THE FORMAL ASYMPTOTIC EXPANSION OF A DARCY-STOKES COUPLED SYSTEM[a]

EXPANSIÓN ASINTÓTICA FORMAL DE UN SISTEMA DARCY-STOKES ACOPLADO

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ABSTRACT: Using the heuristic method of formal asymptotic expansions, we analyze the order of magnitude of the different physical entities involved in the phenomenon of fluid exchange between geometric regions, where the velocity of the fluid flow has different scale. We give the modeling equations, coupling conditions and express the velocity and pressure with formal asymptotic expansions. Then we compute the averages and observe the order of magnitude of each physical effect. Finally, we deduce a Darcy-Brinkman two way coupled system of partial differential equations as the "averaged" or "homogenized" problem.

KEYWORDS: Brinkman, Darcy, Stokes, porous media, multiscale coupled systems.

RESUMEN: Utilizando el método heurístico de expansiones asintóticas formales se analiza el orden de magnitud de las distintas entidades físicas involucradas en el fenómeno de intercambio de fluido entre regiones geométricas, en las cuales las velocidades de flujo son de diferente orden de magnitud. Se presentan las ecuaciones de modelación, condiciones de acoplamiento y se expresan la presión y la velocidad en expansión asintótica formal. Posteriormente se calculan las medias y se observa el orden de magnitud de cada efecto físico. Finalmente, se deduce un sistema Darcy-Brinkman totalmente acoplado de ecuaciones diferenciales parciales como la forma "promediada" u "homogeneizada" del sistema original.

PALABRAS CLAVE: Medios Porosos, Fractura, Sistema Darcy-Stokes acoplado, Brinkmann.

1. INTRODUCTION

Modeling the phenomenon of fluid flow in fissured geological systems is a multiple scale flow problem. It has regions of slow velocity of order $\mathcal{O}(1)$ in a rock matrix and fast flow of order $\mathcal{O}(1/\epsilon)$ in the network of fissures since the fissures are considerably larger $\mathcal{O}(1)$ than the average pore size in

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the rock matrix $\mathcal{O}(\epsilon)$. The most accurate and ambitious way to describe this phenomenon is to model it by a coupled system of partial differential equations, describing flow in porous medium for the rock matrix and free laminar flow for the fissures. Then, there is need of a Darcy-Stokes system of partial differential equations, together with boundary conditions and even more critical coupling, conditions describing fluid exchange and stress balance across the interface. However, the fact that the fissures are considerably smaller than the rock matrix will introduce geometrical singularities in the problem, while the change of scale in the magnitude of the average velocities of both regions will introduce a physical singularity. The presence of such singularities has an impact, from the numerical point of view, in the discretization of the model. Some of these consequences are ill-conditioned matrices, problems of numerical stability, poor quality of the approximate solutions and high computational costs (Arbogast and Brunson, 2007). On the other hand, the presence of the fissures network introduces an important effect (Arbogast and Lehr, 2006) which can not be neglected without falling into unrealistic models. Therefore several attempts have been made in order to handle this issue from the numerical point of view (Weiler and McDonnell, 2007) and from the analytical point of view (Allaire, 2009).

In the present work we analyze the coupling of a Darcy-Stokes system from the heuristic point of view i.e. the method of formal asymptotic formal expansions. This method has been used in several applied problems, where it is clear that some physical entities act at different scale and where the aim is to identify which of them "average out" and which of them "upscale", when subject to an averaging process. The averaging or homogenization process is performed in order to set the problem in a different mathematical framework, but free from the non-desirable geometric and physical singularities. Though there are several rigorous mathematical homogenization techniques in the literature (Sánchez-Palencia, 1980) most of them use formal asymptotic expansion as a first step to gain insight on the phenomenon and later pursue a rigorous mathematical proof. In the homogenization theory, it plays a role analogous to the role that the method of separation of variables plays in theory of partial differential equations. This method constitute a "qualitative modeling" of the phenomenon in order to determine the scale order of the main physical effects involved.

Finally, since Darcy flow and Stokes flow have a very different scale of validity and structure, careful consideration must be given to the coupling conditions describing the fluid exchange and the normal balance stress across both regions (Mikelić and Jager, 2000). This is a very active research field and it is still subject to debate. It will be shown that the formal limit when we letting $\epsilon \to 0$ (i.e. the width of the cracks tends to zero) is a fully-coupled system consisting of Darcy flow in the porous medium and Brinkman tangential flow on cracks.

2.   PRELIMINARIES

Vectors are denoted by boldface letters as are vector-valued functions and corresponding function spaces. We use $\widetilde{\mathbf{x}}$ to denote a vector in $\mathbb{R}^{N-1}$; if $\mathbf{x} \in \mathbb{R}^N$ we identify its $\mathbb{R}^{N-1} \times \{0\}$ projection with $\widetilde{\mathbf{x}} = (x_1, x_2, \ldots, x_N)$ i.e. the first $N-1$ components of the vector $\mathbf{x}$. The $\mathbb{R}^{N-1}$ gradient $\widetilde{\boldsymbol{\nabla}}$ and divergence $\widetilde{\boldsymbol{\nabla}}\cdot$ are described similarly; $\mathbf{n}$ denotes the outward unitary vector normal to a given domain $\Omega \subseteq \mathbb{R}^N$.

Next we describe the geometry of the model. Let Ω_1 be a bounded open region in $\mathbb{R}^N$, for simplicity we assume $x_N < 0$ for each $\mathbf{x} = (\widetilde{\mathbf{x}}, x_N) \in \Omega_1$ and that the portion of its boundary $\partial \Omega_1$ along the top of the domain has the structure $\Gamma \times \{0\}$ for $\Gamma \subseteq \mathbb{R}^{N-1}$ open set in $\mathbb{R}^{N-1}$. This constitutes the interface, and whenever there is no confusion will be simply denoted by Γ. The fissure is defined by the region $\Omega_2^\epsilon \overset{\text{def}}{=} \Gamma \times (0, \epsilon)$ and the full domain is given by $\Omega^\epsilon \overset{\text{def}}{=} \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$, see figure (1). When $\epsilon = 1$ we simply write Ω.

Figure 1: Multiple Scale Flow System
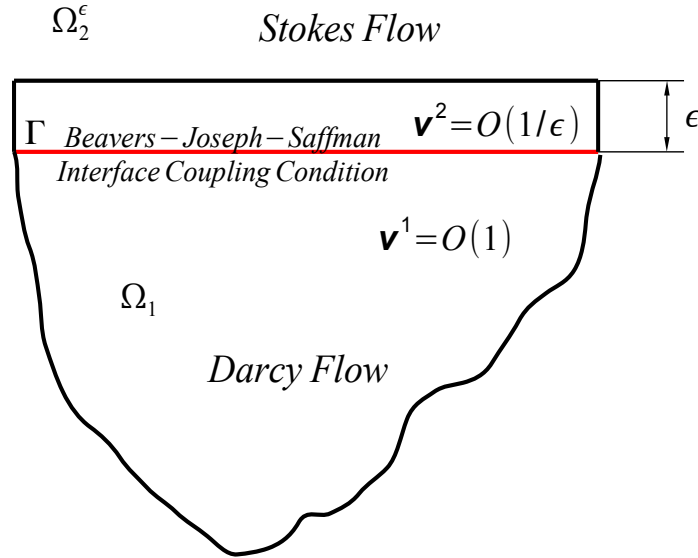
The filtration flow in the porous medium Ω_1 is governed by Darcy's law and the fast flow of the fluid in the narrow channel Ω_2^ϵ by Stokes' law.
The stationary Darcy-Stokes system is described by the set of equations

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1 \,, \tag{1a}$$

$$Q\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0 \quad \text{in } \Omega_1\,. \tag{1b}$$

And

$$\boldsymbol{\nabla} \cdot \mathbf{v}^2 = 0, \tag{1c}$$

$$- \boldsymbol{\nabla} \cdot \epsilon \, \mu \, \boldsymbol{\nabla} \mathbf{v}^2 + \boldsymbol{\nabla} p^2 = f_2 \quad \text{in } \Omega_2^\epsilon. \tag{1d}$$

Where Q is the inverse of permeability and μ denotes the viscosity of the fluid. On the other hand, the stress tensor for a Newtonian fluid given by

$$\sigma^2 = \epsilon \, 2\mu \mathbf{D}(\mathbf{v}^2) \tag{2}$$

has been replaced by $\boldsymbol{\nabla} \cdot \mu \boldsymbol{\nabla} \mathbf{v}^2$ since $\boldsymbol{\nabla} \cdot \mathbf{v}^2 = 0$; which yields

$$\boldsymbol{\nabla} \cdot \sigma^2 = \boldsymbol{\nabla} \cdot 2\mu \, \mathbf{D} \left(\mathbf{v}^2 \right) = \boldsymbol{\nabla} \cdot \mu \boldsymbol{\nabla} \mathbf{v}^2 \tag{3}$$

Finally, Darcy's law (1b) describes the fluid flow in the porous medium. The Newtonian law (2) describes the relationship between fluid stress and velocity in the thin channel. The interface conditions are given by

$$\mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n}, \tag{4a}$$

$$\epsilon \, \mu \, \frac{\partial \, \mathbf{v}^2}{\partial \mathbf{n}} \cdot \mathbf{n} - p^2 + p^1 = \epsilon \, \mu \, \frac{\partial \, v_N^2}{\partial x_N} - p^2 + p^1 = \alpha \, \mathbf{v}^1 \cdot \mathbf{n}, \tag{4b}$$

$$\epsilon \, \mu \, \left[\frac{\partial \, \mathbf{v}^2}{\partial \mathbf{n}} - \left(\frac{\partial \, \mathbf{v}^2}{\partial \mathbf{n}} \cdot \mathbf{n} \right) \mathbf{n} \right] = \epsilon \, \mu \, \frac{\partial \, \widetilde{\mathbf{v}}^2}{\partial \, x_N} = \epsilon^2 \beta \, \sqrt{Q} \, \widetilde{\mathbf{v}}^2 \text{ on } \Gamma. \tag{4c}$$

Remark 2.1. *(i) In the interface balance conditions (4) $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega_1 \cup \partial\Omega_2^\epsilon$, except on Γ where it is directed outside of Ω_1 i.e. $\mathbf{n}|_\Gamma \equiv \widehat{\boldsymbol{e}}_N$ and $\widehat{\boldsymbol{e}}_N$ is the unitary vector in the last direction N.*

(ii) The mass conservation of normal flux across the interface is given by (4a). The balance of normal stress is given by (4b), where $\alpha > 0$ denotes the entry resistance coefficient to normal flow across Γ. The Beavers-Joseph-Saffman (Saffman, 1971) condition (4c), relates the tangential flow along the interface with the tangential stress by means of an experimental tangential friction coefficient $\beta > 0$ and the square root of the permeability $\sqrt{Q}$.

(iii) Equation (2) accounts for the fluid stress of the free flow on the whole fractured medium, since it occurs only on the channel, it has to be scaled in agreement with the width of the channel $\epsilon > 0$. The scale ϵ^2 of the tangential flow resistance represents its dependence on the channel width. These scalings are complementary to the ones introduce by Arbogast and Lehr (Arbogast and Lehr, 2006).

Finally, for the exterior boundary conditions we assume null velocity on the ends of the narrow channel, i.e. $\mathbf{v} = 0$ on $\partial\Gamma \times (0, \epsilon)$. On the outer wall of the channel, $\Gamma \times \{ \epsilon \}$, we assume null-flux

Figure 2: Change of Variable
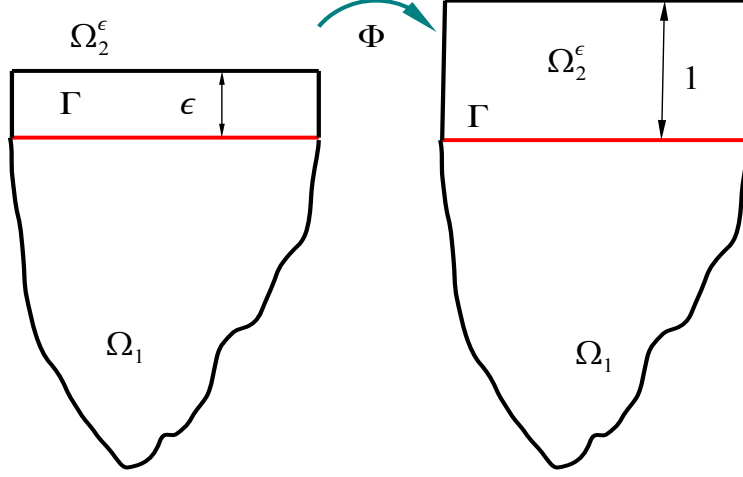
condition i.e. $\mathbf{v}^2 \cdot \mathbf{n} = v_N^2 = 0$ and null tangential stress. Due to (3) this last condition is equivalent to $\partial_N \widetilde{\mathbf{v}}^2 = 0$. On the remaining boundary of the porous medium Ω_1 we declare null-flux condition. These conditions are summarized as follows

$$\mathbf{v}^2 \cdot \mathbf{n} = 0 \text{ and} \tag{5a}$$

$$\partial_N \widetilde{\mathbf{v}}^2 = 0 \quad \text{on } \Gamma \times \{\epsilon\}. \tag{5b}$$

$$\mathbf{v}^2 = 0 \quad \text{on } \partial\Gamma \times (0, \epsilon). \tag{5c}$$

And

$$\mathbf{v}^1 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_1 - \Gamma. \tag{5d}$$

2.1. A common geometric setting

We are to analyze the behavior of the system when letting $\epsilon \to 0$ hence, to make the ϵ-problems and solutions comparable they must be defined on the same geometric region. Therefore, the system must be shifted to a common domain by means of the following change of variables $\Phi : \Omega^\epsilon \to \Omega$

$$\Phi(\widetilde{\mathbf{x}}, x_N) = \begin{cases} \left(\widetilde{\mathbf{x}}, \frac{1}{\epsilon} x_N\right), & (\widetilde{\mathbf{x}}, x_N) \in \Omega_2^\epsilon \\ (\widetilde{\mathbf{x}}, x_N), & (\widetilde{\mathbf{x}}, x_N) \in \Omega_1. \end{cases} \tag{6}$$

The change of variables maps bijectively between domains as figure (2) shows. On the other hand, under this change of variables the only derivative which suffers changes is the one in the

last direction on the channel, i.e. $\frac{\partial}{\partial z} = \frac{1}{\epsilon}\frac{\partial}{\partial x_N}$ whenever $(\widetilde{\mathbf{x}}, x_N) \in \Omega_2^{\epsilon}$. As a result the system of equations (1) transforms in

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1\,, \tag{7a}$$

$$Q\,\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0 \quad \text{in } \Omega_1\,, \tag{7b}$$

and

$$\left(\widetilde{\boldsymbol{\nabla}}, \frac{1}{\epsilon}\partial_z\right) \cdot \mathbf{v}^2 = \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{v}}^2 + \frac{1}{\epsilon}\partial_z v_N^2 = 0 \tag{7c}$$

$$-\epsilon\,\mu\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^2 + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 \widetilde{\mathbf{v}}^2}{\partial z^2} \\[2ex] \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}} v_N^2 + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 v_N^2}{\partial z^2}\end{array}\right\} + \left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}} p^2 \\[2ex] \dfrac{1}{\epsilon}\dfrac{\partial p^2}{\partial z}\end{array}\right\} = f_2 \quad \text{in } \Omega_2 \tag{7d}$$

Where $\Omega_2 = \Gamma \times (0,\,1)$. The interface conditions involving derivatives with respect to x_N must be modified, we have

$$\mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n}\,, \tag{8a}$$

$$\epsilon\,\mu\,\frac{1}{\epsilon}\frac{\partial v_N^2}{\partial z} - p^2 + p^1 = \mu\,\frac{\partial v_N^2}{\partial z} - p^2 + p^1 = \alpha\,\mathbf{v}^1 \cdot \mathbf{n} \quad \text{and} \tag{8b}$$

$$\epsilon\,\mu\,\frac{1}{\epsilon}\frac{\partial \widetilde{\mathbf{v}}^2}{\partial z} = \mu\,\frac{\partial \widetilde{\mathbf{v}}^2}{\partial z} = \epsilon^2\beta\,\sqrt{Q}\,\widetilde{\mathbf{v}}^2 \quad \text{on } \Gamma. \tag{8c}$$

The normal flux conservation across the interface (8a) remains identical because it does not depend on derivatives on the N-th direction, while the other two conditions on normal and tangential stress are affected by a dilation factor. Finally, the boundary conditions are the same as originally proposed but translated to the new boundary. This fact follows directly with one exception, this is $\epsilon^{-1}\partial_z\,\widetilde{\mathbf{v}}^2 = 0$ on $\Gamma \times \{1\}$. However, since the boundary condition is homogeneous it is equivalent to the original one.

3. ASYMPTOTIC EXPANSION

Due to the small width of the channel, the solutions on this region are expected to behave as follows: a robust term or "average" behavior term plus a tail of oscillations about this average of all possible scale orders, relative to the width of the channel (Sánchez-Palencia, 1980). This is intuitive since the small width $\epsilon > 0$ of the fissure constitutes itself a geometric singularity of the phenomenon. Therefore it is natural to propose the following formal asymptotic expansions for the velocity and pressure on the channel

$$\mathbf{v}^2 = \mathbf{v}^{(0)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon\,\mathbf{v}^{(1)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon^2\,\mathbf{v}^{(2)}\left(\widetilde{\mathbf{x}}, z\right) + \ldots = \sum_{j \geq 0}\epsilon^j\,\mathbf{v}^{(j)}\left(\widetilde{\mathbf{x}}, z\right). \tag{9}$$

$$p^2 = p^{(0)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon\, p^{(1)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon^2\, p^{(2)}\left(\widetilde{\mathbf{x}}, z\right) + \ldots = \sum_{j \geq 0} \epsilon^j\, p^{(j)}\left(\widetilde{\mathbf{x}}, z\right). \tag{10}$$

Here, the superscript j indicates the order of scale for each physical entity. With the definitions and the observations on the derivatives behavior mentioned above, we first compute the gradient of the velocity and get

$$\left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} \mathbf{v}^2 = \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} \mathbf{v}^{(0)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} \mathbf{v}^{(1)}\left(\widetilde{\mathbf{x}}, z\right)$$

$$+ \epsilon^2 \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} \mathbf{v}^{(2)}\left(\widetilde{\mathbf{x}}, z\right) + \ldots = \sum_{j \geq 0} \epsilon^j \left(\begin{matrix} \widetilde{\boldsymbol{\nabla}} \widetilde{\mathbf{v}}^{(j)} & \epsilon^{-1}\partial_z\, \widetilde{\mathbf{v}}^{(j)} \\ \left[\widetilde{\boldsymbol{\nabla}} v_N^{(j)}\right]' & \epsilon^{-1}\partial_z\, v_N^{(j)} \end{matrix} \right)\left(\widetilde{\mathbf{x}}, z\right). \tag{11}$$

For the pressure we have:

$$\left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} p^2 = \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} p^{(0)}\left(\widetilde{\mathbf{x}}, z\right) + \epsilon \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} p^{(1)}\left(\widetilde{\mathbf{x}}, z\right)$$

$$+ \epsilon^2 \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} \\ \epsilon^{-1}\partial_z \end{matrix} \right\} p^{(2)}\left(\widetilde{\mathbf{x}}, z\right) + \ldots = \sum_{j \geq 0} \epsilon^j \left\{ \begin{matrix} \widetilde{\boldsymbol{\nabla}} p^{(j)} \\ \epsilon^{-1}\partial_N\, p^{(j)} \end{matrix} \right\}\left(\widetilde{\mathbf{x}}, z\right). \tag{12}$$

3.1. Formal asymptotic expansion of mass conservation law

First we substitute expressions (9), (10), (11) and (12) in (7c), to get

$$\boldsymbol{\nabla}_{\left(\widetilde{\mathbf{x}}, x_N\right)} \cdot \mathbf{v}^2 = \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{v}}^2 + \frac{1}{\epsilon}\partial_z\, v_N^2 = \frac{1}{\epsilon}\partial_z\, v_N^{(0)}\left(\widetilde{\mathbf{x}}, z\right) + \sum_{j \geq 0} \epsilon^j \left(\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{v}}^{(j)} + \partial_z\, v_N^{(j+1)} \right)\left(\widetilde{\mathbf{x}}, z\right) = 0.$$

Experience shows that it is also natural to propose a "separation of scales" (Cioranescu and Donato, 1999) in the terms involved in the expression above. Using this hypothesis we conclude $\partial_z\, v_N^{(0)} = 0$ since it is the only term at scale ϵ^{-1}. On the other hand, the boundary condition (5a) in formal asymptotic expansion yields

$$\mathbf{v}^2 \cdot \mathbf{n} = \sum_{j \geq 0} \epsilon^j\, \mathbf{v}^{(j)} \cdot \mathbf{n} = \sum_{j \geq 0} \epsilon^j\, v_N^{(j)} = 0 \quad \text{on } \Gamma \times \{1\}.$$

Again, due to separation of scales we conclude $v_N^{(j)} = 0$ on $\Gamma \times \{1\}$, for all $j \geq 0$. In particular, since $\partial_N\, v_N^{(0)} = 0$ in Ω_2 and $v_N^{(0)} = 0$ on $\Gamma \times \{1\}$ we conclude

$$v_N^{(0)} = 0 \quad \text{in } \Omega_2. \tag{13}$$

Hence, rearranging the terms of equal scale, the previous expression transforms in

$$\sum_{j \geq 0} \epsilon^j \left[\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{v}}^{(j)} + \partial_z\, v_N^{(j+1)} \right]\left(\widetilde{\mathbf{x}}, z\right) = 0.$$

Next, we compute formally the average of the divergence, with respect to the variable z in order to remove the geometric singularity introduced by the width

$$\int_0^1 \boldsymbol{\nabla} \cdot \mathbf{v}^2 \, dz = \int_0^1 \sum_{j \geq 0} \epsilon^j \left(\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{v}}^{(j)}(\widetilde{\mathbf{x}}, z) + \partial_z v_N^{(j+1)} \right)(\widetilde{\mathbf{x}}, z) \, dz$$

$$= \sum_{j \geq 0} \epsilon^j \left(\widetilde{\boldsymbol{\nabla}} \cdot \int_0^1 \widetilde{\mathbf{v}}^{(j)}(\widetilde{\mathbf{x}}, z) \, dz + \int_0^1 \partial_z v_N^{(j+1)}(\widetilde{\mathbf{x}}, z) \, dz \right) = 0.$$

The width of the artificial domain Ω_2 is one, therefore the expression above is an average in the N-th direction of mass conservation law on the channel. Hence, define the average velocities by

$$\mathbf{V}^{(j)}(\widetilde{\mathbf{x}}) \overset{\mathbf{def}}{=} \int_0^1 \widetilde{\mathbf{v}}^{(j)}(\widetilde{\mathbf{x}}, z) \, dz, \quad \text{for all } j \geq 0. \tag{14}$$

On the other hand we integrate the normal derivative of the last term and combine it with the boundary conditions to get

$$\int_0^1 \partial_z v_N^{(j)}(\widetilde{\mathbf{x}}, z) \, dz = v_N^{(j)}(\widetilde{\mathbf{x}}, 1) - v_N^{(j)}(\widetilde{\mathbf{x}}, 0) = -v_N^{(j)}(\widetilde{\mathbf{x}}, 0), \quad \forall j \geq 0.$$

Combining both previous observations we obtain

$$\sum_{j \geq 0} \epsilon^j \left(\widetilde{\boldsymbol{\nabla}} \cdot \mathbf{V}^{(j)} - v_N^{(j+1)}(\widetilde{\mathbf{x}}, 0) \right) = 0. \tag{15}$$

The normal flux balance on the interface (8a) yields

$$\mathbf{v}^1 \cdot \mathbf{n} = \mathbf{v}^2 \cdot \mathbf{n} = \sum_{j \geq 1} \epsilon^j v_N^{(j)} \quad \text{on } \Gamma.$$

Recalling (13) we conclude $\epsilon v_N^{(1)}(\widetilde{\mathbf{x}}, 0) = \mathbf{v}^1 \cdot \mathbf{n}(\widetilde{\mathbf{x}}, 0)$ for all $\widetilde{\mathbf{x}} \in \Gamma$. This states that the scale magnitude of the normal flux coming from the rock is one level less than that of the average tangential velocity in the channel; which is expected due to the geometry. It is also important to highlight the fact that the velocity and pressure coming from the rock matrix are not subject to asymptotic expansion, since they are defined by Darcy's equation which is already averaged or homogenized. Finally, in order to apply the interface condition above, we modify the expression (15) as follows

$$\sum_{j \geq 0} \epsilon^j \frac{1}{\epsilon} \left(\widetilde{\boldsymbol{\nabla}} \cdot (\epsilon \, \mathbf{V}^{(j)})(\widetilde{\mathbf{x}}) - (\epsilon \, v_N^{(j+1)})(\widetilde{\mathbf{x}}, 0) \right)$$

$$= \frac{1}{\epsilon} \{ \widetilde{\boldsymbol{\nabla}} \cdot (\epsilon \, \mathbf{V}^{(0)})(\widetilde{\mathbf{x}}) - (\epsilon \, v_N^{(1)})(\widetilde{\mathbf{x}}, 0) \} + \sum_{j \geq 1} \epsilon^{j-1} \{ \widetilde{\boldsymbol{\nabla}} \cdot (\epsilon \, \mathbf{V}^{(j)})(\widetilde{\mathbf{x}}) - (\epsilon \, v_N^{(j+1)})(\widetilde{\mathbf{x}}, 0) \}$$

$$= \frac{1}{\epsilon} \left(\widetilde{\boldsymbol{\nabla}} \cdot (\epsilon \, \mathbf{V}^{(0)})(\widetilde{\mathbf{x}}) - \mathbf{v}^1 \cdot \mathbf{n}(\widetilde{\mathbf{x}}, 0) \right) + \sum_{j \geq 1} \epsilon^{j-1} \left(\widetilde{\boldsymbol{\nabla}} \cdot (\epsilon \, \mathbf{V}^{(j)})(\widetilde{\mathbf{x}}) - (\epsilon \, v_N^{(j+1)})(\widetilde{\mathbf{x}}, 0) \right) = 0.$$

Now, the separation of scales gives:

$$\epsilon^{-1}: \quad \widetilde{\boldsymbol{\nabla}} \cdot \left(\epsilon \, \mathbf{V}^{(0)} \right)(\widetilde{\mathbf{x}}) - \mathbf{v}^1 \cdot \mathbf{n}(\widetilde{\mathbf{x}}, 0) = 0, \tag{16a}$$

$$\epsilon^j: \quad \widetilde{\boldsymbol{\nabla}} \cdot \mathbf{V}^{(j)}(\widetilde{\mathbf{x}}) - v_N^{(j+1)}(\widetilde{\mathbf{x}}, 0) = 0 \quad \forall j \geq 1. \tag{16b}$$

3.2. Formal asymptotic expansion of momentum conservation law

In order to analyze the momentum conservation law substitue the expressions (11) and (12) in (7d); we have

$$
f_2 = -\epsilon\,\mu \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{\,2} + \epsilon^{-2}\,\partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{\,2} \\ \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}v_N^{\,2} + \epsilon^{-2}\,\partial_z\,\partial_z\,v_N^{\,2} \end{array} \right\} + \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{\,2} \\ \epsilon^{-1}\,\partial_N\,p^{\,2} \end{array} \right\}
$$

$$
= -\epsilon\,\mu \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(0)} + \epsilon^{-2}\,\partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(0)} \\ 0 \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{(0)} \\ \epsilon^{-1}\,\partial_N\,p^{(0)} \end{array} \right\}(\widetilde{\mathbf{x}}, z)
$$

$$
-\epsilon\,\mu \sum_{j\geq 1}\epsilon^j \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(j)} + \epsilon^{-2}\,\partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(j)} \\ \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}v_N^{(j)} + \epsilon^{-2}\,\partial_z\,\partial_z\,v_N^{(j)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \sum_{j\geq 1}\epsilon^j \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{(j)} \\ \partial_N\,p^{(j)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) \quad (17)
$$

We now group terms according to the order of magnitude

$$
-\frac{1}{\epsilon}\,\mu \left\{ \begin{array}{c} \partial_z\,\partial_z\widetilde{\mathbf{v}}^{(0)} \\ 0 \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \frac{1}{\epsilon} \left\{ \begin{array}{c} 0 \\ \partial_z\,p^{(0)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) - \mu \left\{ \begin{array}{c} \partial_z\,\partial_z\widetilde{\mathbf{v}}^{(1)} \\ \partial_z\,\partial_z v_N^{(1)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{(0)} \\ \partial_z\,p^{(1)} \end{array} \right\}(\widetilde{\mathbf{x}}, z)
$$

$$
- \mu\,\epsilon \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(0)} + \partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(2)} \\ \partial_z\,\partial_z v_N^{(2)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \epsilon \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{(1)} \\ \partial_z\,p^{(2)} \end{array} \right\}(\widetilde{\mathbf{x}}, z)
$$

$$
- \mu\sum_{j\geq 2}\epsilon^j \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(j-1)} + \partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(j+1)} \\ \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}v_N^{(j-1)} + \partial_z\,\partial_z v_N^{(j+1)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) + \epsilon^j \left\{ \begin{array}{c} \widetilde{\boldsymbol{\nabla}}p^{(j)} \\ \partial_z\,p^{(j+1)} \end{array} \right\}(\widetilde{\mathbf{x}}, z) = f_2. \quad (18)
$$

Using the separation of scales hypothesis we conclude

$$
\epsilon^{-1}: \quad \partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(0)} = 0\,, \tag{19a}
$$

$$
\partial_z\,p^{(0)} = 0. \tag{19b}
$$

Notice that (19b) implies that the first order term of the pressure is independent from the variable z as expected: it is a robust term and the changes along the z-axis are oscillations of higher order. On the other hand (19a) gives the structure of the first order tangential velocity, we know it is a linear affine function with respect to the z variable. From the boundary conditions we know $\partial_z\,\widetilde{\mathbf{v}}^{(0)} = 0$ on the top boundary; combined with (19a) we conclude

$$
\partial_z\,\widetilde{\mathbf{v}}^{(0)} = 0\,, \quad \widetilde{\mathbf{v}}^{(0)} = \widetilde{\mathbf{v}}^{(0)}\,(\widetilde{\mathbf{x}})\,. \tag{20}
$$

This conclusion was also expected: it states that the tangential velocity is given by a robust term and the changes along the z-axis are oscillations of higher order. Computing the average of equation

(18)

$$
-\frac{1}{\epsilon}\mu\left\{\begin{array}{c}\int_0^1 \partial_z\,\partial_z\widetilde{\mathbf{v}}^{(0)}\,dz\\ 0\end{array}\right\}+\frac{1}{\epsilon}\left\{\begin{array}{c}0\\ \int_0^1 \partial_z\,p^{(0)}\,dz\end{array}\right\}-\mu\left\{\begin{array}{c}\int_0^1 \partial_z\,\partial_z\widetilde{\mathbf{v}}^{(1)}\,dz\\ \int_0^1 \partial_z\,\partial_z v_N^{(1)}\,dz\end{array}\right\}+\left\{\begin{array}{c}\int_0^1 \widetilde{\boldsymbol{\nabla}}p^{(0)}\,dz\\ \int_0^1 \partial_z\,p^{(1)}dz\end{array}\right\}
$$

$$
-\mu\,\epsilon\left\{\begin{array}{c}\int_0^1 \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(0)}dz+\int_0^1 \partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(2)}dz\\ \int_0^1 \partial_z\,\partial_z v_N^{(2)}dz\end{array}\right\}+\epsilon\left\{\begin{array}{c}\int_0^1 \widetilde{\boldsymbol{\nabla}}p^{(1)}dz\\ \int_0^1 \partial_z\,p^{(2)}dz\end{array}\right\}
$$

$$
-\mu\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}\int_0^1 \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\widetilde{\mathbf{v}}^{(j-1)}dz+\int_0^1 \partial_z\,\partial_z\,\widetilde{\mathbf{v}}^{(j+1)}dz\\ \int_0^1 \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}v_N^{(j-1)}dz+\int_0^1 \partial_z\,\partial_z\,v_N^{(j+1)}dz\end{array}\right\}\epsilon^j\left\{\begin{array}{c}\int_0^1 \widetilde{\boldsymbol{\nabla}}p^{(j)}dz\\ \int_0^1 \partial_z\,p^{(j+1)}dz\end{array}\right\}=\int_0^1 f_2\,dz.\quad(21)
$$

Defining an average pressure analogous to the average velocity defined in (14)

$$
P^{(j)}(\widetilde{\mathbf{x}})\stackrel{\mathbf{def}}{=}\int_0^1 p^{(j)}(\widetilde{\mathbf{x}},z)\,dz\,,\quad\forall\,j\geq 0.\tag{22}
$$

Also recall that due to (20) and (19b) we have that $\widetilde{\mathbf{v}}^{(0)}=\mathbf{V}^{(0)}$ and $p^{(0)}=P^{(0)}$ respectively. Making the substitutions on (21) and computing the antiderivatives whenever possible we get

$$
-\frac{1}{\epsilon}\mu\left\{\begin{array}{c}\partial_z\widetilde{\mathbf{v}}^{(0)}|_0^1\\ 0\end{array}\right\}+\frac{1}{\epsilon}\left\{\begin{array}{c}0\\ p^{(0)}|_0^1\end{array}\right\}-\mu\,\epsilon^0\left\{\begin{array}{c}\partial_z\widetilde{\mathbf{v}}^{(1)}|_0^1\\ \partial_z v_N^{(1)}|_0^1\end{array}\right\}+\epsilon^0\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}}P^{(0)}\\ p^{(1)}|_0^1\end{array}\right\}
$$

$$
-\mu\,\epsilon\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\mathbf{V}^{(0)}+\partial_z\,\widetilde{\mathbf{v}}^{(2)}|_0^1\\ \partial_z v_N^{(2)}|_0^1\end{array}\right\}+\epsilon\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}}P^{(1)}\\ p^{(2)}|_0^1\end{array}\right\}
$$

$$
-\mu\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\mathbf{V}^{(j-1)}+\partial_z\,\widetilde{\mathbf{v}}^{(j+1)}|_0^1\\ \widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\xi^{(j-1)}+\partial_z\,v_N^{(j+1)}|_0^1\end{array}\right\}+\epsilon^j\left\{\begin{array}{c}\widetilde{\boldsymbol{\nabla}}P^{(j)}\\ p^{(j+1)}|_0^1\end{array}\right\}=\int_0^1 f_2\,dz.
$$

Where $\xi^{(j)}$ is defined by

$$
\xi^{(j)}(\widetilde{\mathbf{x}})\stackrel{\mathbf{def}}{=}\int_0^1 v_N^{(j)}(\widetilde{\mathbf{x}},z)\,dz\,,\quad\forall\,j\geq 1.\tag{23}
$$

Evaluating limits of integration and rearranging the terms we get

$$
\frac{1}{\epsilon}\mu\left\{\begin{array}{c}\partial_z\widetilde{\mathbf{v}}^{(0)}\left(\widetilde{\mathbf{x}},0\right)\\0\end{array}\right\}-\frac{1}{\epsilon}\mu\left\{\begin{array}{c}\partial_z\widetilde{\mathbf{v}}^{(0)}\left(\widetilde{\mathbf{x}},1\right)\\0\end{array}\right\}
$$

$$
+\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(1)}\left(\widetilde{\mathbf{x}},0\right)+\widetilde{\boldsymbol{\nabla}}P^{(0)}\\\mu\,\partial_z v_N^{(1)}\left(\widetilde{\mathbf{x}},0\right)-p^{(1)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}-\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(1)}\left(\widetilde{\mathbf{x}},1\right)\\\mu\,\partial_z v_N^{(1)}\left(\widetilde{\mathbf{x}},1\right)-p^{(1)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}
$$

$$
+\epsilon\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(2)}\left(\widetilde{\mathbf{x}},0\right)-\mu\,\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\mathbf{V}^{(0)}+\boldsymbol{\nabla}P^{(1)}\\\mu\,\partial_z v_N^{(2)}\left(\widetilde{\mathbf{x}},0\right)-p^{(2)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}-\epsilon\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(2)}\left(\widetilde{\mathbf{x}},1\right)\\\mu\,\partial_z v_N^{(2)}\left(\widetilde{\mathbf{x}},1\right)-p^{(2)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}
$$

$$
+\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)-\mu\,\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\mathbf{V}^{(j-1)}+\widetilde{\boldsymbol{\nabla}}P^{(j)}\\\mu\,\partial_z v_N^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)-p^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}
$$

$$
-\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}\mu\,\partial_z\widetilde{\mathbf{v}}^{(j+1)}\left(\widetilde{\mathbf{x}},1\right)\\\mu\,\partial_z v_N^{(j+1)}\left(\widetilde{\mathbf{x}},1\right)+\mu\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\xi^{(j-1)}-p^{(j+1)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}=\int_0^1 f_2\,dz.
$$

Applying the interface condition (8c) and the boundary condition (5b) we have

$$
\frac{1}{\epsilon}\left\{\begin{array}{c}\epsilon^2\,\beta\,\sqrt{\mathcal{Q}}\,\widetilde{\mathbf{v}}^{(0)}\\0\end{array}\right\}+\left\{\begin{array}{c}\epsilon^2\,\beta\,\sqrt{\mathcal{Q}}\,\widetilde{\mathbf{v}}^{(1)}\left(\widetilde{\mathbf{x}},0\right)+\widetilde{\boldsymbol{\nabla}}P^{(0)}\\\mu\,\partial_z v_N^{(1)}\left(\widetilde{\mathbf{x}},0\right)-p^{(1)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}-\left\{\begin{array}{c}0\\\mu\,\partial_z v_N^{(1)}\left(\widetilde{\mathbf{x}},1\right)-p^{(1)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}
$$

$$
+\epsilon\left\{\begin{array}{c}\epsilon^2\,\beta\,\sqrt{\mathcal{Q}}\,\widetilde{\mathbf{v}}^{(2)}\left(\widetilde{\mathbf{x}},0\right)-\mu\,\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\mathbf{V}^{(0)}+\boldsymbol{\nabla}P^{(1)}\\\mu\,\partial_z v_N^{(2)}\left(\widetilde{\mathbf{x}},0\right)-p^{(2)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}-\epsilon\left\{\begin{array}{c}0\\\mu\,\partial_z v_N^{(2)}\left(\widetilde{\mathbf{x}},1\right)-p^{(2)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}
$$

$$
+\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}\epsilon^2\,\beta\,\sqrt{\mathcal{Q}}\,\widetilde{\mathbf{v}}^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)-\mu\,\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\mathbf{V}^{(j-1)}+\widetilde{\boldsymbol{\nabla}}P^{(j)}\\\mu\,\partial_z v_N^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)-p^{(j+1)}\left(\widetilde{\mathbf{x}},0\right)\end{array}\right\}
$$

$$
-\sum_{j\geq 2}\epsilon^j\left\{\begin{array}{c}0\\\mu\,\partial_z v_N^{(j+1)}\left(\widetilde{\mathbf{x}},1\right)+\mu\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\boldsymbol{\nabla}}\,\xi^{(j-1)}-p^{(j+1)}\left(\widetilde{\mathbf{x}},1\right)\end{array}\right\}=\int_0^1 f_2\,dz.
$$

Recalling (20), we know $\widetilde{\mathbf{v}}^{(0)}(\widetilde{\mathbf{x}}, 0) = \mathbf{V}^{(0)}$. Regrouping the terms in order to be consistent with the momentum conservation law

$$
\frac{1}{\epsilon} \left\{ \begin{array}{c} 0 \\ \mu\, \partial_z v_N^{(0)}(\widetilde{\mathbf{x}}, 0) - p^{(0)}(\widetilde{\mathbf{x}}, 0) \end{array} \right\} - \frac{1}{\epsilon} \left\{ \begin{array}{c} 0 \\ \mu\, \partial_z v_N^{(0)}(\widetilde{\mathbf{x}}, 1) - p^{(0)}(\widetilde{\mathbf{x}}, 1) \end{array} \right\}
$$

$$
+ \left\{ \begin{array}{c} \beta \sqrt{\mathcal{Q}}\,(\epsilon\, \mathbf{V}^{(0)})(\widetilde{\mathbf{x}}, 0) - \mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \mathbf{V}^{(0)}) + \widetilde{\boldsymbol{\nabla}} P^{(0)} \\ \mu\, \partial_z v_N^{(1)}(\widetilde{\mathbf{x}}, 0) - p^{(1)}(\widetilde{\mathbf{x}}, 0) \end{array} \right\} - \left\{ \begin{array}{c} 0 \\ \mu\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \xi^{(1)}) + \mu\, \partial_z v_N^{(1)}(\widetilde{\mathbf{x}}, 1) - p^{(1)}(\widetilde{\mathbf{x}}, 1) \end{array} \right\}
$$

$$
+ \sum_{j \geq 1} \epsilon^j \left\{ \begin{array}{c} \beta \sqrt{\mathcal{Q}}\,(\epsilon\, \widetilde{\mathbf{v}}^{(j)})(\widetilde{\mathbf{x}}, 0) - \mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \mathbf{V}^{(j)}) + \widetilde{\boldsymbol{\nabla}} P^{(j)} \\ \mu\, \partial_z v_N^{(j+1)}(\widetilde{\mathbf{x}}, 0) - p^{(j+1)}(\widetilde{\mathbf{x}}, 0) \end{array} \right\}
$$

$$
- \sum_{j \geq 1} \epsilon^j \left\{ \begin{array}{c} 0 \\ \mu\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \xi^{(j+1)}) + \mu\, \partial_z v_N^{(j+1)}(\widetilde{\mathbf{x}}, 1) - p^{(j+1)}(\widetilde{\mathbf{x}}, 1) \end{array} \right\} = \int_0^1 f_2\, dz. \quad (24)
$$

In the expression above, we have written the null term corresponding to ϵ^{-1} to emphasize that the normal stress on the interface proposed on (8b) must be further analyzed before it can be applied; we have

$$
\alpha\, \mathbf{v}^1 \cdot \mathbf{n} - p^1 = \mu\, \frac{\partial v_N^2}{\partial z} - p^2 = \sum_{j \geq 0} \epsilon^j \mu\, \frac{\partial v_N^{(j)}}{\partial z} - \sum_{j \geq 0} \epsilon^j p^{(j)} = -P^{(0)} + \sum_{j \geq 1} \epsilon^j \left(\mu\, \frac{\partial v_N^{(j)}}{\partial z} - p^{(j)} \right).
$$

Where the third equality holds due to (13) and $p^{(0)} = P^{(0)}$ due to (19b). Now, the separation of scales yields the following set of interface conditions

$$
\epsilon^0 : \quad -P^{(0)} = \alpha\, \mathbf{v}^1 \cdot \mathbf{n} - p^1 \quad \text{on } \Gamma, \tag{25a}
$$

$$
\epsilon^j : \quad \mu\, \frac{\partial v_N^{(j)}}{\partial z} - p^{(j)} = 0 \quad \forall j \geq 1, \quad \text{on } \Gamma. \tag{25b}
$$

Applying the above on (24) and reordering gives

$$
\left\{ \begin{array}{c} \beta \sqrt{\mathcal{Q}}\,(\epsilon\, \mathbf{V}^{(0)})(\widetilde{\mathbf{x}}, 0) - \mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \mathbf{V}^{(0)}) + \widetilde{\boldsymbol{\nabla}} P^{(0)} \\ 0 \end{array} \right\}
$$

$$
+ \sum_{j \geq 1} \epsilon^j \left\{ \begin{array}{c} \beta \sqrt{\mathcal{Q}}\,(\epsilon\, \widetilde{\mathbf{v}}^{(j)})(\widetilde{\mathbf{x}}, 0) - \mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \mathbf{V}^{(j)}) + \widetilde{\boldsymbol{\nabla}} P^{(j)} \\ \mu\widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}(\epsilon\, \xi^{(j)}) + \mu\, \partial_z v_N^{(j)}(\widetilde{\mathbf{x}}, 1) - p^{(j)}(\widetilde{\mathbf{x}}, 1) \end{array} \right\} = \int_0^1 f_2\, dz.
$$

Finally, we use the separation of scales and regroup the tangential and normal momentum conservation laws of the same order; we have

$$
\beta \sqrt{\mathcal{Q}}\,\big(\epsilon\, \mathbf{V}^{(0)}\big)(\widetilde{x}) - \mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}}\big(\epsilon\, \mathbf{V}^{(0)}\big)(\widetilde{x}) + \widetilde{\boldsymbol{\nabla}} P^{(0)} = \int_0^1 f_2\, dz, \tag{26a}
$$

$$
-P^{(0)} = \alpha\, \mathbf{v}^1 \cdot \mathbf{n} - p^1. \tag{26b}
$$

For the robust terms behavior ϵ^0, and

$$\beta \sqrt{\mathcal{Q}} \left(\epsilon \, \widetilde{\mathbf{v}}^{(j)} \right) (\widetilde{\mathbf{x}}, 0) - \mu \, \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}} \left(\epsilon \, \mathbf{V}^{(j)} \right) + \widetilde{\boldsymbol{\nabla}} P^{(j)} = 0 \,, \tag{26c}$$

$$\mu \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}} \left(\epsilon \, \xi^{(j+1)} \right) + \mu \, \partial_z \, v_N^{(j+1)} (\widetilde{\mathbf{x}}, 1) - p^{(j+1)} (\widetilde{\mathbf{x}}, 1) = 0. \tag{26d}$$

For the higher order oscillations ϵ^j with $j \geq 1$.

4.   A BRINKMAN-STOKES COUPLED SYSTEM

The formal multiscale analysis carried out in sections (3.1) and (3.2) yields averaged and higher order oscillation equations for the interior and the fluid exchange between regions as summarized in (16) and (26) respectively. We focus on the averaged relations only to find the problem which characterizes the average solutions. Recalling that $\mathbf{V}^{(0)}$ and $P^{(0)}$ do not depend on z, they can be considered as functions defined only on Γ. Denoting $\mathbf{V} \overset{\text{def}}{=} \epsilon \mathbf{V}^{(0)}$ and $P \overset{\text{def}}{=} P^{(0)}$ and coupling the averaged equations on the channel with the activity in the rock matrix we get

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h_1, \tag{27a}$$

$$Q \, \mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0 \quad \text{in } \Omega_1. \tag{27b}$$

$$\widetilde{\boldsymbol{\nabla}} \cdot \mathbf{V} - \mathbf{v}^1 \cdot \mathbf{n} = 0, \tag{27c}$$

$$\beta \sqrt{\mathcal{Q}} \, \mathbf{V} - \mu \, \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\boldsymbol{\nabla}} \, \mathbf{V} + \widetilde{\boldsymbol{\nabla}} P = \int_0^1 f_2 \, dz \quad \text{in } \Gamma. \tag{27d}$$

Interface condition

$$p^1 - \alpha \, \mathbf{v}^1 \cdot \mathbf{n} = P \quad \text{in } \Gamma. \tag{27e}$$

Boundary conditions

$$\mathbf{V} = 0 \quad \text{on } \partial\Gamma. \tag{27f}$$

And

$$\mathbf{v}^1 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_1 - \Gamma. \tag{27g}$$

Some comments and conclusions are necessary about the system of equations above.

4.1.   Conclusions and final remarks

(I) All the variables defined in Γ do not depend on z i.e. these physical entities are all tangential since the flow in the small fissure is predominantly parallel to the wall. This phenomenon is know as the *channeling effect* (Bejan and Nield, 1999).

(II) The problem above is no longer a problem between two regions of the same dimension. Now it is a problem describing the fluid exchange between a rock matrix domain of dimension N and a portion of its boundary Γ of dimension $N - 1$ where only tangential flow takes place.

(III) The conservative equation (27a), constitutive equation (27b) and boundary condition (27g) of the rock matrix remain equal to the starting ones (1a), (1b), (5d) since the rock matrix was not subject to multiple scale analysis.

(IV) The variable $\mathbf{V} \stackrel{\mathbf{def}}{=} \epsilon \mathbf{V}^{(0)}$ quantifies discharge rate i.e. this is the physical quantity that is preserved in the averaging process, not the velocity itself.

(V) The equation (27d) is known as the Brinkman equation. It assumes that the viscous and the dragging forces are of the same scale in the phenomenon of momentum conservation. It is considered as an intermediate case between free flow and flow in porous media. Several efforts have been done in order to determine its scale of validity (Lévy, 1983), mainly based on the ratio between the average diameter of the pores and the separation between them.

(VI) Since the interface Γ in the original problem is located between a Stokes and a Darcy flow regions it is intuitive that all the effects on it must be of the same scale. In this particular case, the Beavers-Joseph-Saffman interface condition (4c) relating the fluid tangential stress with tangential drag forces, together with an averaging process, allow to give such conclusion.

(VII) Brinkman flow has already been used successfully to numerically couple Darcy and Stokes flow models (Chen and Gunzburguer and Wang, 2010).

(VIII) The conservation law in the interface Γ (27c) states that the normal flux $\mathbf{v}^1 \cdot \mathbf{n}$ acts as a source in the interface Γ, i.e. the activity on the interface Γ depends on the activity in the rock matrix. It is a consequence of the normal flux balance interface condition (4a).

(IX) The normal stress balance interface (27e) has the same structure as the original one (4b) noticing $\mu \, \partial_z \mathbf{V} = 0$ since $\mathbf{V} = \mathbf{V}(\widetilde{\mathbf{x}})$. On the other hand this normal stress balance states that the fluid pressure on Γ feeds as a boundary condition for the rock matrix. Therefore, the activity in the rock matrix depends on the tangential activity on the interface Γ.

(X) From the two previous parts we deduce the system (27) is a two way coupled Darcy-Brinkman system of partial differential equations.

(XI) The boundary conditions on the interface Γ (27f) follow by computing the average of the original boundary condition (5c).

(XII) Although there is experimental (Chen and Gunzburguer and Wang, 2010) and modeling (Lévy, 1983) evidence that the Brinkman flow is an intermediate case between free flow and flow in porous media and it is useful to couple such phenomenon there is no rigorous mathematical proof for this fact.

(XIII) The present work is only heuristic however, experience has shown that the method of asymptotic expansions is strong enough evidence to pursue a rigorous mathematical treatment, by finding the suitable homogenization technique.

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