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A proof of the Adem relations

Una demostración de las relaciones de Adem

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Abstract. We give an alternative proof of the Bullett-Macdonald identity for the Steenrod squares, which is in turn equivalent to the Adem relations. The main idea is to show that the iterated total squaring operation $S^2 : H^n(X) \to H^{4n}(X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2)$ is the restriction of a total fourth-power operation $T : H^n(X) \to H^{4n}(X \times B\Sigma_4)$.

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Resumen. Damos una demostración alternativa de la identidad de Bullet-Macdonald para los cuadrados de Steenrod, la que a su vez es equivalente a las relaciones de Adem. La idea principal es mostrar que la iteración de la operación cuadrado total $S^2 : H^n(X) \to H^{4n}(X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2)$ es la restricción de una operación total cuarta $T : H^n(X) \to H^{4n}(X \times B\Sigma_4)$.

Palabras y frases clave. Relaciones de Adem, operación cuadrado total.

1. Introduction

The Steenrod squares $Sq^i : H^n(X) \to H^{n+i}(X)$, for $i \geq 0$, are stable operations in mod 2 cohomology that generate the mod 2 Steenrod algebra $\mathcal{A}_2$ and satisfy the well known Adem relations

$$Sq^a Sq^b = \sum_{j=0}^{a/2} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \tag{1}$$

for $a < 2b$. These formulas were obtained in their original form by J. Adem [2]

$$Sq^{2t}Sq^{s} = \sum_{j=0}^{t} \binom{s-t+j-1}{2j} Sq^{s+t+j}Sq^{t-j}$$

for $s > t$, who used them to study the Hopf invariant 1 problem, among other interesting applications. Let $P(t)$ denote the formal power series

$$P(t) = \sum_{i \geq 0} t^i Sq^i.$$

It was shown by Bullett and Macdonald in [3] that the Adem relations (1) are equivalent to the power-series identity

$$P(s^2+st)P(t^2) = P(t^2+st)P(s^2) \tag{2}$$

in the variables s and t. The proof of (2) given in [3] was done by studying the effect of both sides on the cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z}_2, n)$. The purpose of this note is to provide an alternative proof of the Adem relations in the form (2) following an idea of G. Segal, see [3]. Namely, the Steenrod squares are usually defined in terms of a total squaring operation $S : H^n(X) \to H^{2n}(X \times B\mathbb{Z}_2)$ whose iteration gives $S^2 : H^n(X) \to H^{4n}(X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2)$. We show in Section 3 this is the restriction of a total fourth-power operation $T : H^n(X) \to H^{4n}(X \times B\Sigma_4)$, by the cartesian product embedding of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Σ_4, where Σ_4 is the symmetric group on four letters. Using this fact, we show in Section 4 that for any $\xi \in H^n(X)$ the element $S^2(\xi)$ is invariant under the operation of interchanging the factors of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the Adem relations express this invariance. This paper is part of the Master's thesis of the first author written under the supervision of the second author.

2. The definition of the Steenrod squares

Recall the Borel construction $X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is the total space of a fibre bundle

$$X^2 \xrightarrow{j} X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \xrightarrow{\pi} B\mathbb{Z}_2$$

where $E\mathbb{Z}_2$ can be taken to be S^∞ with the $\mathbb{Z}_2$-antipodal action and $B\mathbb{Z}_2$ is $S^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$. This space was originally introduced by Steenrod and used in [5] to explicitly construct the Steenrod squares. Notice that there is a well defined map

$$\Delta \times \mathrm{id} : X \times B\mathbb{Z}_2 \to X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2$$

given by: $(\Delta \times \mathrm{id})(x, [e]) = [(x, x); e]$. Moreover,

Theorem 2.1. *Let X be a CW complex and $\xi \in H^n(X)$, with $n \geq 1$, then there is a unique class $\Gamma(\xi) \in H^{2n}(X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2)$ such that*

(1) $(j^*)\Gamma(\xi) = \xi \otimes \xi$ *in* $H^*(X^2)$,

(2) $\Gamma(\xi)$ *is natural with respect to continuous maps* $f : X \to Y$,

(3) $\Gamma(\xi \cup \eta) = \Gamma(\xi) \cup \Gamma(\eta)$.

Proof. See [1], Theorem IV.7.1. ☑

Thus, if we identify $H^*(X \times B\mathbb{Z}_2)$ with $H^*(X)[t]$ where $t \in H^1(B\mathbb{Z}_2)$, the Steenrod squares of $\xi \in H^n(X)$ are defined by the following formula:

$$(\Delta \times \mathrm{id})^*(\Gamma(\xi)) = \sum_i t^{n-i} \cdot Sq^i(\xi).$$

We define $S(\xi) = (\Delta \times \mathrm{id})^*(\Gamma(\xi))$. The map $S : H^n(X) \to H^{2n}(X \times B\mathbb{Z}_2)$ is known as the total squaring operation.

3. The iterated total squaring operation

Our goal is to show that the iterated total squaring operation $S^2 : H^n(X) \to H^{4n}(X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2)$ can be obtained as the restriction of a total fourth-power operation $T : H^n(X) \to H^{4n}(X \times B\Sigma_4)$, by the cartesian product embedding of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Σ_4. Recall first that the diehedral group of order 8 is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$ and thus it can be generated by three elements of order two α, β, γ subject to the relations

$$\begin{aligned} \alpha\beta &= \beta\alpha \\ \alpha\gamma &= \gamma\beta \\ \beta\gamma &= \gamma\alpha. \end{aligned}$$

Moreover, D_8 can be regarded as a subgroup of Σ_4 by considering the permutations:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

satisfying the above relations. In this situation, the image of the canonical embedding of $\mathbb{Z}_2 \times \mathbb{Z}_2$ into Σ_4 is generated by α and β, and conjugation by $\gamma \in \Sigma_4$ interchanges both generators. Notice that the construction of the total squaring operation $S : H^*(X) \to H^*(X \times B\mathbb{Z}_2)$ can be easily adapted to construct a total fourth-operation, using the groups D_8 or Σ_4 instead of $\mathbb{Z}_2$, by considering the permutation action on X^4. Namely, one can prove the following

Theorem 3.1. *Let X be a CW complex and $G = D_8$ or Σ_4. Then for $\xi \in H^n(X)$ there is a unique class $\tilde{\Gamma}(\xi) \in H^{4n}(X^4 \times_G EG)$ that restricts to $\xi^{\otimes 4}$ in $H^*(X^4)$ and is natural with respect to continuous maps.*

Now, for $G = D_8$ or Σ_4 consider the obvious natural diagonal map

$$(\Delta_4 \times \mathrm{id}) : X \times BG \to X^4 \times_G EG$$

given by $(\Delta_4 \times \mathrm{id})(x; [e]) = [(x, x, x, x); e]$ and define the total fourth-operation $T : H^n(X) \to H^{4n}(X \times BG)$ by $T(\xi) = (\Delta_4 \times \mathrm{id})^*(\tilde{\Gamma}(\xi))$.

Next, as a model for ED_8 we take $E\mathbb{Z}_2 \times E\mathbb{Z}_2 \times E\mathbb{Z}_2 = S^\infty \times S^\infty \times S^\infty$ with the following free action:

$$\begin{aligned} \alpha(x, y, z) &= (-x, y, z) \\ \beta(x, y, z) &= (x, -y, z) \\ \gamma(x, y, z) &= (y, x, -z) \end{aligned}$$

and consider the map

$$\begin{aligned} f : X^4 \times_{D_8} ED_8 &\longrightarrow (X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2)^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \\ [(x_1, x_2, x_3, x_4); (e_1, e_2, e_3)] &\longmapsto [[x_1, x_2; e_1], [x_3, x_4; e_2]; e_3]. \end{aligned}$$

Then, the following diagram is commutative:

$$\begin{tikzcd}
\xi^{\otimes 4} & X^2 \times X^2 \arrow[rr, equal] \arrow[d, "j\times j"'] & & X^4 \arrow[dd, "j_4"] \\
\Gamma(\xi)^{\otimes 2} \arrow[u, mapsto] \arrow[d, mapsto] & (X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2)^2 \arrow[dr, "j_3"] & \Gamma(\Gamma(\xi)) & \\
S(\xi)^{\otimes 2} & (X \times B\mathbb{Z}_2)^2 \arrow[u, "(\Delta\times\mathrm{id})^2"] \arrow[d, "j_2"'] & (X^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2)^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2 & X^4 \times_{D_8} ED_8 \arrow[l, "f"'] \\
\Gamma(S(\xi)) \arrow[u, mapsto] \arrow[d, mapsto] & (X \times B\mathbb{Z}_2)^2 \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \arrow[ur, "(\Delta\times\mathrm{id})^2\times\mathrm{id}"'] & & \\
S(S(\xi)) & X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2 \arrow[u, "\Delta\times\mathrm{id}"] \arrow[rr, "\mathrm{id}_X \times i"] & & X \times BD_8 \arrow[uu, "\Delta_4\times\mathrm{id}"']
\end{tikzcd}$$

Here the maps j, j_2, j_3 and j_4 are the inclusions of the fibers into the corresponding Borel constructions and $i : B\mathbb{Z}_2 \times B\mathbb{Z}_2 \to BD_8$ is the composition of the homotopy equivalence

$$\begin{array}{ccc} B\mathbb{Z}_2 \times B\mathbb{Z}_2 & \xrightarrow{\simeq} & (S^\infty \times S^\infty \times S^\infty)/\mathbb{Z}_2 \times \mathbb{Z}_2 \\ & & \\ ([e_1],[e_2]) & \longmapsto & [e_1, e_1, e_2] \end{array}$$

and the natural map $(S^\infty \times S^\infty \times S^\infty)/\mathbb{Z}_2 \times \mathbb{Z}_2 \to (S^\infty \times S^\infty \times S^\infty)/D_8$. The diagram above allows us to identify the iterated total square $S(S(\xi))$ as the restriction of a total fourth-power operation associated to D_8. Namely, notice that $f^*\Gamma(\Gamma(\xi))$ maps to $\xi^{\otimes 4}$ under $(j_4)^*$, and thus the class $\tilde{\Gamma}(\xi) \in H^{4n}(X^4 \times_{D_8} ED_8)$ from Theorem 3.1 can be taken as $f^*\,\Gamma(\Gamma(\xi))$. Finally, since $T(\xi) \in H^{4n}(X \times BD_8)$ is the pullback of $\tilde{\Gamma}(\xi)$ under $\Delta_4 \times \mathrm{id}$, the commutativity of the diagram implies that the restriction of $T(\xi)$ to $X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2$ is precisely $S(S(\xi))$:

$$S(S(\xi)) = (id_X \times i)^*(\Delta_4 \times id)^*(\tilde{\Gamma}(\xi)).$$

Moreover, the diagram can be further extended up to homotopy by choosing for ED_8 the space $E\Sigma_4$ and considering the commutative diagram

$$\begin{tikzcd} X^4 \arrow[r, equal] \arrow[d] & X^4 \arrow[d] \\ X^4 \times_{D_8} ED_8 \arrow[r, "h"] & X^4 \times_{\Sigma_4} E\Sigma_4 \\ X \times BD_8 \arrow[u, "\Delta_4 \times \mathrm{id}"] \arrow[r, "\bar{h}"] & X \times B\Sigma_4, \arrow[u, "\Delta_4 \times \mathrm{id}"'] \end{tikzcd}$$

where h and $\bar{h}$ are the maps induced by the inclusion of D_8 into Σ_4. Now Theorem 3.1 allows us to identify the iterated total square S^2 with the restriction of a total fourth-power operation associated to Σ_4, as desired.

4. The proof of the Adem relations

Let $C_\gamma : \Sigma_4 \to \Sigma_4$ be conjugation by γ and recall from the previous section that C_γ interchanges the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Σ_4. Notice that if Σ_4 is identified with a category in the obvious way, then conjugation by an element and the identity are related by a natural transformation of functors. Thus they induce homotopic maps once realized, see [4]. As a consequence, the induced morphism $C_\gamma^* : H^*(\Sigma_4) \to H^*(\Sigma_4)$ is the identity and the following diagram is commutative:

$$\begin{array}{ccc} H^*(\Sigma_4) & \xrightarrow{C_\gamma^*=\mathrm{id}} & H^*(\Sigma_4) \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} \\ H^*(\mathbb{Z}_2\times\mathbb{Z}_2) & \xrightarrow[C_\gamma^*]{} & H^*(\mathbb{Z}_2\times\mathbb{Z}_2). \end{array}$$

Thus, the image of the restriction $res : H^*(\Sigma_4) \to H^*(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{F}_2[t,s]$ consists of symmetric polynomials on s and t. We use this fact to prove the Adem relations.

Recall that for $\xi \in H^n(X)$, we have $S(\xi) = \sum_k t^{n-k} Sq^k \xi$. If we identify $H^*(X \times B\mathbb{Z}_2 \times B\mathbb{Z}_2)$ with $H^*(X)[t,s]$, then

$$\begin{aligned} S^2(\xi) &= s^{2n} \sum_{m,k} s^{-m} Sq^m (t^{n-k} Sq^k(\xi)) \\ &= s^{2n} \sum_k P(s^{-1})(t^{n-k} Sq^k(\xi)). \end{aligned}$$

But $P(s^{-1}) = \sum s^{-k} Sq^k$ is a ring homomorphism, and it takes t to $t + s^{-1}t^2$, so

$$\begin{aligned} S^2(\xi) &= s^{2n} \sum_k [P(s^{-1})(t)]^{n-k} \cdot P(s^{-1})(Sq^k(\xi)) \\ &= s^{2n}(t+s^{-1}t^2)^n \sum_k (t+s^{-1}t^2)^{-k} \cdot P(s^{-1})(Sq^k(\xi)) \\ &= s^n t^n (s+t)^n \sum_k P(s^{-1})(t+s^{-1}t^2)^{-k} \cdot P(s^{-1})(Sq^k(\xi)) \\ &= s^n t^n (s+t)^n \cdot P(s^{-1}) \sum_k (t+s^{-1}t^2)^{-k} \cdot Sq^k(\xi) \\ &= s^n t^n (s+t)^n \cdot P(s^{-1}) P((t+s^{-1}t^2)^{-1})(\xi). \end{aligned}$$

Hence $P(s^{-1})P((t+s^{-1}t^2)^{-1})$ is symmetric in s and t. Write $s^{-1} = u(u+v)$, $t^{-1} = v(u+v)$. Then $(t+s^{-1}t^2)^{-1} = v^2$, and we find that

$$P(u(u+v))P(v^2)$$

is symmetric in u and v.

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