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AN EPISODE STARRING THE RESIDUE THEOREM IN THE HISTORY OF ELLIPTIC FUNCTIONS[a]

PAPEL PROTAGÓNICO DEL TEOREMA DEL RESIDUO EN UN EPISODIO DE LA HISTORIA DE LAS FUNCIONES ELÍPTICAS

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ABSTRACT: In this paper we explain how the Residue Theorem was used –perhaps for the first time– to determine the Laurent series development of an elliptic function. This great achievement in the history of Elliptic Functions is due to the French professors Briot and Bouquet. We also draw some conclusions on the role of the historical emergence of Complex Analysis, as a general theory, in the development of Elliptic Functions.

KEYWORDS: Elliptic Functions, Laurent series, Residue Theorem.

RESUMEN: En este artículo se explica la manera cómo el Teorema del Residuo de la Variable Compleja fue usado –quizás por primera vez– para encontrar la serie de Laurent de una función elíptica. Este gran logro en la historia de las funciones elípticas se debe a los profesores franceses Briot y Bouquet. También se presentan algunas conclusiones sobre el surgimiento histórico del análisis complejo, como teoría general, y su influencia en el desarrollo de las funciones elípticas.

PALABRAS CLAVE: Funciones elípticas, series de Laurent, Teorema del Residuo.

1. INTRODUCTION

There was a time in the mid-nineteenth century when Elliptic Functions were not yet recognized as a chapter of Complex Analysis. The research in both areas of knowledge had common elements as the theory of functions of a complex variable provided, increasingly, general and useful theorems. Such new theorems greatly simplified the proofs of the original discoveries of Abel and Jacobi on Elliptic Functions at the beginning of that splendid century.

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One of the most challenging tasks in those days was to find Laurent series expansions for the elliptic functions. Although the expressions given by Abel and Jacobi were regarded as true, the difficulty of their proofs demanded more efficient methods to handle these new functions. A simple look at the *Recherches* (Abel, 1827) or the *Fundamenta nova* (Jacobi, 1829) is enough to discourage the most persevering individual from trying such methods.

In an effort to easily find Taylor series, Gudermann (1839) devised an ingenious method to expand Jacobi elliptic functions using certain differential equations they satisfy. The power series solution to these equations yields an expression for the sought elliptic function far from its poles. Gudermann's preference for this type of methods is related to the rise of the Combinatorial Art in the Germany of the time, as it has been documented by Manning (1975).

In this article we comprehensively explain how Briot and Bouquet (1859) determined the Laurent series of an elliptic function with the assistance of the Residue Theorem. Besides the obvious influence of Cauchy, it is permissible to assume the powerful impact of the influential *Leçons* (Borchardt-Liouville, 1880) in the work of these French professors. Certainly, they still use the concepts of *fonciton monodrome* and *fonction monogène*:

> Concevons que la variable z reste comprise dans une certaine portion du plan: si la fonction u acquiert la même valeur au même point, quel que soit le chemin suivi pour y arriver, sans sortir de la portion de plan considerée, M. Cauchy dit que la fonction est *monodrome* dans cette portion du plan. [···] Lorsque la valeur de la dérivée est indépendante de la direction du déplacement, en d'autres termes lorsque la fonction admet une dérivée unique en chaque point, M. Cauchy dit que la fonction est *monogène*. (Briot and Bouquet, 1859, pp. 3–7).

We recall that path independence of the integral and complex-differentiability are standard properties of holomorphic functions. However, the spirit of Briot and Bouquet goes beyond the basic facts of the theory and actually reaches the heart of the powerful theorems Liouville had shown in the field of Elliptic Functions. What we have in mind is a result like the following.

> Une fonction doublement périodique et à un seul infini est impossible. (Borchardt-Liouville, 1880, p. 283).

That is to say, an elliptic function has at least two poles.

In next section we lay the foundations of the general method employed by Briot and Bouquet (1859) to find the Laurent series of a meromorphic, simply or doubly periodic function. Afterwards, we apply such a method to determine the series of trigonometric and elliptic functions. Along the way, we have reinterpreted the original statements in the light of certain analytical results. Finally, we draw some conclusions regarding the role of this method in the construction of the contemporary notion of elliptic function and discuss the simultaneous or parallel achievements of the general theory (Complex Analysis) along with a part of it (Elliptic Functions).

2. THEORY AND CALCULATIONS

2.1. General method

An elliptic function is a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ with two periods. In other words, there is a pair of complex numbers ω, ω' which are linearly independent as vectors in $\mathbb{R}^2$ such that

$$f(z+p) = f(z), \text{ for all } p = n\omega + m\omega', \text{ where } n, m \in \mathbb{Z}.$$

In this way, the fundamental region of an elliptic function is a parallelogram with sides ω, ω'. As an elliptic function is meromorphic, it has poles. We say that f has a pole of order $m \in \mathbb{N}$ at $c \in \mathbb{C}$ if there is a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $g(c) \neq 0$ and

$$f(z) = \frac{g(z)}{(z-c)^m}, \quad \text{for } z \in \mathbb{C} - \{c\}.$$

Let us recall the well-known Residue Formula. To do this, let D be a complex domain $-i.$ *e.*, an open connected subset of $\mathbb{C}$–, γ a simply closed and null-homologous[d] path in D, A a finite subset of $D \backslash \gamma$ and $f : D \backslash A \to \mathbb{C}$ a holomorphic function. Then,

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\ d\zeta = \sum_{\alpha \in \mathrm{int}\gamma \cap A} \mathrm{res}_\alpha f.$$

Some words about notation are needed. If $\sum_{-\infty}^{\infty} a_\nu (z-\alpha)^\nu$ is the Laurent development of f in a punctured ball $B^\times$ centered at α, the symbol $\mathrm{res}_\alpha f$ denotes the residue of f at α and is defined by

$$\mathrm{res}_\alpha f = a_{-1} = \frac{1}{2\pi i} \int_C f(\zeta)\ d\zeta,$$

for every circle C centered at α and lying entirely in B. *Cf.* Remmert (1991). Also, $\mathrm{int}\gamma$ is the inside (or "interior") of path γ, which can be defined as

$$\mathrm{int}\gamma = \{z \in \mathbb{C} : \mathrm{ind}_\gamma(z) \neq 0\}, \text{ where } \mathrm{ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \in \mathbb{Z}$$

is the index of γ about z. At this point, Briot and Bouquet (1859) pick a $z \in \mathrm{int}\gamma \backslash A$ and calculate

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\ d\zeta = f(z) + \sum_{\alpha \in \mathrm{int}\gamma \cap A} \mathrm{res}_\alpha \tilde{f}, \text{ where } \tilde{f}(\zeta) = \frac{f(\zeta)}{\zeta - z},$$

[d]This name comes from Algebraic Topology. A closed path γ in a domain D is null-homologous in D if for each holomorphic function $f : D \to \mathbb{C}$ the integral theorem holds, that is,

$$\int_\gamma f(\zeta) d\zeta = 0.$$

by virtue of the Cauchy Integral Formula[e]. In search of the Laurent series for $f(z)$, they abridge the remainder as $r(z,\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$ to obtain

$$f(z) = -\left(\sum_{\alpha\in \mathrm{int}\gamma\cap A} \mathrm{res}_\alpha \tilde{f}\right) + r(z,\gamma).$$

Now, if the curve γ is enlarged to the point that it encloses the whole plane ("*de manière qu'elle embrasse tout le plan,*") then –under suitable conditions[f] on $f(z)$– $r(z,\gamma)$ is vanished ("*a pour limite zéro.*") So, "*la somme des résidus, sera convergente.*"

This argument certainly merits further explanation. Let f be a meromorphic function on $\mathbb{C}$, *i. e.*, there exists a discrete subset $P \subset \mathbb{C}$ (depending on f) such that $f : \mathbb{C}\backslash P \to \mathbb{C}$ is holomorphic and each $p \in P$ is a pole of f, in the sense given at the beginning of the section. *Cf.* Remmert (1991). Suppose that there exists

- an increasing sequence $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ of domains,
- a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of paths such that each γ_n is simply closed[g] and null-homologous in D_n and, eventually, they encircle the complex plain –for all $z \in \mathbb{C}$ there exits a $N \in \mathbb{N}$ such that $z \in \mathrm{int}\gamma_N$– and
- an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of P such that each A_n is contained in $D_n\backslash\gamma_n$ and, eventually, they come to be P.

Assume also that, in the sequence $(f_n : D_n\backslash A_n \to \mathbb{C})_{n\in\mathbb{N}}$ of restrictions of f, each member is holomorphic. Then, for $n \in \mathbb{N}$ and $z \in \mathrm{int}\gamma_n\backslash A_n$,

$$f_n(z) = -\left(\sum_{p\in \mathrm{int}\gamma_n\cap A_n} \mathrm{res}_p \tilde{f}_n\right) + r_n(z,\gamma_n), \text{ where } \tilde{f}_n(\zeta) = \frac{f_n(\zeta)}{\zeta - z}.$$

This expression discloses an interpretation for the idea of Briot and Bouquet (1859): if the nature of f implies that the sequence of remainders $(r_n(z,\gamma_n))_{n\in\mathbb{N}}$, $z \in \mathbb{C}\backslash P$, converges –in some sense– to zero, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges –in some sense– to $f(z)$ and so, to the sum of residues with a minus sign in front. In order to analyze the remainders r_n, the French professors invoke the first-order Taylor formula (about $z = 0$)

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} + \left(\frac{1+\epsilon(z)}{\zeta^2}\right) z,$$

[e]If $f : D \to \mathbb{C}$ is holomorphic in a complex domain D and γ is simple and null-homologous in D, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z), \text{ for } z \in D - \{z\}.$$

[f]These conditions are mainly related to the periodicity of f and will be clear in the examples given below.

[g]That is, a connected curve that does not cross itself and ends at the same point where it begins. Examples are circles, ellipses, and polygons.

where $\lim_{z\to 0}\epsilon(z) = 0$ ("ϵ *désignant une quantité infiniment petite.*") Hence,

$$r_n(z,\gamma_n) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{f(\zeta)}{\zeta}\, d\zeta + \frac{z}{2\pi i}\int_{\gamma_n} f(\zeta)\left(\frac{1+\epsilon(z)}{\zeta^2}\right)\, d\zeta.$$

Thus,

- if $|f(\zeta)|$ remains bounded as $n \to \infty$, the standard estimate in conjunction with a proper choice for the paths imply that the second integral on the right vanishes as we pass to the limit (*"Si, le long de la courbe variable, le module de la fonction* $[\cdots]$ *reste moindre qu'une quantité finie, la seconde partie du reste a pour limite zéro.*");
- in the cases of interest to us, the first integral

$$\frac{1}{2\pi i}\int_{\gamma_n} \frac{f(\zeta)}{\zeta}\, d\zeta$$

  is zero when the curves γ_n are chosen according to the symmetries of f.

Such a particular way of thinking produces correct results in the following two examples.

2.2. Example 1: $\csc z$

The pole-set P of the meromorphic function $f(z) = \csc z = 1/\sin z$ consists of the complex values $z = k\pi$, $k \in \mathbb{Z}$. The paths γ_n of integration can be chosen to be the boundaries of the squares (*see* Figure 1)

$$\left[-n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right] \times \left[-n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right], \quad n \in \mathbb{N} \cup \{0\}.$$

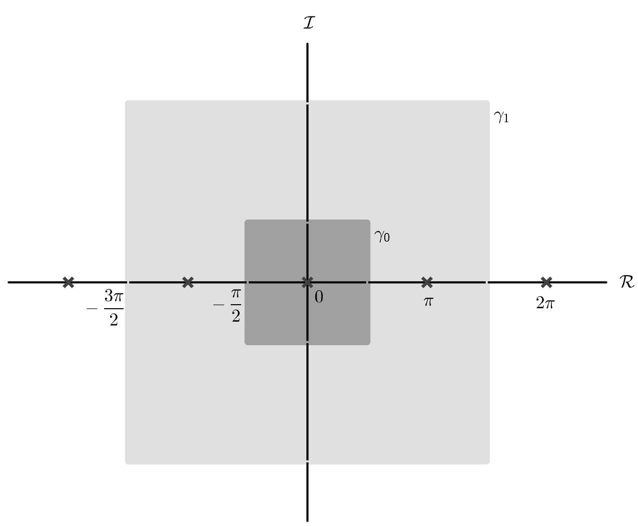

Figure 1: Paths γ_n for $\csc z$. The poles are marked with ×.

As a consequence of the convenient formula

$$\csc(x+iy) = \frac{1}{\sin(x+iy)} = \frac{1}{\sin x \cosh y + i \cos x \sinh y},$$

the modulus $|f(\zeta)|$ stays bounded along the lines

$$x = \pm\left(n+\frac{1}{2}\right)\pi \text{ and } y = \pm\left(n+\frac{1}{2}\right)\pi$$

with a universal bound independent of n. Therefore,

$$\left|\int_{\gamma_n} f(\zeta)\left(\frac{1+\epsilon(z)}{\zeta^2}\right)\,d\zeta\right| \le \frac{M}{n+\frac{1}{2}},$$

for certain constant $M > 0$ and large values of n. In summary, the second integral term of the remainder vanishes at infinity. We are left with

$$\int_{\gamma_n} \frac{f(\zeta)}{\zeta}\,d\zeta.$$

However, f is an odd function and so, this last integral equals zero for all $n \in \mathbb{N} \cup \{0\}$.

The reasoning of Briot and Bouquet (1859) is slightly different from ours, but similar in its essence:

> Intégrons suivant le contour d'un rectangle formé par des parallèles à l'axe des y, menées à la distance $m'\pi + \frac{\pi}{2}$ de l'origine, et par des parallèles à l'axe des x à une distance très-grande arbitraire Y. On reconnaît aisément que le module de la fonction *coséc z* est moindre que l'unité sur les premières parallèles, et très-petit sur les dernières. Il en résulte que la seconde partie du reste a pour limite zéro, quand on fait croître m' et Y indéfiniment. Quant à la première partie, elle est identiquement nulle; car, la fonction étant impaire, les élements différentiels, qui correspondent à deux éléments du rectangle symétriques par rapport au centre, sont égaux et de signes contraires. Ainsi la fonction proposée se développe en une série convergente. (Briot and Bouquet, 1859, p.124).

Thus, the problem is reduced to a calculation of residues. To evaluate

$$-\text{res}_{k\pi}\tilde{f} = \frac{1}{2\pi i}\int_C \frac{\csc\zeta}{z-\zeta}\,d\zeta$$

along a small enough circular path C centered at $k\pi$, $k \in \mathbb{N}$, the French professors cleverly translate $\zeta = k\pi + \vartheta$. In this way, $\sin\zeta = (-1)^k \sin\vartheta$ and

$$-\text{res}_{k\pi}\tilde{f} = \frac{(-1)^k}{2\pi i}\int_D \frac{\csc\vartheta}{z-k\pi-\vartheta}\,d\vartheta = \frac{(-1)^k}{2\pi i}\int_D \frac{g(\vartheta)}{\vartheta}\,d\vartheta,$$

where D is a small circular path about zero and g is holomorphic in a domain containing $\text{int}D \cup D$ with $g(0) = 1/(z - k\pi)$. Finally, by the Cauchy Integral Formula,

$$-\text{res}_{k\pi}\tilde{f} = \frac{(-1)^k}{z - k\pi}$$

and we get the celebrated series

$$\csc z = \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{z - k\pi} = \frac{1}{z} + 2z \sum_{k\in\mathbb{N}} \frac{(-1)^k}{z^2 - k^2\pi^2}.$$

Instead of the auxiliary function g in the last step, Briot and Bouquet seem to use –*mutatis mutandis*– a venturesome passing to the limit:

> Pour évaluer le résidu relatif à un infini quelconque $m\pi$ de la fonction $1/\sin z$, posons $z = m\pi + z'$, et considérons l'intégrale définie
>
> $$\frac{1}{2\pi i}\int \frac{dz}{(t - z)\sin z} = \frac{(-1)^m}{2\pi i}\int \frac{dz'}{(t - m\pi - z')\sin z'},$$
>
> prise le long d'un cercle infiniment petit décrit autour du point $m\pi$; on peut réduire cette integrale à
>
> $$\frac{(-1)^m}{t - m\pi}.\frac{1}{2\pi i}\int \frac{dz'}{z'} = \frac{(-1)^m}{t - m\pi}.$$
>
> (Briot and Bouquet, 1859, pp. 124–125).

2.3. Example 2: $\lambda(z)$

In the first chapter of Book III, Briot and Bouquet (1859, pp. 95–97) define the elliptic function λ as a doubly periodic meromorphic function with complex linearly independent periods ω, ω' –*cf.* our definition at the beginning of 2.1–. Without doubt, the contemporary reader expects a definition of this transcendental function in terms of an infinite series or product. However, such representation was not fully available to our French teachers. Abel (1827) and Jacobi (1829) had certainly constructed functions equivalent to λ through formal inversion of the elliptic integral of the first kind. Gudermann (1839) had also found new methods for proving the properties of such functions more efficiently. Nonetheless, series and infinite products remained as an almost impossible task to perform easily. That might explain why Briot and Bouquet prefer to declare that an elliptic function is a solution of the differential problem

$$\frac{du}{dz} = g\sqrt{(1 - u^2)(1 - k^2u^2)},\ u(0) = 0,$$

where g and k *"sont deux paramètres arbitraires"*. After Jacobi, k is called the modulus of λ. With this, they denote by λ *"la fonction monodrome doublement périodique* $[\cdots]$ *définie par cette*

équation [⋯]*"*. If we remember that *"monodrome"* means path-independent, we must agree that our professors are using nothing but the construction of Abel and Jacobi. Just like them Briot and Bouquet proceed to establish a whole bunch of properties for λ. The proofs run slow and hard as they rely mainly on ingenious algebraic manipulations related to the addition formula.
From now on, g and k are constants which determine relevant features of the function λ. As it was usual in those times, we get a pair of fundamental periods by complex integration :

$$\omega = 4\int_0^1 \frac{du}{g\sqrt{(1-u^2)(1-k^2u^2)}},\ \omega' = -2\int_1^{1/k} \frac{du}{g\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Among other interesting properties, the function is odd with $\lambda(0) = 0$ and has simple poles ("*infinis simples*") at

$$p_{k,l} = k\frac{\omega}{2} + (2l+1)\frac{\omega'}{2}, \quad k,l \in \mathbb{Z}.$$

Since ω, ω' may not be orthogonal, the paths γ_n of integration are in general the boundaries of the parallelograms (*see* Figure 2)

$$\left\{\left(\frac{n}{2}+\frac{1}{4}\right)\omega s + n\omega' t : (s,t) \in [-1,1]\times[-1,1]\right\}.$$

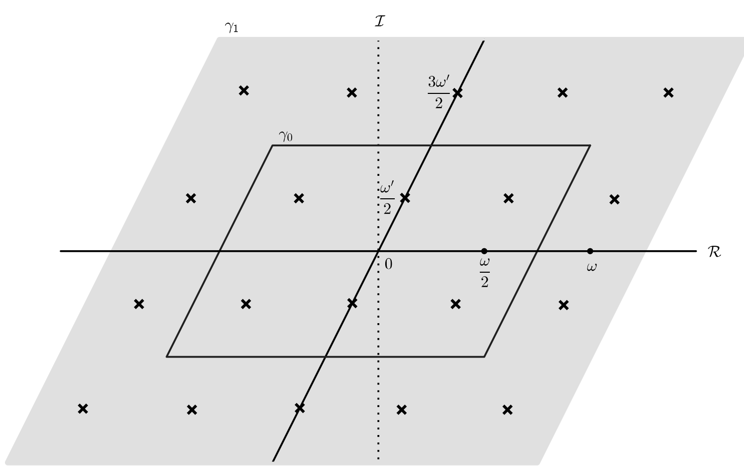

Figure 2: Paths γ_n for $\lambda(z)$. The poles are marked with ×.

As a result of its double periodicity, $|\lambda(\zeta)|$ is bounded along the paths with a bound independent of n. As λ is odd, the first integral in the remainder is equal to zero. Therefore, the Laurent series for $\lambda(z)$ is just but the series of the residues of $\tilde{\lambda}$ (with the opposite sign).

In this way, we need to calculate

$$\mathrm{res}_{p_{k,l}}\tilde{\lambda} = \frac{1}{2\pi i}\int_C \frac{\lambda(\zeta)}{\zeta - z}\, d\zeta,$$

along a small circle C about $p_{k,l}$. The substitution

$$\zeta = k\frac{\omega}{2} + (2l+1)\frac{\omega'}{2} + \vartheta$$

together with a clever change of variables imply

$$\lambda(\zeta) = (-1)^k \times \lambda\left(\frac{\omega'}{2} + \vartheta\right) = \frac{(-1)^k}{m\lambda(\vartheta)}^{\text{h}},$$

m being the modulus of λ[i]. Then, the residue becomes

$$\frac{1}{2\pi i}\int_D \frac{(-1)^k\, d\vartheta}{m(k\frac{\omega}{2} + (2l+1)\frac{\omega'}{2} + \vartheta - z)\lambda(\vartheta)} \ ,$$

where D is a small circular path centered at zero. Near the origin, $1/\lambda(\vartheta) = h(\vartheta)/\vartheta$[j], h holomorphic, and the Cauchy Integral Formula yields

$$-\text{res}_{p_{k,l}}\tilde{\lambda} = \frac{1}{m}\frac{(-1)^k}{z - k\frac{\omega}{2} - (2l+1)\frac{\omega'}{2}}.$$

At this point, we profit the series of $\csc z$ to obtain

$$\begin{aligned}\lambda(z) &= \frac{1}{m}\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{z - k\frac{\omega}{2} - (2l+1)\frac{\omega'}{2}}\\ &= \frac{1}{m}\sum_{l\in\mathbb{Z}}\frac{2\pi}{\omega}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{\left(\frac{2\pi z}{\omega} - (2l+1)\pi\rho\right) - k\pi},\ \rho = \frac{\omega'}{\omega},\\ &= \frac{2\pi}{m\omega}\sum_{l\in\mathbb{Z}}\frac{1}{\sin\left(\frac{2\pi z}{\omega} - (2l+1)\pi\rho\right)}.\end{aligned}$$

By elementary trigonometric identities, *cf. e. g.* Mongua (2013), we get at once

$$\lambda(z) = \frac{8\pi}{m\omega}\ \sin\frac{2\pi z}{\omega}\sum_{l\in\mathbb{N}\cup\{0\}}\frac{\cos(2l+1)\pi\rho}{\cos 2(2l+1)\pi\rho - \cos\frac{4\pi z}{\omega}}.$$

[h]This step is tricky. We read in Briot and Bouquet (1859, Livre III, Chapitre Premier, §87): "We set $z = \omega'/2 + z'$, $u = 1/kv$. The new function v satisfies the differential equation $dv/dz' = -g\sqrt{(1-v^2)(1-k^2v^2)}$ and vanishes at $z' = 0$, the initial value of the radical being ∓ 1. Then, we have $v = \pm\lambda(z')$ according to the radical sign. This provides the relation $\lambda(\omega'/2 + z') = \pm 1/k\lambda(z')$. If we replace $z' = \omega/4$ in this relation, it yields $\lambda(\omega'/2 + \omega/4) = \pm 1/k$. It is easy to determine the right sign: we know that, by rectilinear integration, the value value $1/k$ corresponds to value $z = \omega/4 + \omega'/2$. Therefore, $\lambda(\omega'/2 + \omega/4) = 1/k$. From this, we find that the initial radical value is -1 and so, the derivative dv/dz' has initial value equal to $+1$. From this we conclude the important relation

$$\lambda(\omega'/2 + z) = 1/k\lambda(z),$$

which does not have any analog in the sinus function. However, it has an analog in the tangent function".

[i]Letter $'k'$ has already been used as an exponent. So, we denote the constant modulus by m.

[j]λ has a zero at 0 and so, $1/\lambda$ has a pole at 0.

At last, by setting $q = \exp(2\pi\rho i)$, Euler formula provides

$$\begin{aligned}\lambda(z) &= \frac{8\pi}{m\omega}\,\sin\frac{2\pi z}{\omega}\sum_{l\in\mathbb{N}\cup\{0\}}\frac{q^{2l+1}+q^{-(2l+1)}}{q^{2(2l+1)}+q^{-2(2l+1)}-2\cos\frac{4\pi z}{\omega}}\\ &= \frac{8\pi q}{m\omega}\,\sin\frac{2\pi z}{\omega}\sum_{l\in\mathbb{N}\cup\{0\}}\frac{q^{2l}(1+q^{2l+1})}{1-2q^{2(2l+1)}\cos\frac{4\pi z}{\omega}+q^{4(2l+1)}}.\end{aligned}$$

Briot and Bouquet (1859) were happy to recognize in this expression one of the astonishing series that Jacobi (1829) had found with so much work in pages 36 to 39.

3. CONCLUDING REMARKS

The contemporary theory of Elliptic Functions was given birth without the help of the great theorems of Complex Analysis and, just very slowly along the 19th Century, the latter absorbed the first to the point of making it a part of itself. The process was not linear and was not free of disturbances. The old algebraic methods of the real variable exhibited their difficulties to deal with non-trivial proofs, while the theory of complex functions increasingly supplied vital and powerful theorems from which the results followed easily. The assimilation of Elliptic Functions within the framework of the Complex Analysis reveals that, sometimes, a mathematical theory may appropriate another theory as they change together over time.

In this state of affairs, the Residue Theorem did its bit in the overall process. At the time, this celebrated theorem provided a simple and effective way to develop the elliptic functions into rational fractions. As those exhausting proofs of Abel and Jacobi were simplified, the elliptic functions ceased to be the inverses of the elliptic integrals and turned more and more into doubly periodic meromorphic functions. Briot and Bouquet (1859) were excellent working mathematicians and so, they expressed this very well by saying: "*la double périodicité n'avait été reconnue d'une manière nette*".

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