Omega = R^3$. For the normal vibrations problem we have the system

$$(M^* - \lambda I)\, \widetilde{u} = 0,$$

where the matrix M^* is the same matrix M, and the domain of M^* is defined as

$$D(M^*) = \left\{ \begin{array}{c} \overrightarrow{u} \in L_2(R^3) \,|\exists f \in L_2(R^3) : \\ (\overrightarrow{u}, \nabla\varphi) = (f, \varphi) \forall \varphi \in W_2^2(R^3) \end{array} \right\} \times W_2^2(R^3) \times W_2^2(R^3) .$$

Theorem 3.2. *The essential spectrum of the operator M^* is the whole imaginary axis. Moreover, the points λ such that $\lambda \notin [-iN, iN]$, belong to the continuous spectrum of the operator M^*.*

Proof. Let $\lambda \in [-iN, iN]$. We note first that, due to the inclusion theorem $W_2^2(R^3) \to C(R^3)$, for all $\varphi \in W_2^2(R^3)$ we have the property: $\lim_{|x|\to\infty} \varphi(x) = 0$. Thus, for every $\varphi_1, \varphi_2 \in W_2^2(R^3)$ the integration by parts is valid:

$$\int_{R^3} \frac{\partial \varphi_1}{\partial x_j} \varphi_2 dx = -\int_{R^3} \frac{\partial \varphi_2}{\partial x_j} \varphi_1 dx.$$

Therefore, by Lemma 3.1 we obtain the skew-selfadjointness for the operator M^*, and, using the same Weyl sequence as in Theorem 3.1, we have that $\lambda \in [-iN, iN]$ belongs to the essential spectrum.

It is easy to see that the system (2.2) is equivalent to the scalar equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\lambda^2}{\lambda^2 + N^2} \frac{\partial^2 u}{\partial x_3^2} - \lambda^2 u = 0. \tag{3.7}$$

Now, let us consider $\lambda \in (-i\infty, -iN) \cup (iN, i\infty)$. In this case, the equation (3.7) is elliptic. Thus, changing the scale in x_3, we can perform the equation (3.7) as

$$\Delta u - \lambda^2 u = 0.$$

From [7], [22] we have that the continuous spectrum of the Laplace operator acting in $W_2^2(R^3)$, is composed if the points $\lambda^2 \in (-\infty, 0]$. Thus, the points $\lambda \in (-i\infty, i\infty)$ form the continuous spectrum of the differential operator in (3.7) when it is equivalent to the Laplace operator, in other terms, when $\lambda \in (-i\infty, -iN) \cup (iN, i\infty)$. Finally, we have that the points $\lambda \in [-iN, iN]$ belong to the essential spectrum of M^*, and the points $\lambda \in (-i\infty, -iN) \cup (iN, i\infty)$ belong to the continuous spectrum of M^*, and thus the Theorem is proved. ☑

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