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A result for approximating fixed points of generalized weak contraction of the integral-type by using Picard iteration

Un resultado para aproximar puntos fijos de contracción generalizada débil de la integral-tipo usando iteración de Picard

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Abstract. Following concepts of A. A. Branciari, y B. E. Rhoades, of in this paper, we shall establish a fixed point theorem by using a generalized weak contraction of integral type. Our result is a generalization of the classical Banach's fixed point theorem and other related results.

Key words and phrases. Fixed points, weak contraction of integral type, Picard iteration.

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Resumen. Siguiendo conceptos de A. A. Branciari, y B. E. Rhoades, en este artículo establecemos un teorema de punto fijo usando una contracción débil generalizada de tipo integral. Nuestro resultado es una generalización del clásico teorema del punto fijo de Banach y de otros resultados relacionados.

Palabras y frases clave. Puntos fijos, contracción débil de tipo integral, iteración de Picard.

1. Introduction

Let (X,d) be a complete metric space and $f : X \to X$ a selfmap of X. Suppose that $F_f = \{x \in X \mid f(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is established in Banach [2] by using the following contractive definition: there exists $c \in [0,1)$ (fixed) such that $\forall\, x, y \in X$, we have

$$d\left(f(x), f(y)\right) \leq c\, d(x,y)\,. \tag{1}$$

In a recent paper of Branciari [7], a generalization of Banach [2] is established. In that paper, Branciari [7] employed the following contractive integral inequality condition: there exists $c \in [0,1)$ such that $\forall x, y \in X$, we have

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt\,, \tag{2}$$

where $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$.

Rhoades [13] used the conditions

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt\,, \quad \forall x, y \in X\,, \tag{3}$$

where $m(x,y) = \max\left\{d(x,y), d(x,f(x)), d(y,f(y)), \frac{d(x,f(y))+d(y,f(x))}{2}\right\}$, and

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt\,, \quad \forall x, y \in X\,, \tag{4}$$

with $M(x,y) = \max\{d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))\}$, where $k \in [0,1)$ and $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ in both cases is as defined in (2). Condition (4) is the integral form of Ciric's condition in Ciric [10].

Literature abounds with several generalizations of the classical Banach's fixed point theorem since 1922. For some of these generalizations of the classical Banach's fixed point theorem and various contractive definitions that have been employed, we refer the readers to [1, 4, 5, 6, 3, 9, 10, 12, 14] and other references listed in the reference section of this paper.

In this paper, we shall establish a fixed point result similar to those of Branciari [7] and Rhoades [13] by employing a weak contraction of the integral type.

Our result is a generalization of the classical Banach's fixed point theorem [1, 2, 5, 15] as well as an extension of some results of Berinde [6], Berinde and Berinde [3], Branciari [7], Chatterjea [8], Kannan [11] and Zamfirescu [14].

The following definition is taken from Berinde [6, 3]:

Definition 1.1. *A single-valued mapping $f : X \to X$ is called a* weak contraction *or* $(\delta,\ L)-$weak contraction *if and only if there exist two constants, $\delta \in [0,1)$ and $L \geq 0$, such that*

$$d(f(x), f(y)) \leq \delta d(x,y) + Ld(y, f(x))\,, \quad \forall x, y \in X\,. \tag{5}$$

For the extension of the Banach's fixed point theorem in the sense of multivalued mapping, the reader is referred to Berinde and Berinde [3]. We shall employ the following definition to obtain our result:

Definition 1.2. *We shall say that a single-valued mapping $f : X \to X$ is a* generalized weak contraction of integral type *or* $(\delta,\ L)-$ generalized weak contraction of integral type *if and only if there exist constants $K \geq 0$, $L \geq 0$ and $\delta \in [0,1)$, such that $\forall x, y \in X$,*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \leq \delta \int_0^{d(x,y)} \varphi(t)dt + L\left(\int_0^{d(x,f(x))} \varphi(t)dt\right)^r \left(\int_0^{d(y,f(x))} \varphi(t)dt\right)^{[1-Kd(x,f(x))]}, \tag{6}$$

where $r \geq 0$, $1 - Kd(x, f(x)) > 0$ and $\varphi : I\!R^+ \to I\!R^+$ is a Lebesgue-integrable mapping which is summable, such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$ and nonnegative.

Remark 1.3. *The contractive condition (6) reduces to (5) if $r = K = 0$ and $\varphi(t) = 1, \forall t \in I\!R^+$. Also, if in condition (6) $r = K = 0$, $L = 2\delta$ and $\varphi(t) = 1$, $\forall t \in I\!R^+$, where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $0 \leq \delta < 1$, then we obtain the contractive condition employed by Zamfirescu* [14]. *See also Theorem 2.4 of Berinde* [5] *for the contractive definition of Zamfirescu* [14] *as well as the conditions on α, β* and *γ.*

Remark 1.4. *The contractive condition (6) does not require any additional condition for the uniqueness of the fixed point of f. This is an improvement on the result of Berinde* [6].

2. The main result

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ a $(\delta,\ L)-$generalized weak contraction of integral type. Let $\varphi : I\!R^+ \to I\!R^+$ be a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. Then, f has a unique fixed point $z \in X$ such that for each $x \in X$,* $\lim_{n\to\infty} f^n(x) = z$.

Proof. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = f(x_{n-1}) = f^n x_0$, $n = 1, 2, \ldots$, be the Picard iteration associated to f. From (6), we have that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \int_0^{d(f(x_{n-1}),f(x_n))} \varphi(t)dt \leq \delta \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$
$$+ L \left(\int_0^{d(x_{n-1},f(x_{n-1}))} \varphi(t)dt \right)^r \left(\int_0^{d(x_n,f(x_{n-1}))} \varphi(t)dt \right)^{1-Kd(x_{n-1},f(x_{n-1}))}$$
$$= \delta \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \leq \delta^2 \int_0^{d(x_{n-2},x_{n-1})} \varphi(t)dt \leq \cdots \leq \delta^n \int_0^{d(x_0,x_1)} \varphi(t)dt\,. \tag{7}$$

Taking the limit in (7) as $n \to \infty$ yields

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0\,,$$

since $\int_0^{\epsilon} \varphi(t)dt > 0$ for each $\epsilon > 0$. Therefore, it follows from (7) that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0\,. \tag{8}$$

We now establish that $\{x_n\}$ is a Cauchy sequence. Suppose it is not so. Then, there exists an $\epsilon > 0$ and subsequences $\{x_{m(p)}\}$ and $\{x_{n(p)}\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon\,, \qquad d(x_{m(p)}, x_{n(p)-1}) < \epsilon\,. \tag{9}$$

Again, by using (6), then we have that

$$\int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt = \int_0^{d(f(x_{m(p)-1}),f(x_{n(p)-1}))} \varphi(t)dt$$
$$\leq \delta \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt$$
$$+ L \left(\int_0^{d\left(x_{m(p)-1},x_{m(p)}\right)} \varphi(t)dt \right)^r \left(\int_0^{d(x_{n(p)-1},x_{m(p)})} \varphi(t)dt \right)^{\left[1-Kd(x_{m(p)-1},x_{m(p)})\right]}. \tag{10}$$

By using (8), we have that

$$1 - Kd\left(x_{m(p)-1}, x_{m(p)}\right) \to 1 \text{ as p} \to \infty\,, \tag{11*}$$

and

$$\left(\int_0^{d(x_{m(p)-1},x_{m(p)})} \varphi(t)dt\right)^r \to 0 \text{ as p} \to \infty, \tag{11**}$$

and also from (8), (9) and the triangle inequality, we obtain

$$\begin{aligned} d\left(x_{m(p)-1}, x_{n(p)-1}\right) &\leq d\left(x_{m(p)-1}, x_{m(p)}\right) + d\left(x_{m(p)}, x_{n(p)-1}\right) \\ &< d\left(x_{m(p)-1}, x_{m(p)}\right) + \epsilon \to \epsilon \text{ as p} \to \infty. \end{aligned} \tag{12}$$

Using (9), (11*), (11**) and (12) in (10), then we get

$$\int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d\left(x_{m(p)},x_{n(p)}\right)} \varphi(t)dt \leq \delta \int_0^{\epsilon} \varphi(t)dt, \tag{13}$$

from which we obtain $(1-\delta)\int_0^\epsilon \varphi(t)dt \leq 0$, leading to $1-\delta > 0$. But $\int_0^\epsilon \varphi(t)dt \leq 0$ and this is a contradiction by the condition on φ. Therefore, we must have that $\int_0^\epsilon \varphi(t)dt = 0$, that is, $\epsilon = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and hence convergent. Since (X, d) is a complete metric space, $\{x_n\}$ converges to some $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. Also, from (6), we have that

$$\begin{aligned} \int_0^{d(x_{n+1},f(z))} \varphi(t)dt &= \int_0^{d(f(x_n),f(z))} \varphi(t)dt \\ &\leq \delta \int_0^{d(x_n,z)} \varphi(t)dt + L\left(\int_0^{d(x_n,f(x_n))} \varphi(t)dt\right)^r \left(\int_0^{d(z,f(x_n))} \varphi(t)dt\right)^{[1-Kd(x_n,f(x_n))]} \\ &= \delta \int_0^{d(x_n,z)} \varphi(t)dt + L\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right)^r \left(\int_0^{d(z,x_{n+1})} \varphi(t)dt\right)^{[1-Kd(x_n,x_{n+1}]}. \end{aligned} \tag{14}$$

By taking the limits in (14) as $n \to \infty$, then we get

$$\int_0^{d(f(z),z)} \varphi(t)dt \leq 0, \tag{15}$$

and from (15), we obtain a contradiction again. Therefore, by the condition on φ, we have $\int_0^{d(z,f(z))} \varphi(t)dt = 0$, so that $d(z, f(z)) = 0$, or $z = f(z)$.

We now prove that f has a unique fixed point: Suppose this is not true. Then, there exist w_1, $w_2 \in F_f$, $w_1 \neq w_2$, $d(w_1, w_2) > 0$. Therefore, we obtain by (6) that

$$
\begin{aligned}
\int_0^{d(w_1,w_2)} \varphi(t)dt &= \int_0^{d(f(w_1),f(w_2))} \varphi(t)dt \\
&\leq \delta \int_0^{d(w_1,w_2)} \varphi(t)dt \\
&+ L \left(\int_0^{d(w_1,f(w_1))} \varphi(t)dt \right)^r \left(\int_0^{d(w_2,f(w_1))} \varphi(t)dt \right)^{[1-Sd(w_1,f(w_1))]},
\end{aligned}
$$

leading to $(1-\delta)\int_0^{d(w_1,w_2)} \varphi(t)dt \leq 0$, from which it follows that $1-\delta > 0$, but $\int_0^{d(w_1,w_2)} \varphi(t)dt \leq 0$. Therefore, by the condition on φ again, we get $\int_0^{d(w_1,w_2)} \varphi(t)dt = 0$ so that $d(w_1, w_2) = 0$, or $w_1 = w_2$. Hence, f has a unique fixed point. ☑

Remark 2.2. *Theorem 2.1 is a generalization and extension of the celebrated Banach's fixed point* [1, 2, 5, 15] *as well as an extension of the results of Branciari* [7], *Chatterjea* [8], *Kannan* [11] *and Zamfirescu* [14]. *Theorem 2.1 is also an extension of some results of Berinde* [6] *as well as Theorem 2 of Berinde and Berinde* [3]. *Theorem 2.4 of Berinde* [5] *is the result of Zamfirescu* [14].

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