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Weakly compact cardinals and κ-torsionless modules

Cardinales compacto débiles y módulos κ-sin torsión

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ABSTRACT. We shall prove that every κ-torsionless R-module M of cardinality κ is torsionless whenever κ is weakly compact and $|R| < \kappa$. We also provide some closure properties for ultraproducts and direct products of κ-torsionless modules. We give an example of a κ-torsionless module which is not torsionless, when κ is not weakly compact.

Key words and phrases. Torsionless module, κ-torsionless module, weakly compact cardinal, slender rings.

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RESUMEN. En este trabajo se demuestra que todo R-módulo κ-sin torsión M de cardinalidad κ es sin torsión cuando $|R| < \kappa$. También establecemos algunas propiedades de cerradura para ultraproductos y productos directos de módulos κ-sin torsión. Damos un ejemplo de un módulo κ-sin torsión que no es sin torsión, cuando κ no es compacto débil.

Palabras y frases clave. Módulo sin torsión, módulo κ-sin torsión, cardinal compacto débil, anillo delgado.

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1. Introduction

This paper concerns the theory of κ-torsionless modules. In [3] we find the notion of κ-torsionless group which can be generalized to modules in a natural way: an R-module M is torsionless if it can be embedded in a product of copies of R. An R-module M is κ-torsionless if every R-submodule N of M of cardinality less than κ is torsionless. Clearly, every torsionless module M is κ-torsionless. It is natural to ask whether the converse is true.

In the above mentioned paper it is shown, among other things, that an ultraproduct of κ-torsionless abelian groups is κ-torsionless whenever κ is a strongly compact cardinal. We show in this work that the ultraproduct of a family of torsionless R-modules is torsionless whenever κ is measurable (a strongly compact cardinal is measurable, but the converse is not necessarily true). We prove a similar result for a family of κ-torsionless R-modules.

Wald [10] shows that every κ-torsionless group of cardinality κ, where κ is a weakly compact cardinal, is torsionless. He also gives a counterxample for κ not weakly compact.

In this note we further elaborate this result in the following way. If M is a κ-torsionless module M of cardinality κ and κ is weakly compact, then M is torsionless. Finally, we construct an example of a κ-torsionless R-module of cardinality κ which is not torsionless, where κ is not weakly compact. The latter result holds for slender rings, a large class of rings which contains $\mathbb{Z}$.

In section 2 we gather some auxiliary results about weakly compact cardinals, measurable and $\aleph_0$-measurable, that will be used throughout this paper. §3 is devoted to some characterizations and properties of torsionless modules.

Section 4 has a study of cartesian products and ultraproducts of torsionless and κ-torsionless modules. In §5 we say how to prove the afore mentioned result. Namely: if M is a κ-torsionless R-module, with κ weakly compact, $|M| = \kappa$ and $|R| < \kappa$, then M is torsionless. Finally, in section 6, the mentioned counterexample is constructed when κ is not a weakly compact cardinal following the example of Wald.

We have attempted to make this paper accessible both to algebraists and to set-theoreticians. Thus we have included some well known results with their full proofs, mainly those of set-theoretical nature.

2. Preliminaries

As usual $\aleph_0$ denotes the first infinite cardinal and $\mathbb{Z}$ the set of all integers.

If X is a set, $\wp(X)$ will denote the set of all subsets of X. If $f : X \to Y$ is a function, its image $Im(f)$ is $f[X] = \{f(x) : x \in X\}$.

If f is a module homomorphism, $Ker\, f$ is its kernel. If R is an associative ring which is not necessarily commutative, R_R means we think of R as of a right R-module. For every set x, $|x|$ denotes its cardinality. ZFC represents the usual axiomatization of set theory, namely the Zermelo-Fraenkel-Axiom of Choice system, which is the framework for this paper. The von Neumann hierarchy $\{V_\alpha : \alpha \in Or\}$, where Or is the class of all the ordinals, is defined by transfinite recursion as:

$$
\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \wp(V_\alpha) \\
V_\lambda &= \bigcup_{\beta<\lambda} V_\beta \quad \text{if } \lambda \text{ is a limit ordinal} \\
V &= \bigcup_{\alpha\in Or} V_\alpha,
\end{aligned}
$$

where V is the class (or universe) of all sets. If M is an R-module, $K \subseteq Y$, we denote by $\langle Y\rangle$ the R-submodule of M generated by Y.

Given a family $\{X_\alpha : \alpha \in I\}$ of sets, we form its cartesian product $X = \prod_{\alpha\in I} X_\alpha$, where every element $b \in X$ can be written componentwise as $b = (b(\alpha) : \alpha \in I)$ and $b(\alpha) \in X_\alpha$ for every $\alpha \in I$.

A crucial notion in this work is that of weakly compact cardinal, which we now define.

Definition 1. *Let κ be a cardinal. The language $L_{\kappa\kappa}$ generalizes the first order formal language: it contains predicate, function and constant symbols. It has κ variables and allows conjunction and disjunction of less than κ formulas and quantification of less than κ variables. We say that a set of $L_{\kappa\kappa}$-formulas is κ-satisfiable if every subcollection of less than κ of these formulas is satisfiable. Finally, the cardinal κ is weakly compact if and only if when a collection of $L_{\kappa\kappa}$-predicates is κ-satisfiable, then it is satisfiable, provided the collection has at most κ nonlogical symbols.*

Among the various characterizations for weakly compact cardinals the following two will be those we shall use.

Theorem 2 (Keisler). *The cardinal κ is weakly compact if and only if κ has the extension property: for each $R \subseteq V_\kappa$ there exists a transitive set $X \neq V_\kappa$ and $S \subseteq X$ such that*

$$\langle V_\kappa, \in, R\rangle \prec \langle X, \in, S\rangle,$$

where $\kappa \in X$.

Proof. See, for instance, [6, Theorem 4.5]. ☑

Definition 3. *We recall that for $x \subseteq Or$, $[x]^\gamma = \{y \subseteq x : y$ has ordinal type $\gamma\}$. The partition relation:*

$$\beta \longrightarrow (\alpha)^\gamma_\delta,$$

assures that for any $f : [\beta]^\gamma \to \delta$ there exists a set $H \in [\beta]^\alpha$ homogeneous for f. That is, $|f[[H]^\gamma]| \leq 1$.

Theorem 4. *The cardinal κ is weakly compact if and only if $\kappa \longrightarrow (\kappa)^2_2$.*

Proof. See, for instance, [6, Theorem 7.8]. ☑

We must pay attention to other large cardinals: the measurable ones.

Definition 5. *An ultrafilter $\mathfrak{U}$ is κ-complete if for each $\lambda < \kappa$ and every family $\{U_\alpha : \alpha < \lambda\} \subseteq \mathfrak{U}$, we have that $\bigcap_{\alpha<\lambda} U_\alpha \in \mathfrak{U}$.*

Definition 6. *An uncountable cardinal κ is measurable if there exists a nonprincipal ultrafilter which is κ-complete in κ.*

Proposition 7. *If κ is measurable, then κ is weakly compact.*

Proof. See, for instance, [6, Proposition 4.3]. ☑

Lemma 8. *([6, Exercise 2.7]) An ultrafilter $\mathfrak{U}$ in κ is κ-complete if and only if for every $\lambda < \kappa$ and $\bigcup\{U_\alpha : \alpha < \lambda\} \in \mathfrak{U}$, there exists $\alpha < \lambda$ such that $U_\alpha \in \mathfrak{U}$.*

Proof. We first assume that $\mathfrak{U}$ is κ-complete, that $\lambda < \kappa$ and $\bigcup\{U_\xi : \xi < \lambda\} \in \mathfrak{U}$. Suppose that $U_\xi \notin \mathfrak{U}$ for every $\xi < \lambda$. Since $\mathfrak{U}$ is an ultrafilter, $\kappa - X_\xi \in \mathfrak{U}$ for every $\xi < \lambda$. Therefore,

$$\bigcap_{\xi<\lambda}(\kappa - U_\xi) = \kappa - \bigcup_{\xi<\kappa} U_\xi = U \in \mathfrak{U}.$$

But then $U \cap \bigcup_{\xi<\lambda} U_\xi = \emptyset \in \mathfrak{U}$, a contradiction.

Conversely, suppose that the condition holds. We prove that $\mathfrak{U}$ is κ-complete. To reach a contradiction let us suppose that there are $\lambda < \kappa$ and $\{U_\alpha : \alpha < \lambda\} \subseteq \mathfrak{U}$ such that $\bigcap_{\alpha<\lambda} U_\alpha \notin \mathfrak{U}$. Then,

$$\kappa - \bigcap_{\alpha<\lambda} U_\alpha = \bigcup_{\xi<\lambda}(\kappa - U_\xi) = U \in \mathfrak{U}.$$

But, according to the lemma's condition, $\kappa - U_\xi \in \mathfrak{U}$, for some $\xi < \lambda$, and this yields a contradiction. ☑

Definition 9. *The uncountable cardinal κ is $\aleph_0$-measurable if there exists a nonprincipal ultrafilter which is $\aleph_1$-complete in κ.*

It is clear that every measurable cardinal κ is $\aleph_0$-measurable. In case there were $\aleph_0$-measurable cardinals, we identify the least of them as $\varkappa$.

The following are well known results, but we prove them for the sake of completeness.

Theorem 10. *Let $\mathfrak{U}$ be an $\aleph_1$-complete utrafilter on the uncountable cardinal κ. Then, $\mathfrak{U}$ is $\varkappa$-complete.*

Proof. Let $\lambda < \varkappa$. We shall prove that $\bigcap_{\alpha<\lambda} U_\alpha \in \mathcal{U}$. Let's suppose this is not true, then, according to Theorem 8, there exists a family $W = \{X_\alpha : \alpha < \lambda\}$ whose union belongs to $\mathcal{U}$, but $X_\alpha \notin \mathcal{U}$ for every $\alpha < \lambda$. Without loss of generality we can assume that the X_α are pairwise disjoint.

Set

$$\mathcal{V} = \left\{A \subseteq W : \bigcup A \in \mathcal{U}\right\}.$$

It is clear that $W \in \mathcal{V}$ and that no finite subset of W belongs to $\mathcal{V}$. Let's suppose that $A \in \mathcal{V}$ and that $A \subseteq B \subseteq W$. Then $\bigcup A \in \mathcal{U}$ and $\bigcup A \subseteq \bigcup B$, so that $\bigcup B \in \mathcal{U}$; hence, $B \in \mathcal{V}$. If $A \subseteq W$, then $\bigcup A \cup \bigcup (W - A) = \varkappa$. Therefore, $\bigcup A \in \mathcal{U}$ or $\bigcup (W - A) \in \mathcal{U}$. Thus, $A \in \mathcal{V}$ or $W - A \in \mathcal{V}$.

Finally, suppose that $A_n \in \mathcal{V}$, for each $n \in \omega$. Then, for each $n \in \omega$, we have that $\bigcup A_n \in \mathcal{U}$, which implies, by virtue of the $\aleph_1$-completeness of $\mathcal{U}$, that

$$\bigcap_{n\in\omega} \left(\bigcup A_n\right) \in \mathcal{U}.$$

Since the sets in W are pairwise disjoint, we obtain that

$$\bigcap_{n\in\omega} \left(\bigcup A_n\right) = \bigcup \left(\bigcap_{n\in\omega} A_n\right),$$

from which it follows that $\bigcap_{n\in\omega} A_n \in \mathcal{V}$.

We have proved that $\mathcal{V}$ is a nonprincipal ultrafilter which is $\aleph_1$-complete in W. Since W has cardinality λ and $\lambda < \varkappa$, we have a contradiction due to the definition of $\varkappa$. Consequently, $\mathcal{U}$ is λ-complete. ☑

Lemma 11. *Every cardinal $\lambda > \varkappa$ is $\aleph_0$-measurable.*

Proof. Let $\lambda > \varkappa$ and let $\mathcal{U}$ be an ultrafilter that is $\aleph_1$-complete in $\varkappa$. Take the family

$$\mathcal{F} = \{W \subseteq \lambda : \exists X \in \mathcal{U}(X \subseteq W)\}.$$

Let $\mathcal{V}$ be the ultrafilter generated by $\mathcal{F}$. Then, $\mathcal{V}$ is an ultrafilter which is $\aleph_1$-complete in λ. Therefore, λ is $\aleph_0$-measurable. ☑

We know that if κ is weakly compact, then it is regular and a strong limit. That is, for every $\lambda < \kappa$ we have that $2^\lambda < \kappa$. Besides, if $H(\kappa)$ represents the set of sets whose transitive closure has cardinality less than κ, then $V_\kappa = H(\kappa)$, where V_κ is the κ-th level in von Neumann's hierarchy.

3. Torsionless Modules

In this section we provide the definitions and some important results about torsionless modules.

Definition 12. *Let R be a ring with 1 and let M be a unitary right R-module. The dual of M is the left R-module $M^* = Hom_R(M, R)$. If M is a left R-module, its dual is a right R- module. The dual of M^* is a right R-module M^{**} and there is a natural homomorphism $\sigma : M \to M^{**}$ given by $\sigma(m)(f) = f(m)$ for every $f \in M^*$. If the homomorphism σ is an isomorphism we say that M is a* reflexive *module, while if σ is injective we say that M is* semirreflexive *or a right* torsionless *R-module.*

The following is a well known result (see [7]).

Theorem 13. *For every right R-module M the sequence*

$$0 \longrightarrow M^* \xrightarrow{\sigma} M^{***}$$

is exact and splits, where σ is the natural homomorphism from M^* to its double dual. In particular, M^* is a torsionless module.

Let $X \subset M$. We denote by $l(X)$ the set $l(X) = \{f \in M^* : f(x) = 0, \forall x \in X\}$. If $X \subset M^*$, $r(X)$ is the set $r(X) = \{x \in M : f(x) = 0, \forall f \in X\}$.

We now give several characterizations for torsionless modules.

Proposition 14. *The following conditions for a right R-module M are equivalent.*

(i) M is a torsionless module.

(ii) $r(M^*) = 0$.

(iii) If $0 \neq a \in M$, then there is an $f \in M^*$ such that $f(a) \neq 0$.

(iv) M can be embedded in a direct product of copies of R_R.

(v) For every nontrivial homomorphism of right R-modules $M_0 \longrightarrow M$, there is a homomorphism $M \longrightarrow R$ such that the composite homomorphism $M_0 \longrightarrow M \longrightarrow R$ is not zero.

(vi) M is a submodule of a dual module.

Proof. (i) $\Rightarrow$ (ii). Let $x \in r(M^*)$. That is, $f(x) = 0$ for every $f \in M^*$, so $x \in \bigcap_{f \in M^*} Ker f = (0)$, since M is a torsionless module. Therefore, $x = 0$.

(ii) $\Rightarrow$ (iii). Let $a \in M$, $a \neq 0$, then $a \notin r(M^*)$. Therefore, there is at least one $f \in M^*$ such that $f(x) \neq 0$.

(iii) $\Rightarrow$ *(iv)*. Let us consider the product $\prod_{f\in M^*} R_f$ with $R_f = R$, and define the homomorphism $\lambda : M \to \prod_{f\in M^*} R_f$ given by $\lambda(m)_f = f(m) \in R_f$. Observe that

$$\lambda(m) = 0 \Leftrightarrow f(m) = 0, \qquad \forall f \in M^*.$$

By (iii):

$$\lambda(m) = 0 \Longleftrightarrow m = 0.$$

That is $Ker\,\lambda = (0)$. So λ is injective.

(iv) $\Rightarrow$ *(v)*. Let $\varphi : M_0 \longrightarrow M$ be a nonzero homomorphism and $m_0 \in M_0$ such that $\varphi(m_0) = m \neq 0$. Then, $0 \neq \lambda(m) \in \prod_{f\in M^*} R_f$. We take a nonzero component of $\lambda(m)$, say $\lambda(m)(f_0) \in R_{f_0}$. Then, the homomorphism $\psi : M \longrightarrow R_{f_0}$ given by $\psi = \pi_{f_0} \circ \lambda$, is such that $\psi \circ \varphi$ is nonzero.

(v) $\Rightarrow$ *(i)* Let us suppose that M is not torsionless. That is, $M_0 := Ker\,\sigma \neq (0)$, where σ is the homomorphism from Definition 12. So, the inclusion $M_0 \hookrightarrow M$ is a nonzero homomorphism. Then, by (v), there is a homomorphism $\varphi : M \longrightarrow R$ ($\varphi \in M^*$) such that $\varphi \restriction M_0 : M_0 \longrightarrow R$ is nonzero. That is, there exists $m_0 \in M_0$ such that $\varphi(m_0) \neq 0$; but this contradicts the fact that $m_0 \in Ker\,\varphi$, because in that case $\varphi(m_0) = 0 \in M^{**}$ and $\sigma(m_0)(\varphi) = \varphi(m_0) = 0$.

(i) $\Rightarrow$ *(vi)* . If M is a torsionless module, M is isomorphic to $\sigma(M)$ which is a submodule of the dual of M^*.

(vi) $\Rightarrow$ *(i)*. If M is a submodule of N^*, then invoking Theorem 13 we conclude that M is a submodule of a torsionless module. Hence, M is a torsionless module. ☑

It is now an easy matter to prove the following properties.

(1) If M is a right R-module, we have that $Ker\,\sigma = \bigcap_{b\in M^*} Ker\,b$, where σ is the homomorphism from Definition 12.

(2) M is a torsionless module if and only if $\bigcap_{f\in M^*} Ker\,f = (0)$.

(3) If N is a submodule of M and M is a torsionless module, then N is a torsionless module.

(4) R is a torsionless R-module since $R^{**} = R$.

(5) Quotients of torsionless modules are not necessarily torsionless modules:

Example 15. *The $\mathbb{Z}$-module $\mathbb{Z}$ is torsionles. However, $\mathbb{Z}/n\mathbb{Z}$ is not a torsionless group. Indeed, $(\mathbb{Z}/n\mathbb{Z})^* = (0)$ from which $\sigma = 0$ follows. That is, σ is not injective.*

The following proposition tells us when a quotient module is a torsionless module.

Proposition 16. *Let M be a right R-module and N a submodule of M. Then the following conditions are equivalent:*

(i) M/N *is a torsionless module.*

(ii) *If* $m \in M - N$, *then there is* $f \in M^*$ *such that* $f(m) \neq 0$, *and* $f[N] = 0$.

(iii) $r(l(N)) = N$.

Proof. $(i) \Rightarrow (ii)$. Since M/N is torsionless for $a \in M - N$, that is, $0 \neq \bar{a} = a + N \in M/N$, there is a homomorphism $\bar{f} : M/N \to R$ with $\bar{f}(\bar{a}) \neq 0$. We define $f(m) = \bar{f}(\bar{m})$. It is clear that $f \in M^*$. Then, $f(a) = \bar{f}(\bar{a}) \neq 0$. Besides, $f(n) = \bar{f}(\bar{n}) = 0$ for every $n \in N$. Therefore, $f[N] = 0$.

$(ii) \Rightarrow (iii)$. In general we have that $N \subseteq r(l(N))$. We shall show that $r(l(N)) \subseteq N$. Let $x \in r(l(N))$. If $x \notin N$, then, by (ii), there is $f \in M^*$ such that $f(x) \neq 0$ and $f[N] = 0$. This contradicts the fact that $x \in r(l(N))$ since $f \in l(N)$.

$(iii) \Rightarrow (i)$ Let us suppose that M/N is not a torsionless module, hence there exists $\bar{m} = m + N$, with $m \notin N$ such that for every $f^* \in (M/N)^*$, $f^*(\bar{m}) = 0$.

Claim. $m \in r(l(N))$.

Indeed, if $f \in l(N)$, we define $f^* \in (M/N)^*$ by $f^*(x + N) = f(x)$. This function is well defined since $f \in l(N)$. Then, $f^*(\bar{m}) = f(m) = 0$. That is, $m \in r(l(N))$, in oposition to (iii), since $m \in r(l(N)) - N$. ☑

4. κ-torsionless modules

In this section we investigate some properties of torsionless and κ-torsionless modules mainly related with cartesian products and with ultraproducts module κ-complete ultrafilters.

Definition 17. *Let κ be a regular cardinal and M an R-module. We say that M is a κ-torsionless module if every submodule N of M with $|N| < \kappa$ is torsionless.*

If λ is a singular cardinal, we say that an R-module M is λ-torsionless if M is κ-torsionless for every regular cardinal $\kappa < \lambda$.

Clearly, if M is torsionless, then it is κ-torsionless. The converse, does not necessarily hold as we shall see later on. However, the answer depends on a large cardinal. Namely, on a weakly compact cardinal.

Note that κ-torsionless is not preserved under homomorphic images, since every R-module is the image of a free R-Module, which, being torsionless, is κ-torsionless.

However this class behaves well with respect to cartesian products:

Theorem 18. *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of R-modules that are κ-torsionless. Then $M = \prod_{\alpha<\kappa} M_\alpha$ is κ-torsionless.*

Proof. Let $L < M$ be a submodule of M with $|L| < \kappa$ and $b \in L$, $b \neq 0$. Since $b \neq 0$, there is $\alpha < \kappa$ such that $b(\alpha) \neq 0$. Take the projection $p_\alpha : M \to M_\alpha$ and note that $p_\alpha[L] \leq M_\alpha$ and that $|p[L]| < \kappa$. Then there is, by hypothesis, an $f_\alpha : M_\alpha \to R$ such that $f(b(\alpha)) \neq 0$. Let $f = f_\alpha \circ p_\alpha \restriction L : L \to R$. We have that $f(b) \neq 0$, as we require, and so M is κ-torsionless. ☑

An appeal to this proof establishes a similar result for torsionless modules.

We now turn to ultraproducts of modules. We first investigate the ultraproduct of torsionless modules. In the following result we use ideas from [9]:

Theorem 19. *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of torsionless R-modules with $|R| = \lambda < \kappa$, where κ is a measurable cardinal. If $\mathfrak{U}$ is a κ-complete ultrafilter on κ, then*

$$\overline{M} = \prod_{\alpha<\kappa} M_\alpha/\mathfrak{U}$$

is a torsionless R-module.

Proof. Let $M = \prod_{\alpha<\kappa} M_\alpha$, $\overline{M} = \prod_{\alpha<\kappa} M_\alpha/\mathfrak{U}$, $\overline{a} \in \overline{M}$, $\overline{a} \neq 0$ and let $f : \overline{M} \to M$ be a function that chooses representatives. That is, if $\overline{m} \in \overline{M}$, then $f(\overline{m})$ chooses a representative $m \in M$, in such a way that if $\pi : M \to M/\mathfrak{U}$ is the canonical homomorphism, then $\pi(m) = \overline{m}$. Since π is an R-homomorphism and $\overline{a} \neq 0$, we infere that $f(\overline{a})(\alpha) \neq 0$ for κ coordinates. Actually,

$$I = \{\alpha < \kappa : f(\overline{a})(\alpha) \neq 0\} \in \mathfrak{U}.$$

For each $i \in I$ we choose R-homomorphisms $g_\alpha : M_\alpha \to R$, such that $g_\alpha(a(\alpha)) \neq 0$. Thus,

$$\{\alpha < \kappa : g_\alpha(a(\alpha)) \neq 0\} \in \mathfrak{U}. \tag{1}$$

We define an R-homomorphism $g : M \to R^\kappa$, by:

$$(g(m)(\alpha)) : \alpha < \kappa) = (g_\alpha(m(\alpha)) : \alpha < \kappa),$$

for every $m \in M$. Letting $\overline{g(a)}$ be the class in $R^\kappa/\mathfrak{U}$ of $(g_\alpha(a(\alpha)) : \alpha < \kappa) \in R^\kappa$ and invoking (1) we obtain that $\overline{g(a)} \neq 0$.

We have the maps:

(1) $f : \overline{M} \to M$;

(2) $g : M \to R^\kappa$;

(3) $\nu : R^\kappa \to R^\kappa/\mathfrak{U}$, the canonical R-homomorphism.

Hence, $h_1 = \nu \circ g \circ f : \overline{M} \to R^\kappa/\mathcal{U}$, is a well defined R-homomorphism such that $h_1(\overline{a}) \neq 0$. We need an R-homomorphism $h_2 : R^\kappa/\mathcal{U} \to R$ with $h_2(h_1(\overline{a})) \neq 0$.

For each $\overline{x} \in R^\kappa/\mathcal{U}$, we let $\overline{f}(\overline{x}) = \vec{x} \in R^\kappa$, so that $\vec{x} = (x(\alpha) : \alpha < \kappa)$ and every $x(\alpha) \in R$, where $\overline{f}$ is a function that chooses a representative, like f. Now let

$$U_r^{\overline{x}} = \{\alpha < \kappa : x(\alpha) = r\},$$

hence, $\{U_r^{\overline{x}} : r \in R\}$ is a partition of κ with less than κ elements, since $|R| < \kappa$. By Lema 8, there exists $r \in R$ such that $U_r^{\overline{x}} \in \mathcal{U}$. We now define $h_2(f(\overline{x})) = r$.

It suffices to show that h_2 is an R-homomorphism. Let $\overline{x}, \overline{y} \in R^\kappa/\mathcal{U}$. We must verify that $h_2(f(\overline{x}+\overline{y})) = h_2(f(\overline{x})) + h_2(f(\overline{y}))$. So, let us suppose that $h_2(f(\overline{x})) = r_x$ and $h_2(f(\overline{y})) = r_y$. It is enough to prove that $U_{r_x+r_y}^{\overline{x}+\overline{y}} \in \mathcal{U}$, for which it is sufficient to prove that

$$U_{r_x}^{\overline{x}} \cap U_{r_y}^{\overline{y}} \subseteq U_{r_x+r_y}^{\overline{x}+\overline{y}}.$$

If $\alpha \in U_{r_x}^{\overline{x}} \cap U_{r_y}^{\overline{y}}$, then $x(\alpha) = r_x$ and $y(\alpha) = r_y$, so that $(x+y)(\alpha) = r_x + r_y$. Hence, $\alpha \in U_{r_x+r_y}^{\overline{x}+\overline{y}}$.

Now let $s \in R$ and $\overline{x} \in R^\kappa/\mathcal{U}$, we will show $h_2(sf(\overline{x})) = sh_2(f(\overline{x}))$. Assume that $h_2(f(\overline{x})) = r_x$. If $\alpha \in h_2(\overline{x})$, then $x(\alpha) = r_x$, so $sx(\alpha) = sr_x$, therefore $\alpha \in U_{sr_x}^{s\overline{x}}$. Then, $U_{sr_x}^{s\overline{x}} \in \mathcal{U}$, from which it follows, by definition of h_2, that

$$h_2(sf(\overline{x})) = sr_x = sh_2(f(\overline{x})).$$

Consequently, h_2 is an R-homomorphism. Therefore, we have found an R-homomorphism $h : \overline{M} \to R$ such that $h(\overline{a}) \neq 0$. We apply h_1 and h_2 consecutively to $\overline{a}$ and get $h_2 \circ h_1(\overline{a}) \neq 0$. ☑

We can obtain a similar result for κ-torsionless modules.

Theorem 20. *Let κ be a measurable cardinal and let $\{M_\alpha : \alpha < \kappa\}$ be a family of κ-torsionless R-modules with $|R| = \lambda < \kappa$. If $\mathcal{U}$ is a κ-complete ultrafilter on κ, then*

$$\overline{M} = \prod_{\alpha<\kappa} M_\alpha/\mathcal{U}$$

is a κ-torsionless R-module.

Proof. Let $M = \prod_{\alpha<\kappa} M_\alpha$, and let $\overline{N}$ be an R-submodule of $\overline{M}$ of cardinality less than κ, take $\overline{a} \in \overline{N}$, with $\overline{a} \neq \overline{0}$, let $\pi : M \to \overline{M}$ be the canonical homomorphism, and let $f : \overline{N} \to M$ be a function that chooses representatives in M for each $\overline{n} \in \overline{N}$. Then $f(\overline{a})(\alpha) \neq 0$ for κ coordinates. Otherwise, $\overline{a} = \overline{0}$, since $\mathcal{U}$ is a κ-complete ultrafilter, hence, its members $U \in \mathcal{U}$ have cardinality κ.

Consider the following family of sets:

$$A_\alpha = \{f(\overline{n})(\alpha) : \overline{n} \in \overline{N}\},$$

for each $\alpha < \kappa$. Then $|A_\alpha| < \kappa$ and so, every R-module $N_\alpha = \langle A_\alpha \rangle$ in M_α has cardinality less than κ. Since every M_α is κ-torsionless, it follows that each N_α ($\alpha < \kappa$) is torsionless. For each $\alpha < \kappa$ we have an R-homomorphism $g_\alpha : N_\alpha \to R$ such that $g_\alpha(f(\overline{a})(\alpha)) \neq 0$ whenever $\alpha < \kappa$ with $\overline{a}(\alpha) \neq 0$.

We define a function $g : N \to R^\kappa$, where $N = \prod_{\alpha<\kappa} N_\alpha$, in the following way: if $x \in N$, $g(x) = (g_\alpha(x(\alpha)) : \alpha < \kappa) \in R^\kappa$. Clearly, g is an R-homomorphism. We now define $h_1 = \nu \circ g \circ f : \overline{N} \to R^\kappa/\mathfrak{U}$, where $\nu : R^\kappa \to R^\kappa/\mathfrak{U}$ is the canonical quotient R-homomorphism. We can easily verify that h_1 is a well defined R-homomorphism.

We still have to construct an R-homomorphism $h_2 : R^\kappa/\mathfrak{U} \to R$ with $h_2(h_1(\overline{a})) \neq 0$. But this can be achieved as in the proof of Theorem 18. We apply h_1 and h_2 consecutively to $\overline{a}$ and get $h_2 \circ h_1(\overline{a}) \neq 0$. ☑

5. κ is a weakly compact cardinal

We aim to prove that every κ-torsionless R-module of cardinality κ is torsionless, whenever $|R| < \kappa$ and κ is weakly compact. To start with, we recall several notions for the benefit of the reader. We begin with the notion of elementary substructure.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathfrak{L}$-structures for some first order language $\mathfrak{L}$. We say that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$, in symbols $\mathfrak{A} \prec \mathfrak{B}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for any $\mathfrak{L}$-formula $\varphi(v_0, \ldots, v_n)$ and any elements $x_0, \ldots, x_n$ from the universe of $\mathfrak{A}$ the following condition holds

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_n] \Leftrightarrow \mathfrak{B} \models \varphi[x_0, \ldots, x_n].$$

Recall the definition of ordered pair of sets: if a, b are sets, then

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

We can define the following functions

$$(a, b)_0 = a$$
$$(a, b)_1 = b.$$

Hence,

$$z = (x)_0 \Leftrightarrow \exists\, y(x = (z, y));$$

in a similar way we can define $z = (x)_1$.

We also require to describe an ordinal. That is, a transitive set which is well ordered by $\in$:

$$\begin{aligned} Or(x) \Leftrightarrow & \forall y \forall z (y \in x \wedge z \in y \rightarrow z \in x) \wedge \\ & \forall y \in x \forall z \in x (z = y \vee z \in y \vee y \in z) \wedge \\ & \forall z (z \subseteq x \wedge \neg (z = \emptyset) \rightarrow \exists y \in z \forall u \in z (y = u \vee y \in u)), \end{aligned}$$

while a limit ordinal is described as:

$$\mathrm{Lim}(x) \Leftrightarrow Or(x) \wedge \forall z \in x \exists y \in x (z < y).$$

To continue, we describe a homomorphism between an R-module N and the ring R. We first observe that being a function is described as:

$$\mathrm{Fun}(f) \Leftrightarrow \forall x \in f \exists y \exists z (x = (y, z) \wedge ((y_1, z) \in f \wedge (y_2, z) \in f \rightarrow y_1 = y_2)).$$

As usual we use the notation $f(x) = y$ for $(x, y) \in f$.

We have the following relations associated to the concept of function:

$$\begin{aligned} \mathrm{dom}(f) = z \Leftrightarrow & \mathrm{Fun}(f) \wedge [\forall x \in z \exists y ((x, y) \in f) \wedge ((x, y) \in f \rightarrow x \in z)], \\ \mathrm{ran}(f) = z \Leftrightarrow & \mathrm{Fun}(f) \wedge [\forall y \in z \exists x ((x, y) \in f) \wedge ((x, y) \in f \rightarrow y \in z)]. \end{aligned}$$

Our aim now is to describe an R-homomorphism. We suppose that R is a ring and that N is a left R-module.

Let $\mathrm{Hom}(f, R)$ be the formula:

$$\begin{aligned} \mathrm{Hom}(f, R, N) \Leftrightarrow & \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = N \wedge \mathrm{ran}(f) \subseteq R \wedge \\ & [\forall n_1, n_2 \in N (f(n_1 + n_2) = f(n_1) + f(n_2)) \wedge \\ & \forall r \in R \forall n \in N (f(rn) = r f(n))]. \end{aligned}$$

Now, let us suppose that M is an R-module of cardinality κ, a regular cardinal. We can enumerate M as

$$M = \{m_\alpha : \alpha < \kappa\}.$$

With this we can now define a family of submodules of M in the following way (recall that κ is regular): we define, by transfinite recursion,

$$\begin{aligned} M_0 = & \langle \{m_0\} \rangle, \\ M_{\alpha+1} = & \langle \{m_\beta\} \cup M_\alpha \rangle, \\ M_\alpha = & \bigcup_{\beta < \alpha} M_\beta \qquad \text{if } \beta \text{ is a limit ordinal}, \end{aligned}$$

where m_β in the second equation is the least element, in our enumeration of M, in $M - M_\alpha$.

If $\beta < \alpha$ then M_β is a submodule of M_α. If M is a κ-torsionless R-module, we know that for each $\alpha < \kappa$ and for each $m \in M_\alpha$, $m \neq 0_M$ there is an R-homomorphism $f : M_\alpha \to R$ such that $f(m) \neq 0_R$.

We are ready to prove our main result of this section:

Theorem 21. *Suppose that κ is a weakly compact cardinal, and that M is a κ-torsionless R-module of cardinality κ, where R is a ring of cardinality less than κ. Then, M is torsionless.*

Proof. Without loss of generality we may assume that $R \in V_\kappa$, where V_κ is the κ-th level in von Neumann's hierarchy, and that $M = V_\kappa$. Now consider the following structure in the language $\mathcal{L} = \{\in, T\}$, where T is a unary predicate.

$$W = \langle V_\kappa, \in, \{(\alpha, M_\alpha) : \alpha < \kappa\}\rangle.$$

Let

$$\overline{M} = \{(\alpha, M_\alpha) : \alpha < \kappa\}.$$

Thus, $W \models \overline{M}x$ means that $x \in V_\kappa$ and $x \in \overline{M}$, according to W.

The following claims are easily verified:

The second coordinates of the elements of $\overline{M}$ are R-modules:

$$W \models \forall x(\overline{M}x \to \text{“}(x)_1 \text{ is an } R\text{-module”} \tag{2}$$

The first coordinates of the elements of $\overline{M}$ are ordinals:

$$W \models \forall x(\overline{M}x \to Or((x)_0)) \tag{3}$$

If $\alpha < \beta$, then $M_\alpha < M_\beta$:

$$W \models \forall x \forall y(\overline{M}x \wedge \overline{M}y \wedge (x)_0 < (y)_0 \to (x)_1 \leq (y)_1) \tag{4}$$

If β is limit, M_β is the union of the previous M_α:

$$\begin{aligned} W \models &\forall x(\overline{M}x \wedge Lim((x)_0) \to \\ &\quad \forall z \in (x)_1 \exists y(\overline{M}y \wedge (y)_0 < (x)_0 \wedge z \in (y)_1). \end{aligned} \tag{5}$$

Every ordinal in W enumerates some M_α:

$$W \models \forall \alpha \exists x(Or(\alpha) \wedge \overline{M}x \to (x)_0 = \alpha). \tag{6}$$

Every M_α is torsionless:

$$W \models \psi_1, \tag{7}$$

where

$$\psi_1 \equiv \forall x \forall y (\overline{M}x \wedge y \in (x)_1 \wedge y \neq 0_{(x)_1} \rightarrow \exists f(\mathrm{Hom}(f, R, (x)_1) \wedge \neg(f(y) = 0_R))).$$

We now use Keisler's extension property (Theorem 2). Note that $M = \bigcup_{\alpha<\kappa} M_\alpha$. We know that there exists $\langle X, \in, N\rangle$ with X transitive, $\kappa \in X$, $N \subseteq X$, $V_\kappa \subseteq X$ and

$$\langle V_\kappa, \in, \overline{M}\rangle \prec \langle X, \in, N\rangle.$$

Since $\kappa \in X$ we have that $M = M_\kappa$, by (5) and (6), because κ is limit. From (7) we conclude that M is torsionless, which is what we wanted to prove. ☑

6. κ is not a weakly compact cardinal

In this section we construct an example of an R-module M of cardinality κ which is κ-torsionless, but not torsionless. For that we require a cardinal κ which is neither weakly compact, nor $\aleph_0$-measurable. The reason for κ not to be weakly compact is clear from the result from the previous section. While the reason for it not to be $\aleph_0$-measurable will be a consequence of the theorem stated below. We shall use a nice Wald's example ([10]), but we need several additional facts, because the original example works for abelian groups and we will deal with R-modules.

We recall that if $\{M_\alpha : \alpha < \lambda\}$ is a family of torsionless R-modules, the cartesian product $M = \prod_{\alpha<\lambda} M_\alpha$ is torsionless, so its dual M^* is different from 0 (the 0 homomorphism). However, if $f \in M^*$ is such that $f \restriction \bigoplus_{\alpha<\lambda} M_\alpha = 0$, would it be true that $f = 0$? The following result gives a negative answer to this question, when κ is $\aleph_1$-measurable. In fact, we have the answer for $f \restriction \bigoplus^{(\kappa)}_{\alpha<\kappa} M_\alpha = 0$, where

$$\bigoplus_{\alpha<\kappa}^{(\kappa)} M_\alpha = \left\{ m \in \prod_{\alpha<\kappa} M_\alpha : |\{\alpha < \kappa : m(\alpha) \neq 0\}| < \kappa \right\}.$$

To prove our theorem we use an idea of Fuchs ([4]).

Theorem 22. *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of torsionless R-modules, where κ is a cardinal that is $\aleph_1$-measurable and such that $|R| < \kappa$. Then, there is an R-homomorphism $f : \prod_{\alpha<\kappa} M_\alpha \rightarrow R$ such that $f \restriction \bigoplus^{(\kappa)}_{\alpha<\kappa} M_\alpha = 0$ but $f \neq 0$. In particular, $M^* \neq 0$.*

Proof. Every factor M_α is torsionless, so we can choose an R-homomorphism $f_\alpha : M_\alpha \rightarrow R$ that is not the zero homomorphism. Since κ is $\aleph_0$-measurable, there exists an $\aleph_1$-complete ultrafilter $\mathfrak{U}$ in κ.

We enumerate R as $R = \{r_\alpha : r_\alpha < \lambda\}$, where $\lambda = |R|$. We can assume $r_0 = 0$. For $x \in M$ and for each $\alpha < \lambda$ we define

$$U^x_{r_\alpha} = \{\nu < \kappa : f_\nu(x(\nu)) = r_\alpha\}.$$

The sets $U^x_{r_\alpha}$ form a partition of κ. So, according to Theorem 8 there is $\alpha < \lambda$ such that $U^x_{r_\alpha} \in \mathcal{U}$. We make $f(x) = r_\alpha$. This defines a function $f : M \to R$.

Claim 1. f is an R-homomorphism.

Proof of Claim 1. Let $x, y \in \prod_{\alpha<\kappa} M_\alpha$ and suppose that $f(x) = r_\alpha$ and $f(y) = r_\beta$. Observe that

$$U^x_{r_\alpha} \cap U^y_{r_\beta} \subseteq U^{x+y}_{r_\alpha+r_\beta},$$

because if $\nu \in U^x_{r_\alpha} \cap U^y_{r_\beta}$, we can conclude that $f_\nu(x(\nu)) = r_\alpha$ and $f_\nu(y(\nu)) = r_\beta$, so that $f_\nu(x(\nu) + y(\nu)) = r_\alpha + r_\beta$ and, hence, $\nu \in U^{x+y}_{r_\alpha+r_\beta}$. Then $f(x+y) = f(x) + f(y)$.

Next we shall prove that $f(r_\alpha x) = r_\alpha f(x)$ for every $r_\alpha \in R$ and every $x \in M$. Let $f(x) = r_\beta$. It follows that

$$U^x_{r_\beta} \subseteq U^{r_\alpha x}_{r_\alpha r_\beta},$$

since if $\nu \in U^x_{r_\beta}$, we get $f_\nu(x(\nu)) = r_\beta$, so that $f_\nu(r_\alpha x(\nu)) = r_\alpha f_\nu(x(\nu)) = r_\alpha r_\beta$ and, hence, $\nu \in U^{r_\alpha x}_{r_\alpha r_\beta}$, we obtain that $f(r_\alpha x) = r_\alpha f(x)$.

Claim 2. $f \restriction \bigoplus^{(\kappa)}_{\alpha<\kappa} M_\alpha = 0$.

Proof of Claim 2. Let $x \in \bigoplus^{(\kappa)}_{\alpha<\kappa} M_\alpha$. So, the support of x

$$\mathrm{Supp}(x) = \{\alpha < \kappa : x(\alpha) \neq 0\},$$

has cardinality less than κ. Therefore,

$$U^x_0 = \{\nu < \kappa : x(\nu) = 0\},$$

has cardinality κ. Moreover, its complement has cardinality less than κ, hence it cannot be a member of $\mathcal{U}$. It follows that $U^x_0 \in \mathcal{U}$, so $f(x) = 0$.

Claim 3. f is not the zero homomorphism.

Proof of Claim 3. We must exhibit an element $x \in M$ such that $f(x) \neq 0$. Now, for each $\alpha < \kappa$ we know that $f_\alpha : M_\alpha \to R$ is not zero, so there is an element $x(\alpha) \in M_\alpha$, with $x(\alpha) \neq 0$. Note that, with any of these elements $x \in M$,

$$U^x_0 = \{\nu < \kappa : x(\nu) = 0\}$$

is empty. So $f(x) \neq 0$, as required.

From these three claims the theorem follows at once. ☑

Now we turn to construct the announced example at the beginning of this section. Consider a not weakly compact cardinal κ. According to Theorem 7, κ is not measurable. We will construct an example of an R-module of cardinality κ which is κ-torsionless but $U^* = 0$. Invoking previous results we can assume that κ is not $\aleph_0$-measurable.

We will use the following filter: let $\mathcal{B} = \{X \subseteq \kappa : |\kappa - X| < \kappa\}$. It is clear that $\mathcal{B}$ has the finite intersection property. So it generates a filter $\mathcal{F}$.

Theorem 23. *Let $\{M_\alpha : \alpha < \kappa\}$ be a family of κ-torsionless R-modules, where κ is a cardinal and let $\mathcal{F}$ be the filter described above . Then*

$$\prod_{\alpha<\kappa} M_\alpha/\mathcal{F}$$

is a κ-torsionless R-module.

Proof. Let $M = \prod_{\alpha<\kappa} M_\alpha$, $\overline{M} = M/\mathcal{F}$ and let $\pi : M \to \overline{M}$ be the canonical homomorphism. Now let $\overline{N}$ be an R-submodule of $\overline{M}$ of cardinality less than κ and take $\overline{a} \in \overline{N}$, with $\overline{a} \neq \overline{0}$. We will give an R-monomorphism

$$h : \overline{N} \to R^\kappa.$$

Let $f : \overline{N} \to M$ be a function that chooses representatives in M for each $\overline{n} \in \overline{N}$. For $\overline{n}_1, \overline{n}_2 \in \overline{N}$ and $r \in R$, we define

$$\begin{aligned} A_{\overline{n}_1,\overline{n}_2} &= \{\alpha < \kappa : f(\overline{n}_1 + \overline{n}_2)(\alpha) - f(\overline{n}_1)(\alpha) - f(\overline{n}_2)(\alpha) \neq 0\}, \\ B_{\overline{n},r} &= \{\alpha < \kappa : rf(\overline{n})(\alpha) - f(r\overline{n})(\alpha) \neq 0\}. \end{aligned}$$

Let $A = \bigcup_{\overline{n}_1,\overline{n}_2 \in \overline{N}} A_{\overline{n}_1,\overline{n}_2}$ and let $B = \bigcup_{r\in R, \overline{n}\in \overline{N}} B_{\overline{n},r}$. Since $|R|, |\overline{N}| < \kappa$, it follows that $|A \cup B| < \kappa$. We let $C = A \cup B$ and define $h : \overline{N} \to M$ by

$$h(\overline{n}) = \begin{cases} f(\overline{n})(\alpha), & \text{if } \alpha \in (\kappa - C) \\ 0, & \text{if } \alpha \in C. \end{cases}$$

Claim 1. h is an R-homomorphism.

Proof of Claim 1. It is easily verified that h is well defined. Let $\overline{n}_1, \overline{n}_2 \in \overline{N}$. We shall show that

$$h(\overline{n}_1 + \overline{n}_2)(\alpha) = h(\overline{n}_1)(\alpha) + h(\overline{n}_2)(\alpha) \tag{8}$$

If $\alpha \in C$, (8) does hold. If $\alpha \in \kappa - C$, then

$$\begin{aligned} h(\overline{n}_1 + \overline{n}_2)(\alpha) &= f(\overline{n}_1 + \overline{n}_2)(\alpha), \\ h(\overline{n}_1)(\alpha) &= f(\overline{n}_1)(\alpha), \\ h(\overline{n}_2) &= f(\overline{n}_2)(\alpha), \end{aligned}$$

and, since $\alpha \in \kappa - C$,

$$f(\overline{n}_1 + \overline{n}_2)(\alpha) = f(\overline{n}_1)(\alpha) + f(\overline{n}_2)(\alpha).$$

So, (8) is true.

Now let $\overline{n} \in \overline{N}$ and $r \in R$. We must certify that

$$h(r\overline{n})(\alpha) = rh(\overline{n})(\alpha). \tag{9}$$

If $\alpha \in C$, (9) is immediate. If $\alpha \in \kappa - C$,

$$\begin{aligned} h(r\overline{n})(\alpha) &= f(r\overline{n})(\alpha), \\ rh(\overline{n})(\alpha) &= rf(\overline{n})(\alpha), \end{aligned}$$

and, since $\alpha \in \kappa - C$, it follows that $f(r\overline{n})(\alpha) = rf(\overline{n})(\alpha)$. Therefore (9) is valid.

Claim 2. h is a monomorphism.

Proof of Claim 2. Let $\overline{n}_1$ and $\overline{n}_2$ be two different elements in $\overline{N}$.

Consider the following subset of κ:

$$\text{Diff} = \{\alpha < \kappa : f(\overline{n}_1)(\alpha) \neq f(\overline{n}_2)(\alpha)\}.$$

This set has cardinality κ. Since we have that $|C| < \kappa$ we can find $\alpha^* \in \text{Diff}-C$, so that $f(\overline{n}_1)(\alpha^*) \neq f(\overline{n}_2)(\alpha^*)$. Hence $h(\overline{n}_1)(\alpha^*) \neq h(\overline{n}_2)(\alpha^*)$, from which we conclude that $h(\overline{n}_1) \neq h(\overline{n}_2)$.

We have given an embedding $h : \overline{N} \to R^\kappa$, so $\overline{N}$ is a torsionless R-module. ☑

Let us recall the notion of weak sum:

Definition 24. *Let κ and λ be cardinals. We define:*

$$\kappa^{\underset{\smile}{\lambda}} = \sum_{\rho<\lambda} \kappa^\rho,$$

where the sum runs over the cardinals $\rho < \kappa$.

The following is a well known result, but we did not find an appropriate reference.

Recall that μ is a strong limit cardinal if for every cardinal $\lambda < \mu$, $2^\lambda < \mu$ holds. It follows that every strong limit cardinal is a limit cardinal.

Theorem 25. *Let κ be a cardinal. Then, $\kappa = 2^{\underset{\smile}{\kappa}}$ if and only if $\kappa = \kappa^{\underset{\smile}{\kappa}}$ or κ is a strong limit cardinal.*

Proof. If $\kappa = \kappa^{\underset{\smile}{\kappa}}$ or κ is a strong limit cardinal, it is clear that $\kappa = 2^{\underset{\smile}{\kappa}}$. Conversely, let us supposse that $\kappa = 2^{\underset{\smile}{\kappa}}$. If κ is regular, then

$$\kappa^{\underset{\smile}{\kappa}} \leq \left(2^{\underset{\smile}{\kappa}}\right)^{\underset{\smile}{\kappa}} = 2^{\underset{\smile}{\kappa}} = \kappa.$$

We wish to prove that κ is strong limit, assume that κ is singular. If this were not the case, there would be a cardinal $\mu < \kappa$ with $cf(\kappa) \leq \mu < \kappa$ and $\kappa \leq 2^\mu$. In which case, $2^\mu = \kappa$ and

$$\kappa < \kappa^{cf(\kappa)} \leq \kappa^\mu = (2^\mu)^\mu = 2^\mu = \kappa.$$

☑

We will use as a ring R a slender ring. This notion is due to J. Loś.

Definition 26. *An R-module M is slender if for every R-homomorphism $f : R^{\aleph_0} \to M$ it satisfies the condition that $f(m_l(i)) = 0$ for every $l \in \mathbb{N}$ except for finitely many l's, where*

$$m_l(i) = \begin{cases} 0, & \text{if} \quad l \neq i \\ 1, & \text{if} \quad l = i. \end{cases}$$

As examples of slender R-modules we have $\mathbb{Z}$ and every countable integer domain that is not a field (see [8]). Even more can be said: If R is a pid, R is slender whenever R is not a complete valuation domain, which follows from [5, Lemma 6.6, p.555].

In order to build our example we require the following result which can be obtained from [2] together with [1].

Theorem 27. *Let M be a slender R-module and let κ be a cardinal that is not $\aleph_0$-measurable. For every family $\{M_\alpha : \alpha < \kappa\}$ and for every $f : \prod_{\alpha<\kappa} M_\alpha \to M$, if $f \restriction \bigoplus_{\alpha<\kappa} M_\alpha = 0$, then $f = 0$.* ☑

As we already mentioned the example that we develop here originated in [10]. However, me make it more general, since it shall work for a broader class of rings not only for $\mathbb{Z}$.

Example 28. *There exists an R-module M of cardinality κ, where κ is neither weakly compact nor $\aleph_0$-measurable but weakly inaccessible, such that M is κ-torsionless but not torsionless.*

Recall that a weakly compact cardinal must satisfy the arrow relation:

$$\kappa \longrightarrow (\kappa)^2_2,$$

(Theorem 4), so in our case, given that κ is not weakly compact, there must be a map $p : [\kappa]^2 \to 2$ for which there is no subset of κ of cardinality κ that is homogeneous with respect to p.

Let $\{M_\alpha : \alpha < \kappa\}$ be an arbitrary family of torsionless R-modules with $|M_\alpha| \leq \kappa$ for every $\alpha < \kappa$, where R is a slender ring (viewed as an R-module) and such that $|\{\alpha < \kappa : |M_\alpha| = \kappa\}| = \kappa$. We form the product

$$M = \prod_{\alpha<\kappa} M_\alpha.$$

Let $\mathfrak{F}$ be the filter in κ described above, and let

$$\overline{M} = M/\mathfrak{F}$$

be the reduced product of M module $\mathfrak{F}$. The canonical quotient function is denoted by π, that is to say, $\pi : M \to \overline{M}$. We will build an R-module L such that it is a submodule of $\overline{M}$, with $|L| = \kappa$, and such that L is κ-torsionless, but $L^* = 0$.

For $\alpha < \kappa$ and $i \in \{0, 1\}$, let

$$A^i_\alpha = \{\beta < \kappa : p(\{\alpha, \beta\}) = i\}.$$

If $\mu < \kappa$ and $f : \mu \to \{0, 1\}$, set

$$N_f = \bigcap_{\alpha<\mu} A^{f(\alpha)}_\alpha.$$

If $f : \mu \to \{0, 1\}$ and $g : \nu \to \{0, 1\}$, $f \subseteq g$ occurs when g extends f. We say that f and g are noncomparable when $f \not\subseteq g$ and $g \not\subseteq f$.

Claim 1. If $f \subseteq g$, then $N_g \subseteq N_f$.

Proof of Claim 1. Let $\beta \in N_g$, then $\beta \in A^{g(\alpha)}_\alpha$ for every $\alpha \in \text{dom}(g)$. We must show that $\beta \in A^{f(\alpha)}_\alpha$ for any $\alpha \in \text{dom}(f)$. If $g(\gamma) = i$, then $f(\gamma) = i$, since g extends f. We know that $p(\{\alpha, \beta\}) = i$. Since $A^{f(\alpha)}_\alpha = A^{g(\alpha)}_\alpha$, we have that $\beta \in A^{f(\alpha)}_\alpha$. Therefore, $\beta \in N_f$.

Claim 2. If f, g are noncomparable, then $N_f \cap N_g = \emptyset$.

Proof of Claim 2. Let us assume, to get a contradiction, that $\gamma \in N_f \cap N_g$, then $\gamma \in A^{f(\alpha)}_\alpha$. That is, $p(\{\alpha, \gamma\}) = f(\alpha)$, for every $\alpha \in \text{dom}(f)$ and for every $\gamma \in A^{g(\alpha)}_\alpha$. Hence, $p(\{\alpha, \gamma\}) = g(\alpha)$ for every $\alpha \in \text{dom}(g)$. Suppose that $\text{dom}(f) \leq \text{dom}(g)$. Thus $f(\alpha) = g(\alpha)$ for any $\alpha \in \text{dom}(f)$, so $f \subseteq g$, which is a contradiction.

Claim 3. $N_f \cap \mu = \emptyset$ if $\mu = \text{dom}(f)$.

Proof of Claim 3. Otherwise, there would be a $\gamma \in \mu \cap N_f$. That is, we could calculate $p(\{\gamma, \gamma\}) = p(\{\gamma\})$, which is not possible.

We will use the following notation: if $B \subseteq \kappa$, we define the unitary vector $u_B \in M$ by:

$$u_B(\alpha) = \begin{cases} 1, & \text{if} \quad \alpha \in B \\ 0, & \text{another case.} \end{cases}$$

We write u_f to mean u_{N_f}.

Given $f : \mu \to \kappa$ and $\nu \in \mu$, we define the function $f_\nu : \nu + 1 \to \{0, 1\}$ by

$$f_\nu(\alpha) = \begin{cases} f(\alpha), & \text{si} \quad \alpha < \nu \\ 0, & \text{si} \quad \alpha = \nu \wedge f(\nu) = 1 \\ 1, & \text{si} \quad \alpha = \nu \wedge f(\nu) = 0. \end{cases}$$

We let $f_\mu = f$. By the definition of these functions it is clear that the N_{f_ν} are pairwise disjoint for any $\nu \in \mu$. We now define a homomorphism $F_f : \prod_{\alpha \leq \mu} M_\alpha \to \prod_{\alpha < \kappa} M_\alpha$ by

$$F_f(x) = \sum_{\nu \in \mu + 1} x(\nu) u_{f_\nu}.$$

The composition $F_f \circ \pi$ is an R-homomorphism $\overline{F}_f : \prod_{\alpha \leq \mu} M_\alpha \to \overline{M}$.

Claim 4. Let $\lambda \in \mu$, then

$$N_{f \restriction \lambda} = \bigcup_{\nu \in [\lambda, \mu + 1)} N_{f_\nu} \cup (N_{f \restriction \lambda} \cap (\mu - \lambda)), \tag{10}$$

where $\nu \in [\lambda, \mu + 1)$ means that the union runs over the ordinals $\nu \geq \lambda$ and $\nu < \mu + 1$.

Proof of Claim 4. Since $\nu \geq \lambda$, we have that $f \restriction \lambda \subseteq f_\nu$ and, hence, that $N_{f_\nu} \subseteq N_{f \restriction \lambda}$. Consequently, the right hand side of (10) is contained in the left hand side.

Now, let $\alpha \in N_{f \restriction \lambda}$. First recall that, by definition,

$$N_{f_\mu} = N_f = \bigcap_{\nu \in \mu} A_\nu^{f(\nu)}.$$

Let us suppose that $\alpha \notin N_{f_\mu}$, then there is $\nu \in \mu$ (we can choose the least possible) so that $\alpha \notin A_\nu^{f(\nu)}$. By definition of f_ν and from the fact that $\kappa = A_\nu^0 \cup \{\nu\} \cup A_\nu^1$, it follows that $\alpha = \nu$ or $\alpha \in N_{f_\nu}$. Given that $\alpha \notin A_\nu^{f(\nu)}$, $\alpha \in A_\nu^{f_\nu(\nu)}$ (if $\alpha \neq \nu$). If $\alpha = \nu$, we have that $\nu < \mu$, $\nu > \lambda$. So, $\alpha \in N_{f \restriction \lambda} \cap (\mu - \lambda)$.

Claim 5.

$$\overline{F}_f \left(\sum_{\nu \in [\lambda, \mu + 1)} u_\nu \right) = \overline{u}_{f \restriction \lambda}.$$

Proof of Claim 5. Note that $\mu \in \kappa$, therefore $N_{f\restriction\lambda} \cap (\mu - \lambda)$ has cardinality less than κ. Then, by construction of $\overline{M}$ and by the definition of $\overline{F}_f$, we get

$$\overline{F}_f\left(\sum_{\nu\in[\lambda,\mu+1)} u_\nu\right) = \overline{\sum_{\nu\in[\lambda,\mu+1)} u_{f_\nu}} = \overline{u}_{f\restriction\lambda},$$

where $\overline{u}_{f\restriction\lambda}$ is the class of $u_{f\restriction\lambda}$ in $\overline{M}$.

Given the function $f : \mu \to \{0,1\}$, we develope the functions f^0 and f^1:

$$f^0 = f \cup \{(\mu, 0)\}$$
$$f^1 = f \cup \{(\mu, 1)\},$$

so that $f^1 \restriction \mu = f^0 \restriction \mu = f$ and $f^i(\mu) = i$ for $i \in \{0,1\}$. We already mentioned that $\kappa = A^0_\mu \cup \{\mu\} \cup A^1_\mu$ thus $N_f = N_{f^0} \cup (N_f \cap \{\mu\}) \cup N_{f^1}$. Then,

$$\overline{u}_f = \overline{u}_{f^0} + \overline{u}_{f^1}$$

in $\overline{M}$.

We now define our R-submodule $L < \overline{M}$ as the R-submodule generated by all the images of the homomorphisms $\overline{F}_f$:

$$L = \left\langle \sum_f Im(\overline{F}_f) \right\rangle,$$

where f varies over all the functions $f : \mu \to \{0,1\}$ for $\mu \in \kappa$. For each κ we have $2^{|\mu|}$ functions $f : \mu \to 2$. So, we have $2^{\underset{-}{\kappa}}$ functions $f : \nu \to \{0,1\}$ for some $\nu < \kappa$.

Notice that

$$|\mathrm{Im}(\overline{F}_f)| \leq |\mathrm{dom}(\overline{F}_f)| = \left|\prod_{\alpha\leq\mu} M_\alpha\right| \leq \kappa^\mu \leq \kappa^{\underset{-}{\kappa}} = \kappa.$$

Therefore,

$$|L| \leq 2^{\underset{-}{\kappa}} \sum_{\mu<\kappa} \kappa^\mu = \kappa^{\underset{-}{\kappa}} = \kappa.$$

By hypothesis, we have at least κ R-modules M_α of cardinality κ. This, together with the definition of the R-homomorphisms $\overline{F}_f$, gives $|L| \geq \kappa$. We conclude that $|L| = \kappa$.

Note that L is a κ-torsionless R-module, according to Theorem 23. So, it only remains to be proved that L is not torsionless. In fact, we will prove that $L^* = 0$. That is, that there are no homomorphisms, other than the zero homomorphism, from L to R. So, toward a contradiction suppose that $f \in L^*$ and that f is not the zero homomorphism.

We construct a function $h : \kappa \to \{0,1\}$ such that for some $\mu^* \in \kappa$

$$f(\overline{u}_{h\restriction\mu^*}) \neq 0,$$

for every $\mu \geq \mu^*$, with $\mu \in \kappa$.

By hypothesis there must be a $\mu \in \kappa$ and some $g : \mu \to \{0,1\}$ such that

$$h[Im(\overline{F}_g)] \neq 0.$$

Assume that $h(\overline{u}_{g_\nu}) = 0$ for every $\nu \in \mu+1$. Consider the homomorphism $h \circ \overline{F}_g : \prod_{\alpha\leq\mu} M_\alpha \to R$.

Claim 6. $h \circ \overline{F}_g(u_\nu) = h(\overline{e}_{g_\nu}) = 0$ for every $\nu \in \mu+1$.

Proof of Claim 6. Recall that all the coordinates of u_ν are zero except for the ν-th one which is 1. Therefore, in

$$F_g(u_\nu) = \sum_{\gamma\in\mu+1} u_\nu(\gamma)u_{g_\gamma}$$

only $u_\nu(\nu) = 1$ survives and, hence, $F_g(u_\nu) = u_{g_\nu}$, from which it follows that $\overline{F}_g(u_\nu) = \overline{u}_{g_\nu}$ and $h \circ \overline{F}_g(u_\nu) = h(\overline{u}_\nu) = 0$ for every $\nu \in \mu+1$.

Given that $\mu+1 < \kappa$, we have that $|\mu+1| < \kappa$. In order to apply Theorem 27 we must verify that

$$h \circ \overline{F}_g \restriction \bigoplus_{\nu<\mu+1} M_\nu = 0.$$

Let $z \in \bigoplus_{\nu<\mu+1} M_\nu$, then $z = z_1u_{\nu_1} + \cdots + z_nu_{\nu_n}$, for certain $z_i \in R$ and $\nu_i < \mu+1$. In this case

$$\begin{aligned} h \circ \overline{F}_g(z) =& z_1h \circ \overline{F}_g(u_{\nu_1}) + \cdots + z_nh \circ \overline{F}_g(u_{\nu_n}) \\ =& 0. \end{aligned}$$

So, by theorem 27 ($\mu+1 < \kappa$), $h \circ \overline{F}_g = 0$ holds. This contradicts the fact that $h[Im(\overline{F}_g)] \neq 0$. We can, thus, conclude that $h(\overline{u}_{g_\nu}) \neq 0$ for some $\nu \in \mu+1$. With this ν we make $\mu^* = \text{dom}(g_\nu)$ and $h \restriction \mu^* = g_\nu$.

Let us suppose that $\mu > \mu^*$ and that $k = h \restriction \mu$ is already defined. Under these conditions,

$$g(\overline{u}_k) \neq 0,$$

since $\overline{u}_k = \overline{u}_{k^0} + \overline{u}_{k^1}$, there is an $i \in \{0,1\}$ such that $g(\overline{u}_{k^i}) \neq 0$. We make $h(\mu) = i$. That is,

$$h \restriction \mu+1 = k^i.$$

Suppose $k = h \upharpoonright \mu$ is already defined and let μ be a limit ordinal. We know that

$$g(\overline{u}_{h\upharpoonright\nu}) \neq 0, \quad \forall \nu < \mu, \mu^* \leq \nu.$$

We must show that

$$g(\overline{u}_{h\upharpoonright\mu}) \neq 0.$$

So, let us consider the R-homomorphism $g \circ \overline{F}_k : \prod_{\alpha<\mu+1} M_\alpha \to R$. Since R is slender, almost all the u_ν ($\nu \in \mu + 1$) are mapped into zero under this R-homomorphism. Consequently, there is a $\mu_1 \in \mu$ such that

$$g \circ \overline{F}_k(u_\nu) = 0 \quad \forall \nu \geq \mu_1, \nu < \kappa.$$

Moreover, if $g \circ \overline{F}_k(u_\mu) = 0$, from

$$\overline{F}_k\left(\sum_{\mu_1\in[\nu,\mu+1)} u_\nu\right) = \overline{u}_{h\upharpoonright\mu_1}$$

(Claim 5), together with Theorem 27, it follows that

$$g\left(\overline{u}_{h\upharpoonright\mu_1}\right) = (g \circ \overline{F}_k)\left(\sum_{\mu_1\in[\nu,\mu+1)} u_\nu\right) = 0,$$

which contradicts the hypothesis that $g(\overline{u}_{h\upharpoonright\nu}) \neq 0$ for every $\nu \geq \mu^*$, with $\nu \in \mu$.

Therefore one gets, just as before,

$$0 \neq g \circ \overline{F}_k(u_\mu) = g(\overline{u}_{k_\mu}).$$

But, $k_\mu = k = h \upharpoonright \mu$ and, thus, $g(\overline{u}_{h\upharpoonright\mu}) \neq 0$. Notice that if $X \subseteq \kappa$, then $\overline{u}_X \neq 0$ if and only if $|X| = \kappa$. Otherwise, if $|X| < \kappa$ then $\overline{u}_X$ is in the class of zero. From this it follows that for every $\mu \in \kappa$, $|N_{h\upharpoonright\mu}| = \kappa$: if $g(\overline{u}_{h\upharpoonright\mu}) \neq 0$, then $\overline{u}_{h\upharpoonright\mu} \neq 0$ because g is an R-homomorphism. Therefore,

$$|N_{h\upharpoonright\mu}| = \kappa.$$

To finish, we describe an injective function $b : \kappa \to \kappa$ having the property that

$$b(\mu) = \bigcap_{\nu\in\mu} A_{b(\nu)}^{h(b(\nu))},$$

for each $\mu \in \kappa$. Suppose $b \upharpoonright \mu$ is already defined and let

$$\rho = \sup\{b(\nu) : \nu \in \mu\}.$$

Then, $\rho < \kappa$ since κ is regular.

We can choose $b(\mu) \in N_{h\restriction\rho} - (\rho+1)$ since we know that $|N_{h\restriction\mu}| = \kappa$ for every $\mu \in \kappa$.

Claim 7.

$$N_{h\restriction\rho} - (\rho+1) \subseteq \bigcap_{\nu\in\mu} A_{b(\nu)}^{h(b(\nu))}.$$

Proof of Claim 7. Let $\xi \in N_{h\restriction\rho} - (\rho+1)$, then $\xi \in N_{h\restriction\rho}$ and $\xi > \rho$; besides, $\xi \in A_\eta^{h(\eta)}$ for every $\eta \in \rho$. We must show that $\xi \in A_{b(\nu)}^{h(b(\nu))}$ for every $\nu \in \mu$. Note that $b \restriction \mu : \mu \to \rho$ is injective. Hence,

$$\xi \in \bigcap_{\nu\in\mu} A_{b(\nu)}^{h(b(\nu))}.$$

We are now able to define a subset $H \subseteq \kappa$ of cardinality κ, homogeneous with respect to p. We choose $i \in \{0,1\}$ such that

$$\left|(h \circ b)^{-1}(i)\right| = \kappa.$$

Let $H = b\left((h \circ b)^{-1}(i)\right)$. In this situation $|H| = \kappa$ and for any $\nu, \mu \in H$, $\nu \neq \mu$ there are $\xi, \zeta \in (h \circ b)^{-1}(i)$ such that $b(\xi) = \nu$ and $b(\zeta) = \mu$. Without loss of generality we can assume $\xi < \zeta$ and get

$$b(\zeta) \in A_{b(\xi)}^{h(b(\xi))} = A_{b(\xi)}^{i};$$

this yields $p(\{b(\zeta), b(\xi)\}) = i$ for every $\xi, \zeta \in (h \circ b)^{-1}(i)$. Therefore, H is homogeneous of cardinality κ for p, which is a contradiction. We conclude that $g = 0$ and $L^* = 0$. ☑

To finish we mention some open problems.

Problem 29. *Under $V = L$, can we take κ Mahlo instead of weakly compact in Theorem 21?*

Problem 30. *Does there exist an R-module M which is κ-torsionless but not torsionless and with $M^* \neq 0$?*

Problem 31. *An R-module M is locally projective if for each element $m \in M$, there exist $x_1, \ldots, x_n \in M$ and $f_1, \ldots, f_n \in M^*$ such that $m = \sum_j [x_j, f_j]m$, where $[m, f] = mf(\cdot)$ (for more on locally projective modules see [11]). It is easy to see that every locally projective module is torsionless. Is there an example of a torsionless R-module that is not locally projective?*

Problem 32. *Is it possible to extend example 28 to non-slender rings?*

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