Revista Colombiana de Matemáticas
Volumen 43(2009)1, páginas 43-54

On some properties of the solutions of the problem modelling stratified ocean and atmosphere flows in the half-space

Sobre algunas propiedades asintóticas de las soluciones del problema de modelado estratificado de los flujos del océano y de la atmósfera en el semi-espacio

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Abstract. We obtain a solution of the initial boundary value problem in the 3-dimensional half-space for a system of an exponentially stratified fluid in the gravity field. We prove the uniqueness of the weak solution in the class of growing functions. We also investigate the asymptotic behavior for the solution as $t \to \infty$.

Key words and phrases. Partial differential equations, weak solution, Fourier transform, stratified fluid, internal waves, generalized functions.

2000 Mathematics Subject Classification. 35Q35, 35B05, 76O26.

Resumen. Para el sistema de ecuaciones en derivadas parciales que modela el movimiento del líquido exponencialmente estratificado en el campo gravitacional, se soluciona el problema de frontera de valor inicial para el semi-espacio tridimensional. Se prueba la unicidad de la solución débil en una clase de funciones crecientes. También se investiga el comportamiento asintótico de la solución cuando $t \to \infty$.

Palabras y frases clave. Ecuaciones en derivadas parciales, solución débil, la transformada de Fourier, líquido estratificado, ondas internas, funciones generalizadas.

[a]This work was supported by Fondo de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes.

1. Introduction

The objective of this paper is to study the qualitative properties of the solutions of the system which describes small motions of stratified fluid in the homogeneous gravity field, such as existence, uniqueness and asymptotic behavior. We consider a system of equations of the form

$$\begin{cases} \rho_* \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} & = & 0\,, \\ \rho_* \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} & = & 0\,, \\ \rho_* \frac{\partial v_3}{\partial t} + g\rho + \frac{\partial p}{\partial x_3} & = & 0\,, \\ \frac{\partial \rho}{\partial t} - \frac{N^2 \rho_*}{g} v_3 & = & 0\,, \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} & = & 0\,, \end{cases} \tag{1}$$

in the domain $\left\{\left\{x = (x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, x_3 \geq 0\right\}, t > 0\right\}$, where $\vec{v}(x,t)$ is a velocity field with components $v_1(x,t), v_2(x,t), v_3(x,t)$, $p(x,t)$ is the scalar field of the dynamic pressure, $\rho(x,t)$ is the dynamic density and ρ_*, g, N are positive constants. The function $\rho(x,t)$ describes the deviations of the density from the stationary distribution which are caused by the motion of the fluid. The same is also valid for $p(x,t)$. The equations (1) are deduced in [1] under the assumption that the function of stationary distribution of density is performed by the function $\rho_* e^{-Nx_3}$. The system (1) can be considered as describing linearized motions of three-dimensional fluid in a homogeneous gravity field (atmosphere or ocean). This paper is inspired by the works [7] and [10], where the mathematical properties of rotating (not stratified) fluid were studied. The solutions for a Cauchy problem for (1), for the viscous case of intrusion were constructed in [8], and the uniqueness of the Cauchy problem for the viscous case was studied in [4]. The spectral properties of the differential operator of (1) were studied in [3], [5] and the Cauchy problem for system (1) was considered in [6]. In this work, we consider the system (1) in the semi-space

$$\mathbb{R}^3_+ = \left\{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2,\ x_3 \geq 0\right\},$$

together with the initial conditions

$$\begin{cases} \vec{v}\big|_{t=0} & = & \vec{v}^{\,0}(x)\,, \\ \rho\big|_{t=0} & = & 0\,, \end{cases} \tag{2}$$

and the boundary conditions

$$\left.\frac{\partial v_1}{\partial x_3}\right|_{x_3=0} = \left.\frac{\partial v_2}{\partial x_3}\right|_{x_3=0} = v_3|_{x_3=0} = 0\,. \tag{3}$$

We will construct explicitly the solution of the problem (1)-(3), prove its uniqueness in a class of increasing functions, and establish its velocity of decay for

$t \to \infty$. Without loss of generality, we may assume $\rho_* = g = N = 1$. This can be achieved by the following change of scale, where we modify the unknown functions of velocity and density and use the same notation for modified functions: $\vec{v} = \rho_* \vec{v}, \rho = g\rho$. In that way, instead of system (1), we will consider the following system:

$$\begin{cases} \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} &= 0\,, \\ \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} &= 0\,, \\ \frac{\partial v_3}{\partial t} + \rho + \frac{\partial p}{\partial x_3} &= 0\,, \\ \frac{\partial \rho}{\partial t} - v_3 &= 0\,, \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0\,. \end{cases} \tag{1'}$$

2. Construction of the solutions

First, we would like to recall some properties of the "unilateral" Fourier transforms, which can be found, for example, in [9]. For the functions $f(x) \in C^2 \cap L([0,\infty))$, $x \geq 0$, we can define sine –and cosine– Fourier transforms

$$F_{sin}[f](\xi) = \int_0^\infty f(x) \sin x\xi dx\,,$$

$$F_{cos}[f](\xi) = \int_0^\infty f(x) \cos x\xi dx\,,$$

for which the following relations are valid:

$$F_{\sin}[f'] = -\xi F_{\cos}[f], \qquad F_{\cos}[f'] = f(0) + \xi F_{\sin}[f]\,,$$

$$F_{\sin}[f''] = \xi f(0) - \xi^2 F_{sin}[f], \qquad F_{\cos}[f''] = f'(0) - \xi^2 F_{\cos}[f]\,,$$

$$F^{-1}_{sin}(u(\xi)F_{\sin}[v]) = \frac{1}{\pi}\int_0^\infty v(y)\int_0^\infty u(\xi)[\cos\xi(x-y) - \cos\xi(x+y)]d\xi dy\,,$$

$$F^{-1}_{\cos}(u(\xi)F_{\sin}[v]) = \frac{1}{\pi}\int_0^\infty v(y)\int_0^\infty u(\xi)[\sin\xi(x+y) - \sin\xi(x-y)]d\xi dy\,,$$

$$F^{-1}_{\cos}(u(\xi)F_{\cos}[v]) = \frac{1}{\pi}\int_0^\infty v(y)\int_0^\infty u(\xi)[\cos\xi(x+y) + \cos\xi(x-y)]d\xi dy\,.$$

Theorem 1. *Let $\vec{v}^0(x) \in C_0^\infty(\mathbb{R}^3_+)$ such that $div\,\vec{v}^0 = 0$, and let $\frac{\partial v_1^0}{\partial x_2} - \frac{\partial v_2^0}{\partial x_1} = 0$. Then, the solution of the problem (1'), (2), (3) has the following* *representation:*

$$v_{1,2}(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2} \Delta v_{1,2}^0(y)K_1(x'-y',\xi_3,t)[\cos\xi_3(x_3+y_3)+\cos\xi_3(x_3-y_3)]dy'd\xi_3dy_3,$$

$$v_3(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2} \Delta v_3^0(y)K_1(x'-y',\xi_3,t)[\cos\xi_3(x_3-y_3)-\cos\xi_3(x_3+y_3)]dy'd\xi_3dy_3\,,$$

$$\rho(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2} \Delta v_3^0(y)K_2(x'-y',\xi,t)[\cos\xi_3(x_3-y_3)-\cos\xi_3(x_3+y_3)]dy'd\xi_3dy_3\,,$$

$$p(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2} \frac{\partial v_3^0(y)}{\partial y_3}K_2(x'-y',\xi,t)[\cos\xi_3(x_3-y_3)-\cos\xi_3(x_3+y_3)]dy'd\xi_3dy_3\,,$$

where $x'=(x_1,x_2)$, $y'=(y_1,y_2)$, $\xi'=(\xi_1,\xi_2)$, $\xi=(\xi_1,\xi_2,\xi_3)$;

$$K_1\left(x'-y',\xi_3,t\right) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{i(\xi',x'-y')}\frac{1}{|\xi|^2}\cos\left(\frac{|\xi'|}{|\xi|}t\right)d\xi'\,, \tag{4}$$

$$K_2\left(x'-y',\xi_3,t\right) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{i(\xi',x'-y')}\frac{1}{|\xi||\xi'|}\sin\left(\frac{|\xi'|}{|\xi|}t\right)d\xi'\,. \tag{5}$$

Proof. We apply to the system (1′) the Fourier transform for x_1, x_2 and the Laplace transform for t. Thus we obtain

$$\begin{cases} \lambda\widehat{v}_1(\xi',x_3,\lambda)-\widehat{v}_1^0(\xi',x_3)+i\xi_1\widehat{p}(\xi',x_3,\lambda) &= 0\,,\\ \lambda\widehat{v}_2(\xi',x_3,\lambda)-\widehat{v}_2^0(\xi',x_3)+i\xi_2\widehat{p}(\xi',x_3,\lambda) &= 0\,,\\ \lambda\widehat{v}_3(\xi',x_3,\lambda)-\widehat{v}_3^0(\xi',x_3)+\widehat{\rho}(\xi',x_3,\lambda)+\frac{\partial}{\partial x_3}\widehat{p}(\xi',x_3,\lambda) &= 0\,,\\ \lambda\widehat{\rho}(\xi',x_3,\lambda)-\widehat{v}_3(\xi',x_3,\lambda) &= 0\,,\\ i\xi_1\widehat{v}_1(\xi',x_3,\lambda)+i\xi_2\widehat{v}_2(\xi',x_3,\lambda)+\frac{\partial}{\partial x_3}\widehat{v}_3(\xi',x_3,\lambda) &= 0\,. \end{cases} \tag{6}$$

Now, for the variable x_3, we apply the sine-Fourier transform to the third and the fourth equations of (6), and the cosine-Fourier transform to the rest of the equations of (6). In that way, the system (6) transforms into

$$\begin{cases} \lambda\overset{\bullet}{v}_1-\overset{\bullet}{v}{}_1^0+i\xi_1\overset{\bullet}{p} &= 0\,,\\ \lambda\overset{\bullet}{v}_2-\overset{\bullet}{v}{}_2^0+i\xi_2\overset{\bullet}{p} &= 0\,,\\ \lambda\overset{\bullet\bullet}{v}_3-\overset{\bullet\bullet}{v}{}_3^0+\overset{\bullet\bullet}{\rho}-\xi_3\overset{\bullet}{p} &= 0\,,\\ \lambda\overset{\bullet\bullet}{\rho}-\overset{\bullet\bullet}{v}_3 &= 0\,,\\ i\xi_1\overset{\bullet}{v}_1+i\xi_2\overset{\bullet}{v}_2+\xi_3\overset{\bullet\bullet}{v}_3 &= 0\,, \end{cases} \tag{7}$$

where the following notation is assumed:

$$\overset{\bullet}{f} = \widehat{f}(\xi_1,\xi_2,\xi_3,\lambda) = F^{\cos}_{x_3\to\xi_3}[F_{x_1\to\xi_1}[F_{x_2\to\xi_2}[L_{t\to\lambda}[f(x_1,x_2,x_3,t)]]]],$$
$$\overset{\bullet\bullet}{f} = \widehat{f}(\xi_1,\xi_2,\xi_3,\lambda) = F^{\sin}_{x_3\to\xi_3}[F_{x_1\to\xi_1}[F_{x_2\to\xi_2}[L_{t\to\lambda}[f(x_1,x_2,x_3,t)]]]].$$

Solving the system (7), we obtain

$$\overset{\bullet}{v}_i = \frac{\lambda^2|\xi|^2\,\overset{\bullet}{v}{}_i^0}{\lambda(\lambda^2|\xi|^2+|\xi'|^2)}, \quad i=1,2, \qquad \overset{\bullet\bullet}{v}_3 = \frac{\lambda^2|\xi|^2\,\overset{\bullet\bullet}{v}{}_3^0}{\lambda^2|\xi|^2+|\xi'|^2},$$
$$\overset{\bullet\bullet}{\rho} = \frac{|\xi|^2\,\overset{\bullet\bullet}{v}{}_3^0}{\lambda^2|\xi|^2+|\xi'|^2}, \qquad \overset{\bullet}{p} = \frac{\xi_3\,\overset{\bullet\bullet}{v}{}_3^0}{\lambda^2|\xi|^2+|\xi'|^2}.$$

Applying the inverse Laplace transform to the previous relations, we have

$$\overset{\bullet}{v}_i\,(\xi,t) = \overset{\bullet}{v}{}_i^0\,(\xi)\cos\left(\frac{|\xi'|}{|\xi|}t\right), \ i=1,2, \quad \overset{\bullet\bullet}{v}_3\,(\xi,t) = \overset{\bullet\bullet}{v}{}_3^0\,(\xi)\cos\left(\frac{|\xi'|}{|\xi|}t\right),$$
$$\overset{\bullet}{p}\,(\xi,t) = \overset{\bullet\bullet}{v}{}_3^0\,(\xi)\frac{\xi_3}{|\xi||\xi'|}\sin\left(\frac{|\xi'|}{|\xi|}t\right), \qquad \overset{\bullet\bullet}{\rho}\,(\xi,t) = \overset{\bullet\bullet}{v}{}_3^0\,(\xi)\frac{|\xi|}{|\xi'|}\sin\left(\frac{|\xi'|}{|\xi|}t\right).$$

From the Fourier transform properties, for $i=1,2$, we obtain

$$\overset{\bullet}{v}_i\,(\xi,t) = -\,\overset{\bullet}{\Delta v}{}_i^0\,(\xi)\frac{1}{|\xi|^2}\cos\left(\frac{|\xi'|}{|\xi|}t\right).$$

Using the convolution relation

$$\mathrm{F}^{\cos}_{x_3\to\xi_3}[v_i(x,t)] = -\iint_{\mathbb{R}^2}\overset{\bullet}{\Delta v}{}_i^0\,(y',\xi)\,K_1(x'-y',\xi_3,t)dy',$$

we finally have

$$v_i(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2}\Delta v_i^0(y)K_1\left(x'-y',\xi,t\right)\Big[\cos\xi_3(x_3+y_3)+$$
$$+\cos\xi_3(x_3-y_3)\Big]dy'd\xi_3dy_3,$$

where K_1 is given by (4).

In the same way, we can represent

$$\overset{\bullet\bullet}{v}_3\,(\xi,t) = -\,\overset{\bullet\bullet}{\Delta v}{}_3^0\,(\xi)\frac{1}{|\xi|^2}\cos\left(\frac{|\xi'|}{|\xi|}t\right),$$

and thus obtain

$$v_3(x,t) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty\iint_{\mathbb{R}^2} \triangle v_3^0(y)K_1(x'-y',\xi_3,t)\Big[\cos\xi_3(x_3-y_3)- \\ -\cos\xi_3(x_3+y_3)\Big]dy'd\xi_3dy_3\,.$$

The representations for ρ and p can be obtained analogously. Thus, the theorem is proved. ☑

The initial condition of the absence of curl component in (x_1, x_2) does not restrict the generality, since it was assumed only to represent the Fourier transform of $\overrightarrow{v}$ as a product of the Fourier transform of $\overrightarrow{v}^0$ and the function $\cos\left(\frac{|\xi'|}{|\xi|}t\right)$, rather than a lineal combination of the Fourier images of the coordinates v_i^0, $i = 1,2,3$. It can be easily seen that $K_1 = \frac{\partial K_2}{\partial t}$. To establish the t-asymptotic, in Section 3 we will find an explicit from of the kernel K_1.

3. Uniqueness of the solutions

Dot-multiplying the first three equations of $(1')$ by $\overrightarrow{v}$, and the fourth equation by ρ, we will have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\overrightarrow{v}|^2\right) + \overrightarrow{v}\nabla p + \rho v_3 = 0\,,$$
$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho^2\right) - \rho v_3 = 0\,,$$

from which, using the fifth equation of $(1')$ and the relation $\overrightarrow{v}\nabla p = \operatorname{div}(p\overrightarrow{v}) - p\operatorname{div}\overrightarrow{v}$, we can easily obtain the energy conservation law

$$\frac{\partial}{\partial t}E + \operatorname{div}\overrightarrow{D} = 0\,, \tag{8}$$

where $E = \frac{1}{2}\left(|\overrightarrow{v}|^2 + \rho^2\right)$, $\overrightarrow{D} = p\overrightarrow{v}$. Integrating (8) over $\mathbb{R}_+^3$ and using the Gauss theorem, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}_+^3}\overrightarrow{E}dx - \int_{x_3=0} pv_3dx_1dx_2 = 0\,.$$

By condition (3) the second term in the last equation is zero. Thus, for every $t > 0$ the following relation holds:

$$\int_{\mathbb{R}_+^3} E(x,t)dx = \int_{\mathbb{R}_+^3} E(x,0)dx\,. \tag{9}$$

In terms of L_2-norms, we can write (9) as follows:

$$\|\vec{v}(x,t)\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho(x,t)\|_{L_2(\mathbb{R}^3_+)} = \left\|\vec{v}^0(x)\right\|_{L_2(\mathbb{R}^3_+)}, \tag{10}$$

Since $\vec{v}^0 \in C_0^\infty(\mathbb{R}^3_+)$, all the norms in (10) are finite. The relation (10), evidently, determines the uniqueness of the solutions in $L_2(\mathbb{R}^3_+)$. However, we would like to establish the uniqueness of the solutions of (1′), (2), (3) in a broader class of increasing functions. Now, we will apply the results of [4] to the domain $Q^+ := \{x \in \mathbb{R}^3_+ \mid 0 \le t \le T\}$, using the unilateral Fourier transform.

Definition 1. *Let $g, \varphi \in C([0,\infty)) \longrightarrow \mathbb{R}$ be a pair of positive strictly growing functions such that $\lim\limits_{r\to\infty} \frac{g(r)}{r} \to 0$, $\int\limits_1^\infty \frac{r}{\varphi(r)}dr = \infty$. The set of locally bounded and measurable functions $\{\vec{v}, \rho, p\}$ is called a weak solution of (1′), (2), (3) if it satisfies (1′), (2), (3) in sense of generalized functions. We denote as $K^+_{\varphi,g}$, the class of uniqueness of the weak solutions, which is determined by the inequalities:*

$$\sup_{x_3\in\mathbb{R}^1_+} |\vec{v}(x,t)| \le C_1 \exp\{\varphi(|x'|)\}$$

$$\sup_{x_3\in\mathbb{R}^1_+} |\rho(x,t)| \le C_1 \exp\{\varphi(|x'|^2)\}$$

$$\sup_{x_3\in\mathbb{R}^1_+} |p(x,t)| \le g(|x'|)$$

almost everywhere in $\mathbb{R}^2 \times [0 \le t \le T]$.

Theorem 2. *The weak solution is unique in the class $K^+_{\varphi,g}$.*

Proof. Under the zero initial conditions we will prove that the solution is zero for all $t > 0$. We denote $\overset{*}{f}(x', \xi_3, t) = \mathrm{F}^{\cos}_{x_3\to\xi_3}[f(x,t)]$, $\overset{**}{f}(x', \xi_3, t) = \mathrm{F}^{\sin}_{x_3\to\xi_3}[f(x,t)]$. In this way, after applying the sine-Fourier transform in x_3 to the third and the fourth equations of (1′), and the cosine-Fourier transform to the rest of the equations of (1′), we will have

$$\begin{cases} \dfrac{\partial \overset{*}{v}_1}{\partial t} + \dfrac{\partial \overset{*}{p}}{\partial x_1} = 0, \\ \dfrac{\partial \overset{*}{v}_2}{\partial t} + \dfrac{\partial \overset{*}{p}}{\partial x_2} = 0, \\ \dfrac{\partial \overset{**}{v}_3}{\partial t} + \overset{**}{\rho} - \xi_3 \overset{*}{p} = 0, \\ \dfrac{\partial \overset{**}{\rho}}{\partial t} - \overset{**}{v}_3 = 0, \\ \dfrac{\partial \overset{*}{v}_1}{\partial x_1} + \dfrac{\partial \overset{*}{v}_2}{\partial x_2} + \xi_3 \overset{**}{v}_3 = 0. \end{cases} \tag{11}$$

Without loss of generality, just as in [4], we assume $\{\vec{v},\rho,p\}\in C^3\left(\mathbb{R}^2_{x'}\times\mathbb{R}_t\right)$. By consecutive differentiation and elimination of the functions $\vec{v},\rho$, we can reduce the problem (1′),(2),(3) to the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}\left(\Delta_2 \overset{*}{p} - \xi_3^2 \overset{*}{p}\right) + \Delta_2 \overset{*}{p} &= 0\,, \\ \left(\Delta_2 \overset{*}{p} - \xi_3^2 \overset{*}{p}\right)\Big|_{t=0} = \left(\Delta_2 \overset{*}{p_t} - \xi_3^2 \overset{*}{p_t}\right)\Big|_{t=0} &= 0\,, \end{cases} \tag{12}$$

where $\Delta_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$.

Since $p\in K^+_{\varphi,g}$, then the function $q(\xi,t) := |\xi|^2 \mathrm{F}_{x'\to\xi'}\left[\overset{*}{p}(x',\xi_3,t)\right]$, belongs to $S'\left(\mathbb{R}^2_{\xi'}\right)$. In this way, after applying the Fourier transform (12), we will have

$$\begin{cases} \frac{\partial^2 q(\xi,t)}{\partial t^2} + \frac{|\xi'|^2}{|\xi|^2} q(\xi,t) &= 0\,, \\ q|_{t=0} = q_t|_{t=0} &= 0\,, \quad t\in[0,T]\,. \end{cases} \tag{13}$$

The solution of problem (13) is unique in $S'\left(\mathbb{R}^2_{\xi'}\right)$ if the solution of the adjoint problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}\psi(\xi,t) + \frac{\xi'|^2}{|\xi|^2}\psi(\xi,t) &= 0\,, \\ \psi(\xi,t_0) &= \psi_0(\xi',\xi_3)\,, \\ \psi_t(\xi,t_0) &= \psi_1(\xi',\xi_3)\,, \quad t\in[0,t_0]\,, \quad t_0\in(0,T]\,. \end{cases} \tag{14}$$

belongs to $S'\left(\mathbb{R}^2_{\xi'}\right)$, $\xi_3\in\mathbb{R}^1_+$ for every $\psi_0,\psi_1\in S\left(\mathbb{R}^2_{\xi'}\right)$ ([4]).

Since the function $\exp\left(i\frac{|\xi'|}{|\xi}(t-t_0)\right)$ defines a bounded multiplication operator in $S'\left(\mathbb{R}^2_{\xi'}\right)$, then, evidently, $\psi\in S\left(\mathbb{R}^2_{\xi'}\right)$. By that way, $p(\xi',\xi_3,t)\equiv 0$ and $p^*(x',\xi_3,t)\equiv 0$ for all $t\in[0,T]$, $\xi_3\in\mathbb{R}^1_+$. Following the results of [4], we thus may conclude that $\overset{**}{\rho}\equiv\overset{*}{v}_1\equiv\overset{*}{v}_2\equiv\overset{**}{v}_3\equiv 0$. Thus, the Theorem is proved. ☑

4. Asymptotic behavior for large t

Theorem 3. *Suppose $\vec{v}^{\,0}(x)\in C_0^\infty\left(\mathbb{R}^3_+\right)$. Then, the solution of the problem (1′), (2), (3), decays as $\frac{1}{\sqrt{t}}$ for $t\to\infty$.*

Proof. As it can be seen from Theorem 1, we can limit our study to the investigation of the asymptotic behavior of the kernel function K_3, which can be defined as follows:

$$K_3(z,t) := \int_0^\infty K_1\left(z',\xi_3,t\right)\cos(\xi_3 z_3)\,d\xi_3\,.$$

Using polar coordinates and the formulas

$$\frac{1}{2\pi}\int_0^{2\pi}\cos\left(|x'|\,R\sin\varphi\right)d\varphi = J_0\left(|x'|\,R\right), \qquad \text{and}$$

$$\int_0^{\infty} J_0(aR)\cos(bR)dR = \begin{cases} \frac{1}{\sqrt{a^2-b^2}} & \text{si } a>b\,, \\ 0 & \text{si } a<b\,, \end{cases}$$

we calculate the kernel $K_3(x,t)$ as follows.

$$\begin{aligned} K_3(x,t) =& \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i(\xi,x)}\frac{1}{|\xi|^2}\cos\frac{|\xi'|}{|\xi|}td\xi = \\ =& \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{ix_3\xi_3}\iint_{\mathbb{R}^2}\frac{e^{i(x',\xi')}}{|\xi'|^2+\xi_3^2}\cos\left(\frac{|\xi'|t}{\sqrt{|\xi'|^2+\xi_3}}\right)d\xi' d\xi_3 = \\ =& \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{ix_3\xi_3}\cdot 2\int_0^{\infty}\int_0^{\pi}\left[\frac{R}{R^2+\xi_3^2}\cos(|x'|R\sin\varphi)\right. \\ & \left.\cos\left(\frac{Rt}{\sqrt{R^2+\xi_3}}\right)d\varphi dR d\xi_3\right] = \\ =& \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{R}{R^2+\xi_3^2}J_0(|x'|R)\cos\left(\frac{Rt}{\sqrt{R^2+\xi_3^2}}\right)\cos(\xi_3 x_3)d\xi_3 dR = \\ =& \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\pi/2} J_0(|x'|R)\cos(x_3R\tan\vartheta)\cos(t\cos\vartheta)d\vartheta dR = \\ =& \frac{1}{2\pi^2|x|}\int_{\lambda}^{1}\frac{w}{\sqrt{1-w^2}}\frac{\cos(tw)}{\sqrt{w^2-\lambda^2}}dw\,, \quad \lambda = |x_3|/|x|\,. \end{aligned} \tag{15}$$

Thus,

$$K_2(x,t) = \frac{1}{2\pi^2|x|}\int_{\lambda}^{1}\frac{1}{\sqrt{1-w^2}}\frac{\sin(tw)}{\sqrt{w^2-\lambda^2}}dw, \quad \lambda = |x_3|/|x|. \tag{16}$$

The expressions (15), (16) are convenient for studying the asymptotic behavior of the solution as $t\to\infty$. For obtaining the differentiability properties of solutions, it seems appropriate to use a different expression for the kernels K_1,

K_2:

$$K_2(x-y,t)=\frac{1}{4\pi}\frac{1}{r}\int_0^t J_0(t-\tau)J_0\left(\frac{\rho\tau}{r}\right)d\tau\,, \tag{17}$$

where $\rho^2=(x_3-y_3)^2$, $r^2=\sum_{k=1}^3(x_k-y_k)^2$, and J_0 is the Bessel function of order zero. Let us show first that the expressions (16) and (17) are equivalent.

For the function $T(t,\lambda)=\int_0^t J_0(t-\tau)J_0(\lambda\tau)d\tau$, $0\leq\lambda\leq 1$, we consider the Laplace transform with respect to t and denote it as $\widetilde{T}(p,\lambda)$. Thus we have

$$\widetilde{T}(p,\lambda)=\frac{1}{\sqrt{(p^2+1)}\sqrt{(p^2+\lambda^2)}}\,.$$

On the complex plane of the variable p the function $\widetilde{T}(p,\lambda)$ is multi-valued, having ramification points $\pm i$, $\pm i\lambda$. We cut the plane along the two segments of the imaginary axis, the first connecting the points i and $i\lambda$, and the second, connecting the points $-i$ and $-i\lambda$ and choose the branch of the function $\widetilde{T}(p,\lambda)$ for which $\widetilde{T}(0,\lambda)=1/\lambda$. Thus the chosen branch of $\widetilde{T}(p,\lambda)$ will be a one-valued analytic function of p.

Now let us consider the inverse Laplace transform (the Mellin transform):

$$T(t,\lambda)=\frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty}\widetilde{T}(p,\lambda)e^{-pt}dp\,,\quad a>0\,.$$

Since $\widetilde{T}$ behaves as $1/|p|^2$ for $|p|\to\infty$, we can reduce the domain of above integral to the segments which unite the ramification points of the integrand. Thus, putting $p=ik$ along the cuts, we can transform the last integral as

$$T(t,\lambda)=\frac{1}{i\pi}\int_\lambda^1\frac{e^{ikt}}{\sqrt{1-k^2}\sqrt{k^2-\lambda^2}}\,dk-\frac{1}{i\pi}\int_{-\lambda}^{-1}\frac{e^{ikt}}{\sqrt{1-k^2}\sqrt{k^2-\lambda^2}}\,dk\,.$$

Now, changing the variables from k to $-k$ in the last integral, we finally obtain

$$T(t,\lambda)=\int_0^t J_0(t-\tau)J_0(\lambda\tau)d\tau=\frac{2}{\pi}\int_\lambda^1\frac{1}{\sqrt{1-w^2}}\frac{\sin(tw)}{\sqrt{w^2-\lambda^2}}dw\,.$$

Thus we can conclude that the expressions (16) and (17) represent the same function. We observe that, for $\lambda=1$ the function T is expressed as

$$T(t,1)=\int_0^t J_0(t-\tau)J_0(\lambda\tau)d\tau=\sin t\,. \tag{18}$$

For the integral (15), the asymptotic can be obtained by the stationary phase method. As it can be seen in [2], the main term of the asymptotic expansion has the form

$$K_3(x,t) = \left(\sqrt{\frac{\pi}{2\lambda(1-\lambda^2)}}\right)\left(\frac{\cos\left(t\lambda - \frac{\pi}{4}\right)}{2\pi^2|x|\sqrt{t}}\right) + O\left(t^{-1}\right).$$

Thus, the Theorem is proved. ☑

Remark 1. *For $\lambda = 0$ we have $K_3(x,t) = \frac{1}{2\pi^2|x|}\int_0^1 \frac{\cos(tw)}{\sqrt{1-w^2}}dw$. Using the results of [2], we can extend Theorem (3) to this case.*

Using (17), (18) for $\lambda = 1$ we get

$$K_3(x,t) = \frac{\cos t}{4\pi|x|}, \quad \lambda = 1.$$

The last relation means that on the vertical axis ($\lambda = 1$), the solution acts as a stationary wave with no limit for $t \to \infty$.

Remark 2. *Summing up the obtained results for the solutions as $t \to \infty$, we may conclude that the solution reveals its irregular, non-uniform character. It tends to zero as a stationary wave with vanishing amplitude for $\lambda = 0$. It is a stationary wave which has no limit for $\lambda = 1$. And, finally, it represents the following remarkable wave process for $0 < \lambda < 1$. The equiphase surfaces of the wave (wave peaks), are described by the relation $\lambda = |x_3|/|x| = Const \times t^{-1}$ and are represented by conic surfaces with the vertex in the origin and the vertical axis which increase their opening with the growth of t, approaching the plane $x_3 = 0$. This geometric situation explains the lack of limit of the solution as $t \to \infty$ for $\lambda = 1$ (on the vertical axis).*

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(Recibido en mayo de 2008. Aceptado en marzo de 2009)

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